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UPSCALING OF A SYSTEM OF DIFFUSION-REACTION EQUATIONS COUPLED WITH A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS ORIGINATING IN THE CONTEXT OF CRYSTAL DISSOLUTION AND PRECIPITATION OF MINERALS IN A POROUS MEDIUM

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Abstract. In this paper, we consider diffusion and reaction of mobile chemical species, and dissolution and precipitation of immobile species present inside a porous medium. The transport of mobile species in the pores is modeled by a system of semilinear parabolic partial differential equations. The reactions amongst the mobile species are assumed to be reversible, i.e. both forward and backward reactions are considered. These reversible reactions lead to highly nonlinear reaction rate terms on the right-hand side of the partial differential equations. This system of equations for the mobile species is complemented by flux boundary conditions at the outer boundary. Furthermore, the dissolution and precipitation of immobile species on the surface of the solid parts are modeled by mass action kinetics which lead to a nonlinear precipitation term and a multivalued dissolution term. The model is posed at the pore (micro) scale. The contribution of this paper is two-fold: first we show the existence of a unique positive global weak solution for the coupled systems and then we upscale (homogenize) the model from the micro scale to the macro scale. For the existence of solution, some regularization techniques, Schaefer's fixed point theorem and Lyapunov type arguments have been used whereas the concepts of two-scale convergence and periodic unfolding are used for the homogenization.

1 Introduction

Crystal dissolution and precipitation problems originate in chemical engineering, material sciences, soil mechanics etc., e.g., [Kna86, KvD96, KvDH95, vDP04]. The authors in [Kna86, KvD96, KvDH95, vDP04] have considered two mobile species in the pore space of a porous medium which precipitates to give one molecule of immobile species (blue part in figure 1.1) on the interfaces of the solid parts (gray part in figure 1.1) and vice-versa under the following reaction

$$n_1 X_1 + n_2 X_2 \leftrightharpoons X_{12}, \tag{1.1}$$

where X_1 and X_2 are the mobile species, X_{12} is the immobile species and n_1, n_2 are the number of molecules. At this point we would also like mention the work in [vN09] where the author has considered the similar reaction as (1.1) and obtained a homogenized model via asymptotic expansion. The novelty of this paper is to rigorously treat the multiple species problem. In the present work, the number of mobile and immobile species are assumed to be $I(>2)$ and $\bar{I}(>2)$, respectively. Here, the motivation for studying such multi species problems comes from the concrete carbonation, sulfate attack in sewer pipes or leaching of saline soil where the transport processes of chemical species are considered in a three dimensional pore scale model, cf. [FM12, PB08, Kna86]. The generalization to multiple species gives us highly nonlinear reaction rate terms both for the mobile species in the pore space and for the immobile species on the interfaces. The nonlinear reaction rates in the pore space originate from the reversible reactions amongst the mobile species. The modeling of precipitation and dissolution of crystals lead to multivalued reaction rate terms (see (1.9)-(1.12)). The complete model is a system of semilinear partial differential equations

(PDEs) coupled with nonlinear ordinary differential equations (ODEs) complemented by some inflow and outflow boundary conditions. In this paper we show the existence of a unique positive global weak solution of the system and we also obtain its macroscopic description via periodic homogenization. The model is introduced in section 1.1.

1.1 The Model

To fix the ideas, assume that Ω is a bounded domain (given porous medium) in $\mathbb{R}^n$ s.t. (such that) $\Omega := \Omega_p \cup \Omega_s$ and $\bar{\Omega}_s \cap \Omega_p = \emptyset$, where Ω_p and Ω_s are the (connected) pore space and the union of (disconnected) solid parts, respectively. Γ^* and $\partial\Omega$ denote the union of boundaries of solid parts and the outer boundary of Ω, respectively. Both Γ^* and $\partial\Omega$ are C^2. We assume that $\partial\Omega =: \Gamma_{in} \cup \Gamma_{out}$, where on Γ_{in} and Γ_{out} we prescribe the inflow and outflow boundary conditions. Let $Y := (0,1)^n \subset \mathbb{R}^n$ be the unit representative cell which is composed of a solid part Y_s with boundary Γ and a pore part Y_p s.t. $\bar{Y}_s \subset Y$, $Y_p = Y \setminus \bar{Y}_s$, $Y = \bar{Y}_s \cup Y_p$, and $\bar{Y}_s \cap \bar{Y}_p = \Gamma$. For $k, k_1, k_2 \in \mathbb{Z}^n$, we define $Y^k := Y + k$, $Y_\lambda^k := Y_\lambda + k$ for $\lambda \in \{p, s\}$, $\Gamma^k := \Gamma + k$ s.t. $\bar{Y}_s^k \subset \Omega$, $\bar{Y}_s^{k_1} \cap \bar{Y}_s^{k_2} = \emptyset$. Assume further that Ω is periodic (i.e. the solid parts in Ω are periodically distributed) and is covered by a finite union of the cells $\bar{Y}$. Let $0 < \varepsilon \ll 1$ and $\mathbb{Z}^n \supset w_k := \{k \in \mathbb{Z}^n : \varepsilon Y_k \subset \Omega\}$ s.t. $\Omega \subset \bigcup_{k\in w_k} \varepsilon \bar{Y}^k$, $\Omega_p \subset \bigcup_{k\in w_k} \varepsilon Y_p^k$, $\Omega_s \subset \bigcup_{k\in w_k} \varepsilon \bar{Y}_s^k$ and $\Gamma^* \subset \bigcup_{k\in w_k} \varepsilon \Gamma^k$. We define a connected set $\Omega_p^\varepsilon := \cup_{k\in w_k} \left\{\varepsilon Y_p^k : \varepsilon Y_p^k \subset \Omega\right\}$, a disconnected set $\Omega_s^\varepsilon := \cup_{k\in w_k} \left\{\varepsilon \bar{Y}_s^k : \varepsilon Y_s^k \subset \Omega\right\}$ and $\Gamma^\varepsilon := \cup_{k\in w_k} \left\{\varepsilon \Gamma^k : \varepsilon \Gamma^k \subset \Omega\right\}$, cf. figure 1.1. $S := [0,T)$ is the time interval for $T > 0$; dx, dy as volume elements in Ω and Y; $d\sigma_y$ and $d\sigma_x$ as surface elements on Γ and Γ^ε, respectively. Also,

$$\chi(\frac{x}{\varepsilon}) := \chi^\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \Omega_p^\varepsilon, \\ 0 & \text{for } x \in \Omega \setminus \Omega_p^\varepsilon. \end{cases} \tag{1.2}$$

Ω_p^ε is assumed to be filled by some fluid with a priori known Eulerian velocity $\vec{q^\varepsilon} = \vec{q^\varepsilon}(t,x)$, $(t,x) \in S \times \Omega_p^\varepsilon$, which satisfies

$$\nabla \cdot \vec{q^\varepsilon} = 0 \quad \text{in } \Omega_p^\varepsilon\,, \quad \vec{q^\varepsilon} = \vec{0} \text{ on } \Gamma^\varepsilon \tag{1.3}$$

with

$$-\vec{q^\varepsilon} \cdot \vec{n} \geq 0 \quad \text{on } \Gamma_{in}\,, \quad \vec{q^\varepsilon} \cdot \vec{n} \leq 0 \text{ on } \Gamma_{out}. \tag{1.4}$$

Let $I, \bar{I}, J \in \mathbb{N}$. We denote the mobile species by X^{mob} and minerals (immobile species) by X^{min}. For $i = 1, 2, ..., I$ and $k = 1, 2, ..., \bar{I}$, suppose that X_i^{mob} and X_k^{min} are present in Ω_p^ε and on Γ^ε, respectively. Let $u^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon, ..., u_I^\varepsilon)$ and $w^\varepsilon := (w_1^\varepsilon, w_2^\varepsilon, ..., w_{\bar{I}}^\varepsilon)$ denote the concentration vectors for the mobile and minerals, respectively. The mobile species are transported by diffusion (modelled by Fick's law) and advection in Ω_p^ε, then the flux vector is

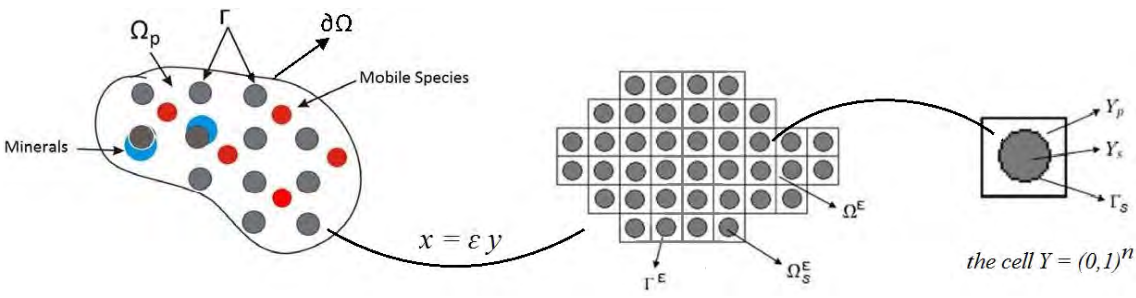

Figure 1.1: Periodic setting of a porous medium containing mobile species in Ω_p^ε and immobile species on Γ^ε.

$$\vec{j}_i^\varepsilon := -D_i \nabla u_i^\varepsilon + \vec{q^\varepsilon} u_i^\varepsilon, \tag{1.5}$$

where D_i is assumed to be a positive scalar and same for all the species, i.e. we set $D := D_i > 0 \ \forall i = 1,2,...,I$. We define the diffusive matrix D_{diff} as $D_{diff} = \underbrace{diag(D,D,...,D)}_{I-times}$.

The reactions amongst X^{mob} 's inside Ω_p^ε are assumed to be reversible and are modelled by mass action kinetics. For $1 \le i \le I$, $1 \le j \le J$ and $\tau_{ij}, \nu_{ij} \in \mathbb{N}_0$, the *j-th* reaction is given by

$$\tau_{1j} X_1^{mob} + \tau_{2j} X_2^{mob} + ... + \tau_{Ij} X_I^{mob} \rightleftharpoons \nu_{1j} X_1^{mob} + \nu_{2j} X_2^{mob} + ... + \nu_{Ij} X_I^{mob} \tag{1.6}$$

with reaction rate

$$R_j(u^\varepsilon) = k_j^f \prod_{m=1}^{I} (u_m^\varepsilon)^{\tau_{mj}} - k_j^b \prod_{m=1}^{I} (u_m^\varepsilon)^{\nu_{mj}}, \tag{1.7}$$

where $k_j^f (> 0)$ and $k_j^b (> 0)$ are forward and backward reaction rate factors, respectively. Thus the reaction rate term for the *i-th* mobile species is

$$\mathcal{S}R(u^\varepsilon)_i = \sum_{j=1}^{J} s_{ij} R_j(u^\varepsilon) = \sum_{j=1}^{J} s_{ij} \left(k_j^f \prod_{m=1}^{I} (u_m^\varepsilon)^{\tau_{mj}} - k_j^b \prod_{m=1}^{I} (u_m^\varepsilon)^{\nu_{mj}} \right) \tag{1.8}$$

where $\mathcal{S} := (s_{ij})_{1 \le i \le I, 1 \le j \le J}$ is the stoichiometric matrix wth entries $s_{ij} := \nu_{ij} - \tau_{ij}$. Besides the reactions (1.6) due to dissolution on Γ^ε, X^{min} 's dissolve to give X^{mob} 's and the reverse reaction is also possible, i.e. following mineral (dissolution-precipitation) reaction is considered

$$\sigma_{ik} X_i^{mob} + \sigma_{jk} X_j^{mob} \rightleftharpoons X_k^{min}, \tag{1.9}$$

where $i,j = 1,2,...,I$, $i \ne j$ and $k = 1,2...,\bar{I}$. The precipitation rate can be modeled via mass action kinetics, namely, $(R_p(u^\varepsilon))_k := k_k^p \prod_{m=1}^{I} (u_m^\varepsilon)^{\sigma_{mk}}$, where

$$\sigma_{mk} \;=\; \begin{cases} 1 & \text{if } m=i \text{ or } m=j, \\ 0 & \text{if } m \neq i \text{ or } m \neq j, \end{cases} \tag{1.10}$$

for $k=1,2,...,\bar{I}$ and $m=1,2,...,I$. It is well-known that minerals have "constant activity", i.e. the dissolution rate is constant, if the mineral is present. If the mineral is absent, dissolution rate can not be stronger than precipitation in order to maintain the non-negativity of the surface concentration, cf. [KvDH95, vDP04]. This leads to a multivalued dissolution term $(R_d(w^\varepsilon))_k \in k_k^d \psi(w_k^\varepsilon)$, where

$$\psi(c) \;=\; \begin{cases} 0 & \text{if } c<0, \\ [0,1] & \text{if } c=0, \\ 1 & \text{if } c>0. \end{cases} \tag{1.11}$$

Here k_k^p, $k_k^d > 0$ are the reaction rate factors. The $k-th$ reaction rate term for minerals on Γ^ε is

$$R(u^\varepsilon, w^\varepsilon)_k = \underbrace{k_k^p \prod_{m=1}^{I} (u_m^\varepsilon)^{\sigma_{mk}} -}_{precipitation} \underbrace{k_k^d z_k^\varepsilon}_{dissolution}, \; z_k^\varepsilon \in \psi(w_k^\varepsilon),\; k=1,2,...,\bar{I}. \tag{1.12}$$

This contributes to the flux for u_i^ε on Γ^ε as

$$D_{diff}\,\nabla u_i^\varepsilon \cdot \vec{n} = \varepsilon \sum_{k=1}^{\bar{I}} \sigma_{ik} \frac{\partial w_k^\varepsilon}{\partial t} = \varepsilon \sum_{k=1}^{\bar{I}} \sigma_{ik} R(u^\varepsilon, w^\varepsilon)_k = \varepsilon \sum_{k=1}^{\bar{I}} \sigma_{ik} \left(k_k^p \prod_{m=1}^{I} (u_m^\varepsilon)^{\sigma_{mk}} - k_k^d z_k^\varepsilon \right), \tag{1.13}$$

where $z_k^\varepsilon \in \psi(w_k^\varepsilon)$, σ 's are defined in (1.10) and $i=1,2,...,I$. Therefore the required micro (pore) scale model is given by

$$\frac{\partial u^\varepsilon}{\partial t} - \nabla\cdot(D_{diff}\nabla u^\varepsilon - \vec{q^\varepsilon} u^\varepsilon) = \mathcal{S}R(u^\varepsilon) \qquad \text{in } S\times\Omega_p^\varepsilon, \tag{1.14a}$$

$$-\left(D_{diff}\nabla u^\varepsilon - \vec{q^\varepsilon} u^\varepsilon\right)\cdot\vec{n} = d \qquad \text{on } S\times\Gamma_{in}, \tag{1.14b}$$

$$-\left(D_{diff}\nabla u^\varepsilon - \vec{q^\varepsilon} u^\varepsilon\right)\cdot\vec{n} = 0 \qquad \text{on } S\times\Gamma_{out}, \tag{1.14c}$$

$$-\left(D_{diff}\nabla u_i^\varepsilon - \vec{q^\varepsilon} u_i^\varepsilon\right)\cdot\vec{n} = \varepsilon \sum_{k=1}^{\bar{I}} \sigma_{ik} \frac{\partial w_k^\varepsilon}{\partial t} \qquad \text{on } S\times\Gamma^\varepsilon, \tag{1.14d}$$

$$u^\varepsilon(0,x) = u_0(x) \qquad \text{in } \Omega_p^\varepsilon, \tag{1.14e}$$

$$\frac{\partial w^\varepsilon}{\partial t} = R(u^\varepsilon, w^\varepsilon) \qquad \text{on } S\times\Gamma^\varepsilon, \tag{1.14f}$$

$$z^\varepsilon \in \psi(w^\varepsilon) \qquad \text{on } S\times\Gamma^\varepsilon, \tag{1.14g}$$

$$w^\varepsilon(0,x) = w_0(x) \qquad \text{on } \Gamma^\varepsilon, \tag{1.14h}$$

where $i = 1,2,...,I$ and the velocity $\vec{q^\varepsilon}$ satisfies (1.3)-(1.4). Here $d := d(t,x) \le 0$ componentwise which signifies the inflow on Γ_{in} whereas (1.14c) shows the outflow on Γ_{out}. We denote the problem (1.14a)-(1.14h) by (P^ε). Note that $R(u^\varepsilon)$ in (1.14a) and $R(u^\varepsilon, w^\varepsilon)$ in (1.14f) are two-different types of reaction rate terms.

Remark 1.1. *We note that* $\lim_{\varepsilon\to 0}\varepsilon|\Gamma^\varepsilon| = |\Gamma|\frac{|\Omega|}{|Y|}$. *Since the surface area of* Γ^ε *increases proportionally to* $\frac{1}{\varepsilon}$, *i.e.* $|\Gamma^\varepsilon| \to \infty$ *as* $\varepsilon \to 0$, *The appearance of* ε *in the boundary flux in* (1.14d) *allows us to control this growth. We would also like to mention that for technical reasons our analysis required* ε *independent* d *and* $\varepsilon, x, u^\varepsilon$ *independent (i.e. species-independent) diffusion co-efficients* D.

In the next section we define the neccessary function spaces and collect some mathematical tools to investigate the above problem in sections 3 and 4.

2 Mathematical Preliminaries

2.1 Function Spaces

Define $\mathbb{R}_0^+ := \{x \in \mathbb{R} : x \ge 0\}$. Let $\theta \in [0,1]$, $p > n+2$ and q be s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Note that p as the suffix in Ω_p^ε is used to denote the pore space and should not be confused with the exponents of the function spaces defined here. Assume that $\Xi \in \{\Omega, \Omega_p^\varepsilon\}$, then as usual $L^p(\Xi)$, $H^{1,p}(\Xi)$, $C^\theta(\bar{\Xi})$, $(\cdot,\cdot)_{\theta,p}$ and $[\cdot,\cdot]_\theta$ are the Lebesgue, Sobolev, Hölder, real- and complex-interpolation spaces respectively endowed with their standard norms. $C^\gamma_{per}(Y)$ denotes the set of all *Y-periodic* γ*-times* continuously differentiable functions in y for $\gamma \in \mathbb{N}$. In particular, $C_{per}(Y)$ is the space of all the *Y-periodic* continuous function in y. For a Banach space X, X^* denotes its dual and the duality pairing is denoted by $\langle . \, , \, . \rangle_{X^*\times X}$. The symbols $\hookrightarrow$, $\hookrightarrow\hookrightarrow$ and $\overset{d}{\hookrightarrow}$ denote the continuous, compact and dense embeddings, respectively. We define $L^p(\Xi) \hookrightarrow H^{1,q}(\Xi)^*$ as

$$\langle f, v\rangle_{H^{1,q}(\Xi)^*\times H^{1,q}(\Xi)} = \langle f, v\rangle_{L^p(\Xi)\times L^q(\Xi)} := \int_\Xi f v \, dx \text{ for } f \in L^p(\Xi),\ v \in H^{1,q}(\Xi). \tag{2.1}$$

We introduce the $L^p(\Gamma^\varepsilon)$-$L^q(\Gamma^\varepsilon)$ duality as

$$\langle \zeta_1, \zeta_2\rangle := \varepsilon \int_{\Gamma^\varepsilon} \zeta_1\zeta_2 \, d\sigma_x \text{ for } \zeta_1 \in L^p(\Gamma^\varepsilon),\ \zeta_2 \in L^q(\Gamma^\varepsilon) \tag{2.2}$$

and the space $L^p(\Gamma^\varepsilon)$ is furnished with the norm

$$||\zeta||^p_{L^p(\Gamma^\varepsilon)} := \varepsilon \int_{\Gamma^\varepsilon} |\zeta|^p \, d\sigma_x \text{ and } ||\zeta||_{L^\infty(\Gamma^\varepsilon)} := \operatorname*{ess\,sup}_{x\in\Gamma^\varepsilon}|\zeta(x)|. \tag{2.3}$$

The *Sobolev-Bochner spaces* used here are:

$$\mathcal{F}_p(\Xi) := \left\{ u \in L^p(S; H^{1,p}(\Xi)) : \frac{\partial u}{\partial t} \in L^p(S; H^{1,q}(\Xi)^*) \right\}$$
$$:= H^{1,p}(S; H^{1,q}(\Xi)^*) \cap L^p(S; H^{1,p}(\Xi)), \tag{2.4a}$$
$$\mathcal{M}_p(\Gamma^\varepsilon) := \left\{ w \in L^p(S; L^p(\Gamma^\varepsilon)) : \frac{\partial w}{\partial t} \in L^p(S; L^p(\Gamma^\varepsilon)) \right\} = H^{1,p}(S; L^p(\Gamma^\varepsilon)), \tag{2.4b}$$
$$\mathcal{N}_p(\Gamma^\varepsilon) := L^p(S; L^p(\Gamma^\varepsilon)) = L^p(S \times \Gamma^\varepsilon), \tag{2.4c}$$
$$\mathcal{X}_p(\Xi) := (H^{1,q}(\Xi)^*, H^{1,p}(\Xi))_{1-\frac{1}{p},p}, \tag{2.4d}$$

where $(H^{1,q}(\Xi)^*, H^{1,p}(\Xi))_{1-\frac{1}{p},p}$ is the real-interpolation space between $H^{1,q}(\Xi)^*$ and $H^{1,p}(\Xi)$, and $\frac{d}{dt}$ is the distributional time derivative. These spaces are endowed with the norms:

$$||u||_{\mathcal{F}_p(\Xi)} := ||u||_{L^p(S;H^{1,p}(\Xi))} + \left|\left| \frac{\partial u}{\partial t} \right|\right|_{L^p(S;H^{1,q}(\Xi)^*)}, \tag{2.5a}$$
$$||w||_{\mathcal{M}_p(\Gamma^\varepsilon)} := ||w||_{L^p(S;L^p(\Gamma^\varepsilon))} + \left|\left| \frac{\partial w}{\partial t} \right|\right|_{L^p(S;L^p(\Gamma^\varepsilon))}, \tag{2.5b}$$
$$||\zeta||_{\mathcal{N}_p(\Gamma^\varepsilon)} := \left[\varepsilon \int_0^T \int_{\Gamma^\varepsilon} |\zeta|^p \, d\sigma_x \, dt \right]^{\frac{1}{p}} \quad \text{for } p < \infty,$$
$$:= \operatorname*{ess\,sup}_{(t,x) \in S \times \Gamma^\varepsilon} |\zeta(t,x)| \quad \text{for } p = \infty. \tag{2.5c}$$

and the norm on $\mathcal{X}_p$ can be defined as Definition 1.2.8 in [Lun95]. Let $I \in \mathbb{N}$. For $a, b \in \mathbb{R}^I$, the scalar product and Euclidean norm are given by $\langle a, b \rangle_I := \sum_{i=1}^I a_i b_i$ and $|a|_I := \sqrt{\sum_{i=1}^I |a_i|^2}$, respectively. We introduce the norms on the vector-valued function spaces. Assume $u : \Omega \to \mathbb{R}^I$. We define

$$[L^p(\Xi)]^I := \underbrace{L^p(\Xi) \times L^p(\Xi) \times \ldots \times L^p(\Xi)}_{I\text{-times}} \tag{2.6}$$

and for $u \in [L^p(\Xi)]^I$,

$$|||u|||_{[L^p(\Xi)]^I} := \left[\sum_{i=1}^I ||u_i||^p_{L^p(\Xi)} \right]^{\frac{1}{p}} \quad \text{for } p < \infty,$$
$$:= \max_{1 \le i \le I} ||u_i||_{L^\infty(\Xi)} \quad \text{for } p = \infty. \tag{2.7a}$$

We also define

$$\mathcal{F}_p^I(\Xi) := \left[H^{1,p}(S;H^{1,q}(\Xi)^*) \cap L^p(S;H^{1,p}(\Xi))\right]^I, \tag{2.7b}$$

$$\mathcal{X}_p^I(\Xi) := \left[(H^{1,q}(\Xi)^*, H^{1,p}(\Xi))_{1-\frac{1}{p},p}\right]^I, \tag{2.7c}$$

$$\mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon) := \left[H^{1,p}(S;L^p(\Gamma^\varepsilon))\right]^{\bar{I}}, \tag{2.7d}$$

$$\mathcal{N}_p^{\bar{I}}(\Gamma^\varepsilon) := [L^p(S \times \Gamma^\varepsilon)]^{\bar{I}}, \tag{2.7e}$$

$$\mathcal{M}_p^{\bar{I}}(\Gamma) := H^{1,p}(S;L^p(\Omega \times \Gamma))^{\bar{I}}, \tag{2.7f}$$

$$\mathcal{N}_p^{\bar{I}}(\Gamma) := L^p(S \times \Omega \times \Gamma)^{\bar{I}}. \tag{2.7g}$$

The norms on $\left[H^{1,p}(\Xi)\right]^I$, $[H^{1,q}(\Xi)^*]^I$, $\mathcal{F}_p^I(\Xi)$, $\mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)$, $\mathcal{M}_p^{\bar{I}}(\Gamma)$, $\mathcal{N}^{\bar{I}}(\Gamma^\varepsilon)$, $\mathcal{N}^{\bar{I}}(\Gamma)$ and $\mathcal{X}_p^I(\Xi)$ are defined in the similar fashion as in (2.6) and (2.7a). The vector-valued duality pairing is defined by $\langle \alpha, \beta\rangle_{[X^*]^I \times X^I} := \sum_{i=1}^I \langle \alpha_i, \beta_i\rangle_{X^* \times X}$ for $\alpha \in [X^*]^I$ and $\beta \in X^I$. By $\xrightarrow{2}$, $\xrightarrow{w}$ and $\to$ we denote the two-scale, weak and strong convergence of a sequence respectively. Finally, throughout this paper C and C_i denote the generic non-negative constants but may be different at different parts of the inequalities and, λ and Λ_λ are the constants of the *Young's inequality*. To analyze the problem (1.14a)-(1.14h), we make following assumptions: (i) $p > n+2$, (ii) $u_0, w_0 \geq 0$ componentwise, (iii) $u_{0_i} \in \mathcal{X}_p(\Omega_p^\varepsilon)$, $w_{0_k} \in L^\infty(\Omega)$ for $i = 1,2,...,I$ and $k = 1,2,...,\bar{I}$ s.t. $\sup_{\varepsilon>0} ||u_0||_{\mathcal{X}_p^I(\Omega_p^\varepsilon)}, \sup_{\varepsilon>0} ||w_0||_{L^\infty(\Omega)^{\bar{I}}} < \infty$, (iv) All the reactions amongst mobile species are linearly independent s.t. the stoichiometric matrix $\mathcal{S} = (s_{ij})_{\substack{1\leq i\leq I\\ 1\leq j\leq J}}$ has the maximal column rank, i.e., $rank(\mathcal{S}) = J$, (v) $\vec{q^\varepsilon}$ satisfies (1.3)-(1.4), $\vec{q^\varepsilon} \in L^\infty(S \times \Omega)$ and $\vec{q^\varepsilon} \cdot \vec{n} \in L^\infty(S \times \Gamma_{in})$. Define $Q := \sup_{\varepsilon>0} \left|\left|\vec{q^\varepsilon}\right|\right|_{L^\infty(S\times\Omega)} < \infty$ and $Q_1 := \sup_{\varepsilon>0} \left|\left|\vec{q^\varepsilon} \cdot \vec{n}\right|\right|_{L^\infty(S\times\Gamma_{in})} < \infty$, (vi) $d_i \in L^\infty(S \times \Gamma_{in})$ and $d_i \leq 0$ for all $i = 1,2,...,I$.

Definition 2.1 (Weak solution)**.** *A triple* $(u^\varepsilon, w^\varepsilon, z^\varepsilon) \in \mathcal{F}_p^I(\Omega_p^\varepsilon) \times \mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon) \times \mathcal{N}_\infty^{\bar{I}}(\Gamma^\varepsilon)$ *is said to be a weak solution of the problem* (1.14a)-(1.14h) *if it satisfies* $(u^\varepsilon(0,x), w^\varepsilon(0,x)) = (u_0(x), w_0(x))$ *and*

$$\langle \frac{\partial u^\varepsilon}{\partial t}, \phi\rangle + \sum_{i=1}^I \int_{\Omega_p^\varepsilon} \left(D_{diff} \nabla u_i^\varepsilon - \vec{q^\varepsilon} u_i^\varepsilon\right) \cdot \nabla \phi_i \, dx + \sum_{i=1}^I \int_{\Gamma_{in}} d_i \phi_i \, ds$$
$$+ \sum_{i=1}^I \varepsilon \int_{\Gamma^\varepsilon} \sum_{k=1}^{\bar{I}} \sigma_{ik} \frac{\partial w_k^\varepsilon}{\partial t} \phi_i \, d\sigma_x = \langle \mathcal{S}R(u^\varepsilon), \phi\rangle, \tag{2.8a}$$

$$\sum_{k=1}^{\bar{I}} \int_{\Gamma^\varepsilon} \frac{\partial w_k^\varepsilon}{\partial t} \zeta_k \, d\sigma_x = \sum_{k=1}^{\bar{I}} \int_{\Gamma^\varepsilon} R(u^\varepsilon, w^\varepsilon)_k \zeta_k \, d\sigma_x, \tag{2.8b}$$

$$z^\varepsilon \in \psi(w^\varepsilon), \tag{2.8c}$$

for a.e. t *and for all* $(\phi, \zeta) \in H^{1,q}(\Omega_p^\varepsilon)^I \times L^q(\Gamma^\varepsilon)^{\bar{I}}$.

We now state the two main theorems:

Theorem 2.1. *Suppose that the assumptions (i)-(vi) are satisfied. Then there exists an unique positive global weak solution* $(u^\varepsilon, w^\varepsilon, z^\varepsilon) \in \mathcal{F}_p^I(\Omega_p^\varepsilon) \times \mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon) \times \mathcal{N}_\infty^{\bar{I}}(\Gamma^\varepsilon)$ *of the problem* (P^ε).

Theorem 2.2. *Asssume that (i)-(vi) hold. Then there exists* $(u,v,w) \in \mathcal{F}_p^I(\Omega) \times \mathcal{M}_p^{\bar{I}}(\Gamma) \times \mathcal{N}_\infty^{\bar{I}}(\Gamma)$ *which satisfies the homogenized problem (4.23a)-(4.23g) of* (P^ε).

3 Proof of theorem 2.1

To obtain the existence of solution of (P^ε) we first regularize ψ in (1.14g). For a parameter $\delta > 0$ (cf. [vDP04, Hof10])

$$\psi_\delta(w_\delta^\varepsilon) := \begin{cases} 0 & \text{if } w_\delta^\varepsilon \leq 0, \\ \dfrac{w_\delta^\varepsilon}{\delta} & \text{if } 0 < w_\delta^\varepsilon < \delta, \\ 1 & \text{if } w_\delta^\varepsilon \geq \delta, \end{cases} \tag{3.1}$$

where $0 \ll \varepsilon < \delta \ll 1$. We denote this regularized problem by (P_δ^ε) and is given by

$$\frac{\partial u_\delta^\varepsilon}{\partial t} - \nabla \cdot (D_{diff} \nabla u_\delta^\varepsilon - \vec{q}^\varepsilon u_\delta^\varepsilon) = \mathcal{S}R(u_\delta^\varepsilon) \quad \text{in } S \times \Omega_p^\varepsilon, \tag{3.2a}$$

$$-\left(D_{diff} \nabla u_\delta^\varepsilon - \vec{q}^\varepsilon u_\delta^\varepsilon\right) \cdot \vec{n} = d \quad \text{on } S \times \Gamma_{in}, \tag{3.2b}$$

$$-(D_{diff} \nabla u_\delta^\varepsilon - \vec{q}^\varepsilon u_\delta^\varepsilon) \cdot \vec{n} = 0 \quad \text{on } S \times \Gamma_{out}, \tag{3.2c}$$

$$-(D_{diff} \nabla u_{\delta_i}^\varepsilon - \vec{q}^\varepsilon u_{\delta_i}^\varepsilon) \cdot \vec{n} = \varepsilon \sum_{k=1}^{\bar{I}} \sigma_{ik} \frac{\partial w_{\delta_k}^\varepsilon}{\partial t} \quad \text{on } S \times \Gamma^\varepsilon, \tag{3.2d}$$

$$u_\delta^\varepsilon(0,x) = u_0(x) \quad \text{in } \Omega_p^\varepsilon, \tag{3.2e}$$

$$\frac{\partial w_\delta^\varepsilon}{\partial t} = \left(R_p(u_\delta^\varepsilon) - k^d \psi_\delta(w_\delta^\varepsilon)\right) \quad \text{on } S \times \Gamma^\varepsilon, \tag{3.2f}$$

$$w_\delta^\varepsilon(0,x) = w_0(x) \quad \text{on } \Gamma^\varepsilon, \tag{3.2g}$$

for $i = 1,2,...,I$. The idea here is to first show the existence of solution of (P_δ^ε). Second step would be to let the regularization parameter $\delta \to 0$. In the third step we let the scale parameter $\varepsilon \to 0$ to obtain the homogenized problem. For technicality we modify the rate terms in (3.2a) and (3.2f) by introducing new rate function $\bar{R} : \mathbb{R}^I \to \mathbb{R}^J$ and $\bar{R}_p : \mathbb{R}^I \to \mathbb{R}^{\bar{I}}$ as

$$\bar{R}(u_\delta^\varepsilon) := R(u_\delta^{\varepsilon^+}) \text{ and } \bar{R}_p(u_\delta^\varepsilon) := R_p(u_\delta^{\varepsilon^+}), \tag{3.3}$$

where $u^\varepsilon = u_\delta^{\varepsilon^+} - u_\delta^{\varepsilon^-}$, $u_\delta^{\varepsilon^+} := \max(u_\delta^\varepsilon, 0)$ and $u_\delta^{\varepsilon^-} := \max(-u_\delta^\varepsilon, 0)$. We replace the reaction rate terms $\mathcal{S}R(u_\delta^\varepsilon)$ and $R_p(u_\delta^\varepsilon)$ in (P_δ^ε) by $\mathcal{S}\bar{R}(u_\delta^\varepsilon)$ and $\bar{R}_p(u_\delta^\varepsilon)$, respectively, and denote this modified problem by $(P_\delta^{\varepsilon^+})$. At first we show the existence of a global solution of $(P_\delta^{\varepsilon^+})$ and since it will be shown in lemma 3.1 that the solution of $(P_\delta^{\varepsilon^+})$ is nonnegative, it also solves (P_δ^{ε}). We commence our investigation with the positivity of solutions of $(P_\delta^{\varepsilon^+})$. The problem $(P_\delta^{\varepsilon^+})$ is

$$\frac{\partial u_\delta^\varepsilon}{\partial t} - \nabla\cdot(D_{diff}\nabla u_\delta^\varepsilon - \vec{q^\varepsilon} u_\delta^\varepsilon) = \mathcal{S}\bar{R}(u_\delta^\varepsilon) \qquad \text{in } S\times\Omega_p^\varepsilon, \tag{3.4a}$$

$$-\left(D_{diff}\nabla u_\delta^\varepsilon - \vec{q^\varepsilon} u_\delta^\varepsilon\right)\cdot\vec{n} = d \qquad \text{on } S\times\Gamma_{in}, \tag{3.4b}$$

$$-(D_{diff}\nabla u_\delta^\varepsilon - \vec{q^\varepsilon} u_\delta^\varepsilon)\cdot\vec{n} = 0 \qquad \text{on } S\times\Gamma_{out}, \tag{3.4c}$$

$$-(D_{diff}\nabla u_{\delta_i}^\varepsilon - \vec{q^\varepsilon} u_{\delta_i}^\varepsilon)\cdot\vec{n} = \varepsilon\sum_{k=1}^{\bar{I}}\sigma_{ik}\frac{\partial w_{\delta_k}^\varepsilon}{\partial t} \qquad \text{on } S\times\Gamma^\varepsilon, \tag{3.4d}$$

$$u_\delta^\varepsilon(0,x) = u_0(x) \qquad \text{in } \Omega_p^\varepsilon, \tag{3.4e}$$

$$\frac{\partial w_\delta^\varepsilon}{\partial t} = \left(\bar{R}_p(u_\delta^\varepsilon) - k^d\psi_\delta(w_\delta^\varepsilon)\right) \qquad \text{on } S\times\Gamma^\varepsilon, \tag{3.4f}$$

$$w_\delta^\varepsilon(0,x) = w_0(x) \qquad \text{on } \Gamma^\varepsilon, \tag{3.4g}$$

for $i = 1, 2, ..., I$.

Lemma 3.1. *Let the assumptions (i)-(vi) be satisfied and* $(u_\delta^\varepsilon, w_\delta^\varepsilon) \in \mathcal{F}_p^I(\Omega_p^\varepsilon)\times\mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)$ *be a solution of* $(P_\delta^{\varepsilon^+})$*. Then* $u_\delta^\varepsilon \geq 0$ *componentwise in* $S\times\Omega_p^\varepsilon$ *and* $w_\delta^\varepsilon \geq 0$ *componentwise on* $S\times\Gamma^\varepsilon$*.*

Proof. Since $p > n+2$ and $u_\delta^\varepsilon(t) \in H^{1,p}(\Omega_p^\varepsilon)^I$, $u_\delta^\varepsilon(t) \in H^{1,q}(\Omega_p^\varepsilon)^I$ for a.e. $t \in S$ and consequently we have $u_\delta^{\varepsilon^-}(t), u_\delta^{\varepsilon^+}(t) \in H^{1,q}(\Omega_p^\varepsilon)^I$ for a.e. $t \in S$. Testing (3.4a) by $-u_\delta^{\varepsilon^-}(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left|\left|\left|u_\delta^{\varepsilon^-}(t)\right|\right|\right|^2_{L^2(\Omega_p^\varepsilon)^I} + D\left|\left|\left|\nabla u_\delta^{\varepsilon^-}(t)\right|\right|\right|^2_{L^2(\Omega_p^\varepsilon)^I} - \sum_{i=1}^{I}\int_{\Omega_p^\varepsilon}\vec{q^\varepsilon}\cdot\nabla u_{\delta_i}^{\varepsilon^-}(t)u_{\delta_i}^{\varepsilon^-}(t)\,dx$$
$$-\sum_{i=1}^{I}\int_{\Gamma_{in}} d_i u_{\delta_i}^{\varepsilon^-}(t)\,ds + \sum_{i=1}^{I}\int_{\Gamma_{out}}\vec{q^\varepsilon}\cdot\vec{n}\left|u_{\delta_i}^{\varepsilon^-}(t)\right|^2 ds$$
$$-\sum_{i=}^{I}\varepsilon\int_{\Gamma^\varepsilon}\left(\sum_{k=1}^{\bar{I}}\sigma_{ik}\frac{\partial w_{\delta_k}^\varepsilon}{\partial t}\right)u_{\delta_i}^{\varepsilon^-}\,d\sigma_x = -\sum_{i=1}^{I}\int_{\Omega_p^\varepsilon}\mathcal{S}\bar{R}(u_\delta^\varepsilon(t))_i u_{\delta_i}^{\varepsilon-}(t)\,dx,$$

$$\frac{1}{2}\frac{d}{dt}\left|\left|\left|u_{\delta_i}^{\varepsilon^-}(t)\right|\right|\right|^2_{L^2(\Omega)^I} + D\left|\left|\left|\nabla u_{\delta_i}^{\varepsilon^-}(t)\right|\right|\right|^2_{L^2(\Omega)^I} \underbrace{+\sum_{i=1}^{I}\int_{\Gamma_{in}} -d_i u_{\delta_i}^{\varepsilon^-}(t)\,ds}_{\geq 0}$$
$$\underbrace{+\sum_{i=1}^{I}\int_{\Gamma_{out}}\vec{q^\varepsilon}\cdot\vec{n}\left|u_{\delta_i}^{\varepsilon^-}(t)\right|^2 ds}_{\geq 0}\ \underbrace{+\sum_{i=1}^{I}\sum_{k=1}^{\bar{I}}\varepsilon\sigma_{ik}\int_{\Gamma^\varepsilon}\left(\bar{R}_p(u_\delta^\varepsilon)_k - k_k^d\psi_\delta(w_\delta^\varepsilon)_k\right)u_{\delta_i}^{\varepsilon^-}(t)\,d\sigma_x}_{\geq 0}$$

$$\leq \frac{Q^2}{2D}\sum_{i=1}^{I}\int_{\Omega_p^\varepsilon}\left|u_{\delta_i}^{\varepsilon^-}(t)\right|^2 dx + \sum_{i=1}^{I}\frac{D}{2}\int_{\Omega_p^\varepsilon}\left|\nabla u_{\delta_i}^{\varepsilon^-}(t)\right|^2 dx - \sum_{i=1}^{I}\int_{\Omega_p^\varepsilon}\mathcal{S}\bar{R}(u_\delta^\varepsilon(t))_i u_{\delta_i}^{\varepsilon^-}(t)\,dx.$$

Note that $-\sum_{i=1}^{I}\varepsilon\int_{\Gamma^\varepsilon}\sum_{k=1}^{\bar{I}}\sigma_{ik}\left(\bar{R}_p(u_\delta^\varepsilon)_k - k_k^d\psi_\delta(w_\delta^\varepsilon)_k\right)u_{\delta_i}^{\varepsilon^-}\,d\sigma_x = -\sum_{i=1}^{I}\varepsilon\int_{\Gamma^\varepsilon}\sum_{k=1}^{\bar{I}}\sigma_{ik}\bar{R}_p(u_\delta^\varepsilon)_k u_{\delta_i}^{\varepsilon^-}\,d\sigma_x +$ $\sum_{i=1}^{I}\varepsilon\int_{\Gamma^\varepsilon}\sum_{k=1}^{\bar{I}}\sigma_{ik}k_k^d\psi_\delta(w_\delta^\varepsilon)_k u_{\delta_i}^{\varepsilon^-}\,d\sigma_x$ where the second term is non-negative. Due to (3.3), the term $-\sum_{i=1}^{I}\int_{\Omega_p^\varepsilon}\mathcal{S}\bar{R}(u_\delta^\varepsilon(t))_i u_{\delta_i}^{\varepsilon^-}(t)\,dx = 0$. This gives

$$\frac{1}{2}\frac{d}{dt}\left|\left|\left|u_\delta^{\varepsilon^-}(t)\right|\right|\right|^2_{L^2(\Omega_p^\varepsilon)^I} \leq \frac{Q^2}{2D}\left|\left|\left|u_\delta^{\varepsilon^-}(t)\right|\right|\right|^2_{L^2(\Omega_p^\varepsilon)^I}. \tag{3.5}$$

Since $u_{\delta_i}^\varepsilon(0) = u_i(0) \geq 0$ for all i. By Gronwall's inequality it follows $u_\delta^\varepsilon(t,x) \geq 0$ componentwise for a.e. in $S\times\Omega_p^\varepsilon$. To show the positivity of w_δ^ε, we test (3.4f) by $-w_\delta^{\varepsilon^-}(t)$ which gives

$$\frac{1}{2}\sum_{k=1}^{\bar{I}}\frac{d}{dt}\int_{\Gamma^\varepsilon}|w_{\delta_k}^{\varepsilon^-}(t)|^2\,d\sigma_x = -\sum_{k=1}^{\bar{I}}\int_{\Gamma^\varepsilon}\bar{R}_p(u_\delta^\varepsilon(t))_k w_{\delta_k}^{\varepsilon^-}(t)\,d\sigma_x + \sum_{k=1}^{\bar{I}}\int_{\Gamma^\varepsilon}k_k^d\psi_\delta(w_\delta^\varepsilon)_k w_{\delta_k}^{\varepsilon^-}(t)\,d\sigma_x,$$

$$\frac{d}{dt}|||w_\delta^{\varepsilon^-}(t)|||^2_{L^2(\Gamma^\varepsilon)^{\bar{I}}} \leq 2\varepsilon\sum_{k=1}^{\bar{I}}\int_{\Gamma^\varepsilon}k_k^d\psi_\delta(w_\delta^\varepsilon)_k w_{\delta_k}^{\varepsilon^-}(t)\,d\sigma_x = 0. \tag{3.6}$$

Since $w_\delta^\varepsilon(0) \geq 0$, it follows from (3.6) that $w_\delta^{\varepsilon^-}(t,x) = 0$, i.e. $w_\delta^\varepsilon(t,x) \geq 0$ for a.e. $(t,x) \in S\times\Gamma^\varepsilon$.

□

Lemma 3.2. *For a given* $\hat{u}_\delta^\varepsilon \in \mathcal{F}_p^I(\Omega_p^\varepsilon)$, *there exists a unique positive global weak solution* $w_\delta^\varepsilon \in \mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)$ *of* (3.4f)-(3.4g).

Proof. Note that $\bar{R}_p(\hat{u}_\delta^\varepsilon)$ is constant in w_δ^ε and $\psi_\delta(.)$ is *Lipschitz* in w_δ^ε, i.e. $R(u_\delta^\varepsilon, w_\delta^\varepsilon)$ is Lipschitz continuous w.r.t. w_δ^ε. Therefore for any arbitrary $x \in \Omega_p^\varepsilon$, by *Picard-Lindelöf theorem* there exists a unique local solution $w_\delta^\varepsilon \in C^1(0, T_1(x))$ of the problem (3.4f)-(3.4g), where $T_1(x) \leq T$.

For all $1 \leq i \leq I$, it can be shown that $\hat{u}_{\delta_i}^\varepsilon \in \mathcal{F}_p(\Omega_p^\varepsilon)$ implies $\hat{u}_{\delta_i}^\varepsilon \in C([0,T]\times\bar{\Omega}_p^\varepsilon)$ (for details see theorem 2.2 in [MB13a]), i.e. $\hat{u}_{\delta_i}^\varepsilon$ is continuous and bounded up to the boundary Γ^ε. Since for all k, $\bar{R}_p(\hat{u}_\delta^\varepsilon)_k$ is the product of $\hat{u}_{\delta_i}^\varepsilon$'s, hence $\bar{R}_p(\hat{u}_\delta^\varepsilon)_k \in L^\infty(S\times\Gamma^\varepsilon)$ $\forall k = 1,2,...,\bar{I}$. By partial integration we have

$$\begin{aligned}|w_{\delta_k}^\varepsilon(t,x)| &\leq w_{0_k}(x) + \int_0^t [|\bar{R}_p(\hat{u}_\delta^\varepsilon)| + k_k^d|\psi_\delta(w_\delta^\varepsilon)|]\,dt \\ &\leq ||w_{0_k}||_{L^\infty(\Omega)} + T\left(||\bar{R}_p(\hat{u}_\delta^\varepsilon)_k||_{L^\infty(S\times\Gamma^\varepsilon)} + k_k^d\right) \quad \forall t \text{ and } x.\end{aligned} \tag{3.7}$$

The estimate (3.7) shows that for every $x \in \Gamma^\varepsilon$ the solution w_δ^ε of (3.4f)-(3.4g) exists globally on $[0,T)$, i.e. $T_1(x) = T$. Next,

$$\sum_{k=1}^{\bar{I}} \varepsilon \int_0^T \int_{\Gamma^\varepsilon} |\frac{\partial w_{\delta_k}^\varepsilon}{\partial t}|^p \, d\sigma_x \, dt \leq \sum_{k=1}^{\bar{I}} \varepsilon \, 2^{p-1} \int_0^T \int_{\Gamma^\varepsilon} \left[\operatorname*{ess\,sup}_{(t,x)\in S\times\Gamma^\varepsilon} |\bar{R}_p(\hat{u}_\delta^\varepsilon)_k|^p + (k_k^d)^p \right] d\sigma_x \, dt$$
$$\leq T\,|\Gamma| \frac{|\Omega|}{|Y|} \sum_{k=1}^{\bar{I}} 2^{p-1} \left(||\bar{R}_p(\hat{u}_\delta^\varepsilon)_k||^p_{L^\infty(S\times\Gamma^\varepsilon)} + (k_k^d)^p \right) < \infty. \quad (3.8)$$

From (3.7) and (3.8) it follows that $w_\delta^\varepsilon, \frac{\partial w_\delta^\varepsilon}{\partial t} \in L^p(S; L^p(\Gamma^\varepsilon))^{\bar{I}}$, i.e. $w_\delta^\varepsilon \in \mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)$. □

3.1 A-priori estimates

In this section we obtain some L^r, global in time, *a-priori estimates* of the solutions u_δ^ε and w_δ^ε which are independent of ε, δ and t.

3.1.1 Introduction of a Lyapunov Function

Let $r \in \mathbb{N}$ and $\mu^0 \in \mathbb{R}^I$ be a solution of the linear system

$$\mathcal{S}^T \mu^0 = -\log K, \quad (3.9)$$

where $K \in \mathbb{R}^J$ is the vector of equilibrium constants $K_j = \frac{k_j^f}{k_j^b}$ related to the J kinetic reactions. Note that all the logarithms considered in this paper are natural logarithms with base e. Due to assumption (iv), the system (3.9) has a solution μ^0. For $i = 1,2,...,I$, we define $g_i : \mathbb{R}_0^+ \to \mathbb{R}$, $g : \mathbb{R}_0^{+I} \to \mathbb{R}$, $f_r : \mathbb{R}_0^{+I} \to \mathbb{R}$ and $F_r : L_+^\infty(\Omega_p^\varepsilon)^I \to \mathbb{R}$ as

$$g_i(u_{\delta_i}^\varepsilon) := (\mu_i^0 - 1 + \log u_{\delta_i}^\varepsilon) u_{\delta_i}^\varepsilon + e^{1-\mu_i^0}, \quad (3.10a)$$
$$g(u_\delta^\varepsilon) := \sum_{i=1}^I g_i(u_{\delta_i}^\varepsilon), \quad (3.10b)$$
$$f_r(u_\delta^\varepsilon) := [g(u_\delta^\varepsilon)]^r \text{ and} \quad (3.10c)$$
$$F_r(u_\delta^\varepsilon) := \int_{\Omega_p^\varepsilon} f_r(u_\delta^\varepsilon(x)) \, dx. \quad (3.10d)$$

Clearly $g_i(u_{\delta_i}^\varepsilon) \geq (e-1)e^{-\mu_i^0} \geq (e-1)e^{-\max_i |\mu_i^0|} > 0$ for $i = 1,2,...,I$, which implies g, f_r, $F_r > 0$. For an $\alpha > 0$, there exist a constant C depending on α and μ_i^0, but independent of $u_{\delta_i}^\varepsilon$, ε and δ, s. t.

$$g(u_\delta^\varepsilon) \geq g_i(u_{\delta_i}^\varepsilon) \geq u_{\delta_i}^\varepsilon \text{ and } F_r(u_\delta^\varepsilon) \geq \left|\left| u_{\delta_i}^\varepsilon \right|\right|^r_{L^r(\Omega_p^\varepsilon)}. \quad (3.11a)$$

$$g_i(u_{\delta_i}^\varepsilon) \leq C(1 + u_{\delta_i}^\varepsilon)^{1+\alpha}), \; g(u_\delta^\varepsilon) \leq C(1 + |u_\delta^\varepsilon|_I^{1+\alpha}) \text{ and } f_r(u_\delta^\varepsilon) \leq C(1 + |u_\delta^\varepsilon|_I^{r(1+\alpha)}) \quad (3.11b)$$

for $i = 1, 2, ..., I$. Let us consider the derivative (in the classical sense) of $f_r : \mathbb{R}_0^{+I} \to \mathbb{R}^I$ which is given as

$$\begin{aligned}\partial f_r(u_\delta^\varepsilon) &= \nabla_{u_\delta^\varepsilon} f_r(u_\delta^\varepsilon) \\ &= r[g(u_\delta^\varepsilon)]^{r-1} \nabla_{u_\delta^\varepsilon} g(u_\delta^\varepsilon) \\ &= r f_{r-1}(u_\delta^\varepsilon) \left(\mu^0 + \log u_\delta^\varepsilon\right).\end{aligned}$$

We see that $\partial f_r(u_\delta^\varepsilon)$ is undefined for $u_\delta^\varepsilon = 0$ whereas $f_{r-1}(u_\delta^\varepsilon)$ is defined for all $u_\delta^\varepsilon \geq 0$. Since we only know the nonnegativity of u_δ^ε, we define, for any $\tau > 0$,

$$u_{\delta,\tau}^\varepsilon := u_\delta^\varepsilon + \tau. \tag{3.12}$$

Clearly, $u_{\delta,\tau}^\varepsilon \geq \tau > 0$, $u_{\delta,\tau}^\varepsilon \in \mathcal{F}_p^I(\Omega_p^\varepsilon)$ and $\partial f_r(u_{\delta,\tau}^\varepsilon)$ is well-defined for all $u_{\delta,\tau}^\varepsilon > 0$. From here on we work with the function $u_{\delta,\tau}^\varepsilon$ unless stated otherwise.

Remark 3.1. *In [Krä11] (see also [MB13a]), it is shown that $F_r : L_+^\infty(\Omega_p^\varepsilon)^I \to \mathbb{R}$ and $u_{\delta,\tau}^\varepsilon \mapsto \partial f_r(u_{\delta,\tau}^\varepsilon) : \mathcal{F}_p^I(\Omega_p^\varepsilon) \to L^\infty(S \times \Omega_p^\varepsilon)^I$ are continuous. We will use $\partial f_r(u_{\delta,\tau}^\varepsilon)$ as a test function in the weak formulation of* (1.14a)*, see (5.12) in section 5. It can be shown that $\nabla_x(\partial f_r(u_{\delta,\tau}^\varepsilon)) \in L^q(S \times \Omega_p^\varepsilon)^I$ and $\partial f_r(u_{\delta,\tau}^\varepsilon) \in L^q(S; H^{1,q}(\Omega_p^\varepsilon))^I$.*

From (3.11a) the L^r-norm of $u_{\delta_i}^\varepsilon$ will be finite if we can obtain an upper bound of $F_r(u_\delta^\varepsilon)$. This is addressed in the next theorem.

3.1.2 *A-priori* Estimate

The main result of this section is the next theorem:

Theorem 3.1. *Let $r \in \mathbb{N}$ ($r \geq 2$) and $0 \leq t < T$. Further assume that $(u_\delta^\varepsilon, w_\delta^\varepsilon) \in \mathcal{F}_p^I(\Omega_p^\varepsilon) \times \mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)$ is a solution of $(P_\delta^{\varepsilon^+})$. Then the following inequality holds good:*

$$F_r(u_\delta^\varepsilon(t)) \leq e^{Ct} F_r(u_\delta^\varepsilon(0)) \qquad \textit{for a.e. } t \in S, \tag{3.13}$$

where C is independent of u_δ^ε, w_δ^ε, ε, δ and t but it depends on r.

To prove the above theorem we require the following lemma:

Lemma 3.3. *Let $p > n+2$ and $r \in \mathbb{N}$ ($r \geq 2$). Assume that $(u_\delta^\varepsilon, w_\delta^\varepsilon) \in \mathcal{F}_p^I(\Omega_p^\varepsilon) \times \mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)$ is a solution of $(P_\delta^{\varepsilon^+})$. Then the following inequality holds:*

$$\int_0^t \left\langle \frac{\partial u_\delta^\varepsilon}{\partial \theta}, \partial f_r(u_{\delta,\tau}^\varepsilon) \right\rangle d\theta \leq h(t,\tau,u_{\delta,\tau}^\varepsilon) + C \int_0^t F_r(u_{\delta,\tau}^\varepsilon)\, d\theta \textit{ for a.e. } t \in S, \tag{3.14}$$

where $h(t,\tau,u_{\delta,\tau}^\varepsilon) \to 0$ as $\tau \to 0$ for a.e. t, and C is independent of ε, δ, τ, t, $u_{\delta,\tau}^\varepsilon$ and $w_{\delta,\tau}^\varepsilon$ but it depends on r.

We use theorem 3.1 to obtain L^r-estimates of the solution u^ε_δ for $r > 2$.

Corollary 3.1. *For any arbitrary solution* $(u^\varepsilon_\delta, w^\varepsilon_\delta) \in \mathcal{F}^I_p(\Omega^\varepsilon_p) \times \mathcal{M}^{\bar{I}}_p(\Gamma^\varepsilon)$ *of* $(P^{\varepsilon^+}_\delta)$*, the following estimate holds true:*

$$\sup_{\delta,\varepsilon>0}\Bigg(\sup_{t>0}|||u^\varepsilon_\delta(t)|||_{L^r(\Omega^\varepsilon_p)^I} + |||u^\varepsilon_\delta|||_{L^r(S\times\Omega^\varepsilon_p)^I} + |||\nabla u^\varepsilon_\delta|||_{L^2(S\times\Omega^\varepsilon_p)^I} + |||\frac{\partial u^\varepsilon_\delta}{\partial t}|||_{L^2(S;H^{1,2}(\Omega^\varepsilon_p)^*)^I}$$
$$+\varepsilon\sum_{k=1}^{\bar{I}}\int_0^T\int_{\Gamma^\varepsilon}|w^\varepsilon_{\delta_k}|^p\,d\sigma_x\,dt + \varepsilon\sum_{k=1}^{\bar{I}}\int_0^T\int_{\Gamma^\varepsilon}\left|\frac{\partial w^\varepsilon_{\delta_k}}{\partial t}\right|^p d\sigma_x\,dt\Bigg) \le C < \infty, \tag{3.15}$$

where C *is independent of* ε, δ, u^ε_δ, w^ε_δ *and* t *but it depends on* r *and* p.

Just to get to the main results of this paper the proofs of theorem 3.1, lemma 3.3 and corollary 3.1 are postponed for the moment and will be given in appendix.

Corollary 3.2. *Let* $r \in \mathbb{N}$ *and the assumptions (i) - (vi) be satisfied. Then there exists a constant* C *independent of* $u^\varepsilon_\delta, w^\varepsilon_\delta$ *and* t *such that any arbitrary solution* $(u^\varepsilon_\delta, w^\varepsilon_\delta) \in \mathcal{F}^I_p(\Omega^\varepsilon_p) \times \mathcal{M}^{\bar{I}}_p$ *of* $(P^{\varepsilon^+}_\delta)$ *satisfies the following estimate*

$$|||u^\varepsilon_\delta|||_{\mathcal{F}^I_p(\Omega^\varepsilon_p)} \le C. \tag{3.16}$$

Proof. We reformulate (3.4a) - (3.4e) as

$$\frac{\partial u^\varepsilon_\delta}{\partial t} + Au^\varepsilon_\delta = f_{bound}(u^\varepsilon_\delta) + f_{reac}(u^\varepsilon_\delta), \tag{3.17a}$$
$$u^\varepsilon_\delta(0,x) = u_0(x), \tag{3.17b}$$

where $f_{bound}(u^\varepsilon_\delta)_i := Q^2_{in}(-d)_i + \varepsilon\sum_{k=1}^{\bar{I}}\sigma_{ik}R_{\Gamma^\varepsilon}(u^\varepsilon_\delta, w^\varepsilon_\delta)_k$, $R_{\Gamma^\varepsilon}(u^\varepsilon_\delta, w^\varepsilon_\delta)_k := -\frac{\partial w^\varepsilon_{\delta_k}}{\partial t}$ and $f_{reac}(u^\varepsilon_\delta)_i = \mathcal{S}R(u^\varepsilon_\delta)_i - \vec{q}^{\,\varepsilon}\cdot\nabla u^\varepsilon_{\delta_i}$. Set $f(u^\varepsilon_\delta) := f_{bound}(u^\varepsilon_\delta) + f_{reac}(u^\varepsilon_\delta)$. The operator $A : H^{1,p}(\Omega^\varepsilon_p)^I \to {H^{1,q}(\Omega^\varepsilon_p)^*}^I$ is defined by

$$Au^\varepsilon_\delta := (A_1u^\varepsilon_{\delta_1}, A_2u^\varepsilon_{\delta_2}, ..., A_Iu^\varepsilon_{\delta_I}), \tag{3.18a}$$
$$\langle A_iu^\varepsilon_{\delta_i}, \varphi_i\rangle := \int_{\Omega^\varepsilon_p} D\nabla u^\varepsilon_{\delta_i}\cdot\nabla\varphi_i\,dx + \int_{\Gamma_{in}}(-\vec{q}^{\,\varepsilon}\cdot\vec{n})u^\varepsilon_{\delta_i}\varphi_i\,ds, \tag{3.18b}$$

for $u^\varepsilon_\delta \in H^{1,p}(\Omega^\varepsilon_p)^I, \varphi \in {H^{1,q}(\Omega^\varepsilon_p)}^I$. The operator A has maximal (parabolic) regularity on ${H^{1,q}(\Omega^\varepsilon_p)^*}^I$, cf. [RDR09, CL94]. From lemma 5.15 in [RDR09], it follows that $Q^2_{in}(-d)_i$, $\varepsilon\sum\limits_{k=1}^{\bar{I}}\sigma_{ik}R_{\Gamma^\varepsilon}(u^\varepsilon_\delta, w^\varepsilon_\delta)_k \in L^p(S; H^{1,q}(\Omega^\varepsilon_p)^*)$ which gives $f_{bound} \in L^p(S; H^{1,q}(\Omega^\varepsilon_p)^*)^I$. Moreover, it can be seen that $f_{reac}(u^\varepsilon_\delta) \in L^p(S; H^{1,q}(\Omega^\varepsilon_p)^*)^I$ and, by assumption (iii), we have $u_0 \in \mathcal{X}_p^I(\Omega_p^\varepsilon)$. Therefore, by Theorem 2.5 in [PS01], the solution u_δ^ε of (3.17a) - (3.17b) in $\mathcal{F}_p^I(\Omega_p^\varepsilon)$ satisfies

$$|||u_\delta^\varepsilon|||_{\mathcal{F}_p^I(\Omega_p^\varepsilon)} \leq C\left(|||u_0|||_{\mathcal{X}_p^I(\Omega_p^\varepsilon)} + |||f|||_{L^p(S;H^{1,q}(\Omega_p^\varepsilon)^*)^I}\right) \leq C < \infty,$$

where C is independent of t, u_δ^ε and w_δ^ε. □

3.1.3 Existence of solution of $(P_\delta^{\varepsilon^+})$

We employ Schaefer's fixed point theorem to prove the existence of solution of $(P_\delta^{\varepsilon^+})$. Let (i)-(vi) hold. We define an operator $\mathcal{Z} : \mathcal{F}_p^I(\Omega_p^\varepsilon) \to \mathcal{F}_p^I(\Omega_p^\varepsilon)$ by $u_\delta^\varepsilon := Z(\hat{u}_\delta^\varepsilon)$, where u_δ^ε is the solution of

$$\frac{\partial u_\delta^\varepsilon}{\partial t} - \nabla \cdot (D_{diff}\nabla u_\delta^\varepsilon - \vec{q}^\varepsilon u_\delta^\varepsilon) = \mathcal{S}\bar{R}(\hat{u}_\delta^\varepsilon) \quad \text{in } S \times \Omega_p^\varepsilon, \tag{3.19a}$$

$$-\left(D_{diff}\nabla u_\delta^\varepsilon - \vec{q}^\varepsilon u_\delta^\varepsilon\right) \cdot \vec{n} = d \quad \text{on } S \times \Gamma_{in}, \tag{3.19b}$$

$$-\left(D_{diff}\nabla u_\delta^\varepsilon - \vec{q}^\varepsilon u_\delta^\varepsilon\right) \cdot \vec{n} = 0 \quad \text{on } S \times \Gamma_{out}, \tag{3.19c}$$

$$-\left(D_{diff}\nabla u_{\delta_i}^\varepsilon - \vec{q}^\varepsilon u_{\delta_i}^\varepsilon\right) \cdot \vec{n} = \varepsilon \sum_{k=1}^{\bar{I}} \sigma_{ik} \frac{\partial w_{\delta_k}^\varepsilon}{\partial t} \quad \text{on } S \times \Gamma^\varepsilon, \tag{3.19d}$$

$$u_\delta^\varepsilon(0,x) = u_0(x) \quad \text{in } \Omega_p^\varepsilon, \tag{3.19e}$$

$$\frac{\partial w_\delta^\varepsilon}{\partial t} = R(\hat{u}_\delta^\varepsilon, w_\delta^\varepsilon) = \left(\bar{R}_p(\hat{u}_\delta^\varepsilon) - k^d\psi_\delta(w_\delta^\varepsilon)\right) \quad \text{on } S \times \Gamma^\varepsilon, \tag{3.19f}$$

$$w_\delta^\varepsilon(0,x) = w_0(x) \quad \text{on } \Gamma^\varepsilon, \tag{3.19g}$$

for $i = 1, 2, ..., I$. Note that for a given $\hat{u}_\delta^\varepsilon \in \mathcal{F}_p^I(\Omega_p^\varepsilon)$, w_δ^ε is the solution of (3.19f)-(3.19g) (see Lemma 3.2). Every fixed point of $\mathcal{Z}$ is a solution of $(P_\delta^{\varepsilon^+})$. Let us verify that $\mathcal{Z}$ is well defined. The abstract formulation of (3.19a)-(3.19e) is given by

$$\frac{\partial u_\delta^\varepsilon}{\partial t} + Au_\delta^\varepsilon = f_{bound}(u_\delta^\varepsilon) + f_{reac}(u_\delta^\varepsilon), \tag{3.20a}$$

$$u_\delta^\varepsilon(0,x) = u_0(x), \tag{3.20b}$$

where the terms f_{bound}, f_{reac} and the operator A are defined in the same way as in corollary 3.2. Following the arguments of Corollary 3.2 there exists a unique $u_\delta^\varepsilon \in \mathcal{F}_p^I(\Omega_p^\varepsilon)$ which solves (3.20a)-(3.20b). Thus $\mathcal{Z}$ is well defined. The same steps show that $\mathcal{Z}$ is continuous and compact, cf. [MB14].

Lemma 3.4. *There exists a positive global weak solution* $(u_\delta^\varepsilon, w_\delta^\varepsilon) \in \mathcal{F}_p^I(\Omega_p^\varepsilon) \times \mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)$ *of* $(P_\delta^{\varepsilon^+})$.

Proof. We employ Schaefer's fixed point theorem. From section 3.1.3 (i.e. from the existence of solution of $(P_\delta^{\varepsilon^+})$) it is clear that the operator $\mathcal{Z}$ is continuous and compact. It remains to check that the set $\{u^\varepsilon_\delta \in \mathcal{F}^I_p(\Omega^\varepsilon_p) : \exists\lambda \in [0,1] \text{ s.t. } u^\varepsilon_\delta = \lambda\mathcal{Z}(u^\varepsilon_\delta)\}$ is bounded, i.e., we need to obtain an estimate of the solution of

$$\frac{\partial u^\varepsilon_\delta}{\partial t} - \nabla\cdot(D_{diff}\nabla u^\varepsilon_\delta - \vec{q^\varepsilon} u^\varepsilon_\delta) \;=\; \lambda\,\mathcal{S}\bar{R}(u^\varepsilon_\delta), \tag{3.21}$$

where initial and boundary values u_0, d and $\varepsilon\sum_{k=1}^{\bar{I}}\sigma_{ik}\frac{\partial w^\varepsilon_{\delta_k}}{\partial t}$ are replaced by λu_0, λd and $\lambda\varepsilon\sum_{k=1}^{\bar{I}}\sigma_{ik}\frac{\partial w^\varepsilon_{\delta_k}}{\partial t}$, respectively. Clearly the estimates in section 3.1.2, derived for $\lambda = 1$, hold for $0 \le \lambda \le 1$. Thus $\mathcal{Z}$ has a fixed point, i.e. there exists a solution u^ε_δ of (3.2a)-(3.2e). This implies that $(P^{\varepsilon^+}_\delta)$ has a solution $(u^\varepsilon_\delta, w^\varepsilon_\delta) \in \mathcal{F}^I_p(\Omega^\varepsilon_p)\times\mathcal{M}^{\bar{I}}_p(\Gamma^\varepsilon)$.

□

Hence, we have shown that $(P^{\varepsilon^+}_\delta)$ has a positive global weak solution in $\mathcal{F}^I_p(\Omega^\varepsilon_p)\times\mathcal{M}^{\bar{I}}_p(\Gamma^\varepsilon)$. Since the solution of $(P^{\varepsilon^+}_\delta)$ is nonnegative, it also solves the problem (P^ε_δ). This finally implies the existence of a global positive weak solution $(u^\varepsilon_\delta, w^\varepsilon_\delta) \in \mathcal{F}^I_p(\Omega^\varepsilon_p)\times\mathcal{M}^{\bar{I}}_p(\Gamma^\varepsilon)$ of (P^ε_δ).

Lemma 3.5. *There exists a unique positive global weak solution $(u^\varepsilon, w^\varepsilon, z^\varepsilon)$ of the following problem (P^ε) in $\mathcal{F}^I_p(\Omega^\varepsilon_p)\times\mathcal{M}^{\bar{I}}_p(\Gamma^\varepsilon)\times\mathcal{N}^{\bar{I}}_\infty(\Gamma^\varepsilon)$:*

$$\frac{\partial u^\varepsilon}{\partial t} - \nabla\cdot(D_{diff}\nabla u^\varepsilon - \vec{q^\varepsilon}u^\varepsilon) = \mathcal{S}R(u^\varepsilon) \qquad \text{in } S\times\Omega^\varepsilon_p, \tag{3.22a}$$

$$-\Big(D_{diff}\nabla u^\varepsilon - \vec{q^\varepsilon}u^\varepsilon\Big)\cdot\vec{n} = d \qquad \text{on } S\times\Gamma_{in}, \tag{3.22b}$$

$$-\Big(D_{diff}\nabla u^\varepsilon - \vec{q^\varepsilon}u^\varepsilon\Big)\cdot\vec{n} = 0 \qquad \text{on } S\times\Gamma_{out}, \tag{3.22c}$$

$$-(D_{diff}\nabla u^\varepsilon_i - \vec{q^\varepsilon}u^\varepsilon_i)\cdot\vec{n} = \varepsilon\sum_{k=1}^{\bar{I}}\sigma_{ik}\frac{\partial w^\varepsilon_k}{\partial t} \qquad \text{on } S\times\Gamma^\varepsilon, \tag{3.22d}$$

$$u^\varepsilon(0,x) = u_0(x) \qquad \text{in } \Omega^\varepsilon_p, \tag{3.22e}$$

$$\frac{\partial w^\varepsilon}{\partial t} = R(u^\varepsilon, w^\varepsilon) = \Big(R_p(u^\varepsilon) - k^d z^\varepsilon\Big) \qquad \text{on } S\times\Gamma^\varepsilon, \tag{3.22f}$$

$$z^\varepsilon \in \psi(w^\varepsilon) \qquad \text{on } S\times\Gamma^\varepsilon, \tag{3.22g}$$

$$w^\varepsilon(0,x) = w_0(x) \qquad \text{on } \Gamma^\varepsilon, \tag{3.22h}$$

for $i = 1,2,...,I$. Moreover, the solution $(u^\varepsilon, w^\varepsilon, z^\varepsilon)$ satisfies

$$||u^\varepsilon_i||_{L^r((0,T)\times\Omega^\varepsilon_p)} + ||\nabla u^\varepsilon_i||_{L^2((0,T)\times\Omega^\varepsilon_p)} + ||\frac{\partial u^\varepsilon_i}{\partial t}||_{L^2((0,T);H^{1,2}(\Omega^\varepsilon_p)^*)} + ||w^\varepsilon_k||_{L^p((0,T)\times\Gamma^\varepsilon)}$$
$$+||\frac{\partial w^\varepsilon_k}{\partial t}||_{L^p((0,T)\times\Gamma^\varepsilon)} \le C < \infty, \tag{3.23}$$

for all $r \in \mathbb{N}$, for all $i = 1,2,...,I$ and for all $k = 1,2,...,\bar{I}$ and the constant C is independent of δ, ε and t.

Proof. We divide the whole proof in to three steps.

Step 1: With the help of weak convergence, the bounds in (3.22h) follows from the estimate (3.15) and theorem 4.6 in [MB13b]. Also by corollary 3.1 and corollary 4 in [Sim86], we have

$$u_\delta^\varepsilon \to u^\varepsilon \text{ strongly in } L^2((0,T);L^2(\Omega_p^\varepsilon))^I \text{ and} \tag{3.24}$$

$$u_\delta^\varepsilon \to u^\varepsilon \text{ strongly in } [L^2((0,T);H^{s,2}(\Omega_p^\varepsilon))\cap C([0,T];H^{-s,2}(\Omega_p^\varepsilon))]^I \tag{3.25}$$

as $\delta\to 0$ for any $0<s<1$. By trace theorem 8.7 in [Wlo87], we get the strong convergence of u_δ^ε to u^ε in $L^2(S;H^{s-\frac{1}{2}}(\Gamma^\varepsilon))^I$ for $0<s<1$. Now we follow the similar arguments given in theorem 2.21 of [vDP04] to obtain the equations (3.22a)-(3.22h).

Step 2. In step 1 we have shown that a triple $(u^\varepsilon,w^\varepsilon,z^\varepsilon)\times\mathcal{F}_2^I(\Omega_p^\varepsilon)\cap L^r((0,T);L^r(\Omega_p^\varepsilon))^I\times\mathcal{M}_2^{\bar{I}}(\Gamma^\varepsilon)\times\mathcal{N}_\infty^{\bar{I}}(\Gamma^\varepsilon)$ solves (P^ε). As shown in corollary 3.1, it can be proved that $w^\varepsilon,\frac{\partial w^\varepsilon}{\partial t}\in L^p((0,T);L^p(\Gamma^\varepsilon))^{\bar{I}}$. Next the reformulation of (3.22a)-(3.22e) is

$$\frac{\partial u^\varepsilon}{\partial t}+Au^\varepsilon=f_{bound}(u^\varepsilon)+f_{reac}(u^\varepsilon), \tag{3.26a}$$

$$u^\varepsilon(0,x)=u_0(x), \tag{3.26b}$$

where $f_{bound}(u^\varepsilon),f_{reac}(u^\varepsilon)$ and the operator A are defined as in corollary 3.2. A has maximal parabolic regularity on $[H^{1,q}(\Omega_p^\varepsilon)^*]^I$ and $f_{bound},f_{reac}\in L^p((0,T);H^{1,q}(\Omega_p^\varepsilon)^*)^I$. Proceeding with the similar arguments as that of corollary 3.2, we obtain a solution $u^\varepsilon\in\mathcal{F}_p^I(\Omega_p^\varepsilon)$ of (3.26a)-(3.26b) which satisfies the estimate of type (3.23). Therefore we obtain the existence of a global positive weak solution $(u^\varepsilon,w^\varepsilon,z^\varepsilon)\in\mathcal{F}_p^I(\Omega_p^\varepsilon)\cap L^r((0,T);L^r(\Omega_p^\varepsilon))^I\times\mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)\times\mathcal{N}_\infty^{\bar{I}}(\Gamma^\varepsilon)$ of the problem (P^ε).

Step 3. Uniqueness: Now we show the uniqueness of solution of (P^ε). We first consider the reaction rate term on Γ^ε which is $R(u^\varepsilon,w^\varepsilon)=(R(u^\varepsilon,w^\varepsilon)_k)_{1\le k\le\bar{I}}$. By lemma 3.4 $(u^\varepsilon,w^\varepsilon)$ is a nonnegative solution of (P^ε) in componentwise sense. Clearly, $u_i^\varepsilon\mapsto R(u^\varepsilon,w^\varepsilon)_k$ is monotonically increasing and $w_k^\varepsilon\mapsto R(u^\varepsilon,w^\varepsilon)_k$ is monotonically decreasing by definition. For a $\delta>0$ we define two functions ϕ and Φ as

$$\Phi_\delta(x):=\begin{cases}-x-\dfrac{\delta}{2} & \text{if } x\le-\delta,\\ \dfrac{x^2}{2\delta} & \text{if } -\delta\le x<\delta,\\ x-\dfrac{\delta}{2} & \text{if } x\ge\delta\end{cases} \tag{3.27a}$$

and

$$\phi_\delta(x):=\begin{cases}-1 & \text{if } x\le-\delta,\\ \dfrac{x}{\delta} & \text{if } -\delta\le x<\delta,\\ 1 & \text{if } x\ge\delta\end{cases} \tag{3.27b}$$

such that $\frac{d\Phi_\delta}{dx} := \Phi_\delta^{'}(x) = \phi_\delta(x)$. On the contrary, let $(u_1^\varepsilon, w_1^\varepsilon, z_1^\varepsilon)$ and $(u_2^\varepsilon, w_2^\varepsilon, z_2^\varepsilon)$ be the solution of (3.22a)-(3.22h) such that $u_1^\varepsilon(0) = u_2^\varepsilon(0)$ and $w_1^\varepsilon(0) = w_2^\varepsilon(0)$. Set $\bar{u}^\varepsilon := u_1^\varepsilon - u_2^\varepsilon$ and $\bar{w}^\varepsilon := w_1^\varepsilon - w_2^\varepsilon$. We substract the equation satisfied by $(u_1^\varepsilon, w_1^\varepsilon, z_1^\varepsilon)$ with the equation satisfied by $(u_2^\varepsilon, w_2^\varepsilon, z_2^\varepsilon)$ and consider the scalar system then

$$\frac{\partial \bar{u}_i^\varepsilon}{\partial t} - \nabla\cdot(D_{diff}\nabla\bar{u}_i^\varepsilon - \vec{q^\varepsilon}\bar{u}_i^\varepsilon) = \mathcal{S}R(u_1^\varepsilon)_i - \mathcal{S}R(u_2^\varepsilon)_i \quad \text{in } S\times\Omega_p^\varepsilon, \tag{3.28a}$$

$$-\left(D_{diff}\nabla\bar{u}_i^\varepsilon - \vec{q^\varepsilon}\bar{u}_i^\varepsilon\right)\cdot\vec{n} = 0 \quad \text{on } S\times\Gamma_{in}, \tag{3.28b}$$

$$-\left(D_{diff}\nabla\bar{u}_i^\varepsilon - \vec{q^\varepsilon}\bar{u}_i^\varepsilon\right)\cdot\vec{n} = 0 \quad \text{on } S\times\Gamma_{out}, \tag{3.28c}$$

$$-\left(D_{diff}\nabla\bar{u}_i^\varepsilon - \vec{q^\varepsilon}\bar{u}_i^\varepsilon\right)\cdot\vec{n} = \varepsilon\sum_{k=1}^{\bar{I}}\sigma_{ik}\frac{\partial\bar{w}_k^\varepsilon}{\partial t} \quad \text{on } S\times\Gamma^\varepsilon, \tag{3.28d}$$

$$\bar{u}_i^\varepsilon(0,x) = 0 \quad \text{in } \Omega_p^\varepsilon, \tag{3.28e}$$

$$\frac{\partial\bar{w}_k^\varepsilon}{\partial t} = [R_p(u_1^\varepsilon)_k - R_p(u_2^\varepsilon)_k - \{R_d(w_1^\varepsilon)_k - R_d(w_2^\varepsilon)_k\}] \quad \text{on } S\times\Gamma^\varepsilon, \tag{3.28f}$$

$$\bar{w}_k^\varepsilon(0,x) = 0 \quad \text{on } \Gamma^\varepsilon. \tag{3.28g}$$

For a.e. t, $u_i^\varepsilon(t) \in H^{1,p}(\Omega_p^\varepsilon)$ and $\phi_\delta(u_i^\varepsilon(t)) \in H^{1,p}(\Omega_p^\varepsilon)$. We test (3.28a) with $\phi_\delta(\bar{u}_i^\varepsilon(t))$ and (3.28f) by $\varepsilon\phi_\delta(\bar{w}_k^\varepsilon(t))$ and, the addition of resulting two equations will yield

$$\begin{aligned}
&\int_0^t\int_{\Omega_p^\varepsilon}\frac{\partial\bar{u}_i^\varepsilon}{\partial\theta}\phi_\delta(\bar{u}_i^\varepsilon)\,dx\,d\theta + \varepsilon\sum_{k=1}^{\bar{I}}\int_0^t\int_{\Gamma^\varepsilon}\frac{\partial\bar{w_k}^\varepsilon}{\partial\theta}\phi_\delta(\bar{w}_k^\varepsilon)\,d\sigma_x\,d\theta \\
&+\int_0^t\int_{\Omega_p^\varepsilon}D\nabla\bar{u_i}^\varepsilon\nabla(\phi_\delta(\bar{u}_i^\varepsilon))\,dx\,d\theta = \int_0^t\int_{\Omega_p^\varepsilon}\vec{q^\varepsilon}\cdot\nabla(\phi_\delta(\bar{u}_i^\varepsilon))\bar{u}_i^\varepsilon\,dx\,d\theta \\
&\quad+\int_0^t\int_{\Omega_p^\varepsilon}(\mathcal{S}R(u_1^\varepsilon)_i - \mathcal{S}R(u_2^\varepsilon)_i)\,\phi_\delta(\bar{u}_i^\varepsilon)\,dx\,d\theta \\
&-\varepsilon\sum_{k=1}^{\bar{I}}\int_0^t\int_{\Gamma^\varepsilon}\{(R_p(u_1^\varepsilon)-R_d(w_1^\varepsilon))_k - (R_p(u_2^\varepsilon)-R_d(w_2^\varepsilon))_k\}\,(\sigma_{ik}\phi_\delta(\bar{u}_i^\varepsilon) - \phi_\delta(\bar{w}_k^\varepsilon))\,d\sigma_x\,d\theta.
\end{aligned} \tag{3.29}$$

We simplyfy each term in (3.29) one by one.

$$\begin{aligned}
\lim_{\delta\to0} I_\delta^1 = \lim_{\delta\to0}\int_0^t\int_{\Omega_p^\varepsilon}\frac{\partial\bar{u_i}^\varepsilon}{\partial\theta}\phi_\delta(\bar{u}_i^\varepsilon)\,dx\,d\theta &= \lim_{\delta\to0}\int_0^t\frac{d}{d\theta}\int_{\Omega_p^\varepsilon}\Phi_\delta(\bar{u}_i^\varepsilon)\,dx\,d\theta \\
&= \lim_{\delta\to0}[\int_{\Omega_p^\varepsilon}\Phi_\delta(\bar{u}_i^\varepsilon(t,x))\,dx - \int_{\Omega_p^\varepsilon}\Phi_\delta(\bar{u}_i^\varepsilon(0,x))\,dx]
\end{aligned}$$

Note that $|\Phi_\delta(\bar{u}_i^\varepsilon)| \le |\bar{u}_i^\varepsilon| + \dfrac{\delta}{2}$ and $\bar{u}_i^\varepsilon(t) \in L^2(\Omega_p^\varepsilon)$, then by Lebesgue dominated convergence theorem

$$\lim_{\delta\to0} I_\delta^1 = \int_{\Omega_p^\varepsilon}|\bar{u}_i^\varepsilon(t,x)|\,dx - \int_{\Omega_p^\varepsilon}|\bar{u}_i^\varepsilon(0,x)|\,dx = \int_{\Omega_p^\varepsilon}|\bar{u}_i^\varepsilon(t,x)|\,dx, \text{ since } \bar{u}_i^\varepsilon(0,x) = 0. \tag{3.30}$$

By similar techniques, it gives

$$\lim_{\delta\to 0} I_\delta^2 = \lim_{\delta\to 0}\int_0^t\int_{\Gamma^\varepsilon}\frac{\partial \bar{w}_k{}^\varepsilon}{\partial\theta}\phi_\delta(\bar{w}_k^\varepsilon)\,d\sigma_x\,d\theta = \int_{\Gamma^\varepsilon}|w_k^\varepsilon(t)|\,d\sigma_x - \int_{\Gamma^\varepsilon}|w_k^\varepsilon(0)|\,d\sigma_x = \int_{\Gamma^\varepsilon}|w_k^\varepsilon(t,x)|\,d\sigma_x. \tag{3.31}$$

$$\lim_{\delta\to 0} I_\delta^3 = \lim_{\delta\to 0} D\int_0^t\int_{\Omega_p^\varepsilon}\phi_\delta'(\bar{u}_i^\varepsilon)|\nabla u_i^\varepsilon|^2\,dx\,d\theta \geq 0, \text{ since } \phi_\delta'(\bar{u}_i^\varepsilon)\geq 0. \tag{3.32}$$

$$\lim_{\delta\to 0} I_\delta^4 = \int_0^t\int_{\Omega_p^\varepsilon}\vec{q^\varepsilon}\cdot\nabla(\phi_\delta(\bar{u}_i^\varepsilon))\bar{u}_i^\varepsilon\,dx\,d\theta \leq 0, \quad \text{by (1.4) and (3.27).} \tag{3.33}$$

$$\begin{aligned}\lim_{\delta\to 0} I_\delta^5 &= \lim_{\delta\to 0}\int_0^t\int_{\Omega_p^\varepsilon}\{\mathcal{S}R(u_1^\varepsilon)_i - \mathcal{S}R(u_2^\varepsilon)_i\}\phi_\delta(\bar{u}_i^\varepsilon)\,dx\,d\theta\\ &\leq \lim_{\delta\to 0}\sum_{j=1}^J|s_{ij}|\int_0^t\int_{\Omega_p^\varepsilon}|R(u_1^\varepsilon)_j - R(u_2^\varepsilon)_j||\phi_\delta(\bar{u}_i^\varepsilon)|\,dx\,d\theta\end{aligned}$$

Each term in the expansion of $R(u_1^\varepsilon)_j - R(u_2^\varepsilon)_j$ yields a factor $u_{1_i}^\varepsilon - u_{2_i}^\varepsilon$ whereas the rest of terms are bounded in $L^\infty((0,T)\times\Omega_p^\varepsilon)$ by theorem 2.2 in [MB13a] and $|s_{ij}| \leq \max_{\substack{1\leq i\leq I\\ i\leq j\leq J}}|s_{ij}|$. This gives

$$\lim_{\delta\to 0} I_\delta^5 \leq C\int_0^t\int_{\Omega_p^\varepsilon}|u_{1_i}^\varepsilon - u_{2_i}^\varepsilon|\,dx\,d\theta. \tag{3.34}$$

We exploit the monotone property of $R(u^\varepsilon, w^\varepsilon)_k$ to show that $\lim_{\delta\to 0} I_\delta^6 =$

$$\lim_{\delta\to 0}\varepsilon\sum_{k=1}^{\bar{I}}\int_0^t\int_{\Gamma^\varepsilon}\{(R_p(u_1^\varepsilon) - R_d(w_1^\varepsilon))_k - (R_p(u_2^\varepsilon) - R_d(w_2^\varepsilon))_k\}\{\sigma_{ik}\phi_\delta(\bar{u}_i^\varepsilon) - \phi_\delta(\bar{w}_k^\varepsilon)\}\,d\sigma_x\,d\theta \geq 0, \tag{3.35}$$

see [vNPR07] for details. We combine all the equations (3.29)-(3.35), we obtain

$$\int_{\Omega_p^\varepsilon}|u_{1_i}^\varepsilon - u_{2_i}^\varepsilon|\,dx + \varepsilon\sum_{k=1}^{\bar{I}}\int_{\Gamma^\varepsilon}|w_{1_i}^\varepsilon - w_{2_i}^\varepsilon|\,d\sigma_x \leq C\int_0^t\int_{\Omega_p^\varepsilon}|u_{1_i}^\varepsilon - u_{2_i}^\varepsilon|\,dx\,d\theta. \tag{3.36}$$

By *Gronwall's inequality* $u_1^\varepsilon = u_2^\varepsilon$ a.e. in $(0,T)\times\Omega_p^\varepsilon$ and $w_1^\varepsilon = w_2^\varepsilon$ a.e. in $(0,T)\times\Gamma^\varepsilon$. In other words, we have shown the existence of a unique positive global weak solution $(u^\varepsilon, w^\varepsilon, z^\varepsilon)\in\mathcal{F}_p^I(\Omega_p^\varepsilon)\times\mathcal{M}_p^{\bar{I}}(\Gamma^\varepsilon)\times\mathcal{N}_\infty^{\bar{I}}(\Gamma^\varepsilon)$ of the problem (P^ε) which satisfies the estimate (3.23). □

4 Proof of theorem 2.2

This section is dedicated to the homogenization of (P^ε) where we obtain the limit problem (macro problem) as $\varepsilon\to 0$.

4.1 ε independent a-priori estimates

Our point of departure is the following lemma from Meirmanov and Zimin (cf. theorem 2.1[MZ11]):

Lemma 4.1. *Let (c^ε) be a bounded sequence in $L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);H^{1,2}(\Omega))$ and weakly convergent in $L^2((0,T);L^2(\Omega))\cap L^2((0,T);H^{1,2}(\Omega))$ to a function c. Suppose further that the sequence $(\chi^\varepsilon \frac{\partial}{\partial t}c^\varepsilon)$ is bounded in $L^2((0,T);H^{1,2}(\Omega)^*)$. Then the sequence (c^ε) is strongly convergent to the function c in $L^2((0,T);L^2(\Omega))$.*

Let $\tilde{u}^\varepsilon(t,.)$ denote the extension of $u^\varepsilon(t,.)$ in to all of Ω s.t. $\tilde{u}^\varepsilon|_{\Omega_p^\varepsilon} = u^\varepsilon$. However, for sake of notation, we denote the extended function also by u^ε.

Lemma 4.2. *Let $r\in\mathbb{N}$ $(r\geq 2)$ and the solution u^ε of (P^ε) be extended in to all of $S\times\Omega$. Then there exists a positive constant C independent of ε such that the following estimate hold true:*

$$\sup_{\varepsilon>0}\left[|||u^\varepsilon|||_{L^\infty(S;L^r(\Omega))^I} + |||u^\varepsilon|||_{L^r(S;L^r(\Omega))^I} + |||\nabla u^\varepsilon|||_{L^2(S;L^2(\Omega))^I} + \left|\left|\left|\chi^\varepsilon \frac{\partial u^\varepsilon}{\partial t}\right|\right|\right|_{L^2(S;H^{1,2}(\Omega)^*)^I}\right] \leq C < \infty. \tag{4.1}$$

Proof. This follows directly from the estimate (3.23), lemma 5.2 and some scaling arguments, for details see [MB13b]. □

Lemma 4.3. *Let (u^ε) satisfies the estimates (4.1). Then there exists a function $u\in L^2((0,T);H^{1,2}(\Omega))^I$ and a function $u^1\in L^2((0,T)\times\Omega;H^{1,2}_{per}(Y)/\mathbb{R})^I$ such that up to a subsequence, still denoted by same subscript, the following convergence results hold:*

(i) (u^ε) is weakly convergent to u in $L^2((0,T);H^{1,2}(\Omega))^I$.

(ii) (u^ε) is strongly convergent to u in $L^2((0,T);L^2(\Omega))^I$.

(iii) (u^ε) and $(\nabla_x u^\varepsilon)$ are two-scale convergent to u and $\nabla_x u+\nabla_y u^1$ in the sense of (5.8) respectively.

Proof. The proof follows from the estimate (4.1) and lemmas 5.5, 5.6 and 5.7. □

Lemma 4.4. *The weak limit u belongs to $L^r(S\times\Omega)^I$.*

Proof. Since (u^ε) is strongly convergent to u in $L^2(S;L^2(\Omega))^I$, (u_i^ε) is strongly convergent to u_i in $L^2(S;L^2(\Omega))$ for all $i=1,2,...,I$. There exists a subsequence $(u^{\varepsilon'})_{\varepsilon'>0}$ which is pointwise convergent to u almost everywhere in $S\times\Omega$ (see corollary on page 53 in [Yos70]), i.e.,

$$\lim_{\varepsilon'\to 0} u_i^{\varepsilon'}(t,x) = u_i(t,x) \qquad \text{a.e.} \qquad (t,x)\in S\times\Omega. \tag{4.2}$$

Using this pointwise convergence and L^r-estimate of u_i^ε, it follows that $||u_i||_{L^r(S\times\Omega)} < \infty$. □

Lemma 4.5. *The source term* $(\mathcal{S}R(u^\varepsilon))$ *is strongly convergent to* $\mathcal{S}R(u)$ *in* $L^2(S\times\Omega)^I$.

Proof. The strong convergence of (u^ε) and L^r-estimates of u^ε and u lead to the conclusion. □

Lemma 4.6. *Let* (w^ε) *satisfies (3.23), the following convergence results hold:*

(i) (w^ε) *is two-scale convergent to* w *in* $L^2((0,T)\times\Omega\times\Gamma)^{\bar{I}}$.

(ii) $\left(\dfrac{\partial w^\varepsilon}{\partial t}\right)$ *is two-scale convergent to* $\dfrac{\partial w}{\partial t}$ *in* $L^2((0,T)\times\Omega\times\Gamma)^{\bar{I}}$.

(iii) (z^ε) *is two-scale convergent to* z *in* $L^2((0,T)\times\Omega\times\Gamma)^{\bar{I}}$.

Proof. Note that from (3.23) it follows that $\varepsilon\int_0^T\int_{\Gamma^\varepsilon}|z_k^\varepsilon|^2\,d\sigma_x\,dt \le ||z_k^\varepsilon||^2_{L^\infty((0,T)\times\Gamma^\varepsilon)}\,\varepsilon\,|\Gamma^\varepsilon|T = T||z_k^\varepsilon||^2_{L^\infty((0,T)\times\Gamma^\varepsilon)}|\Gamma|\dfrac{|\Omega|}{|Y|} < \infty$. From this and lemma 5.8, the convergence results follow in the sense (5.9). □

We notice that the assumption (v) implies $\int_0^T\int_{\Omega_p^\varepsilon}|\vec{q^\varepsilon}|^p\,dx\,dt \le \sup_{\varepsilon>0}||\vec{q^\varepsilon}||^p_{L^\infty(S\times\Omega)}\,T\,|\Omega| < \infty$. Let $\vec{Q^\varepsilon}$ be extension of $\vec{q^\varepsilon}$ defined as follows:

$$\vec{Q^\varepsilon} \;=\; \begin{cases} \vec{q^\varepsilon} & \text{in } S\times\Omega_p^\varepsilon, \\ 0 & \text{in } S\times\Omega_s^\varepsilon. \end{cases}$$

For the sake of brevity, we still denote the extension of $\vec{q^\varepsilon}$ by $\vec{q^\varepsilon}$. We see that the extended velocity is bounded in $L^p(S\times\Omega)$, hence in $L^2(S\times\Omega)$. Therefore $\vec{q^\varepsilon}$ is two-scale convergent to the limit $\vec{q_1}$ in $L^2(S\times\Omega\times Y)$ and weakly convergent to $\vec{q}=\int_Y\vec{q_1}\,dy$ in $L^2(S\times\Omega)$. We use the convergence of u^ε and $\vec{q^\varepsilon}$ to show that $\vec{q^\varepsilon}u_i^\varepsilon$ two-scale converges to $\vec{q_1}u_i$ for each $i=1,2,...,I$. This means that the unfolded sequence $\mathcal{T}^\varepsilon\vec{q^\varepsilon}u_i^\varepsilon = \mathcal{T}^\varepsilon\vec{q^\varepsilon}\mathcal{T}^\varepsilon u_i^\varepsilon$ is weakly convergent. Let $\varphi\in C_0^\infty(S\times\Omega;C^\infty_{per}(Y))$, then

$$\begin{aligned}
&\left|\int_S\int_\Omega\int_Y \mathcal{T}^\varepsilon\vec{q^\varepsilon}\mathcal{T}^\varepsilon u_i^\varepsilon\varphi\,dx\,dy\,dt - \int_S\int_\Omega\int_Y\vec{q_1}u_i\varphi\,dx\,dy\,dt\right| \\
&\le ||\mathcal{T}^\varepsilon\vec{q^\varepsilon}||_{L^2(S\times\Omega\times Y)}||\varphi||_{L^\infty(S\times\Omega\times Y)}||\mathcal{T}^\varepsilon u_i^\varepsilon - u_i||_{L^2(S\times\Omega\times Y)} + \left|\int_S\int_\Omega\int_Y(\mathcal{T}^\varepsilon\vec{q^\varepsilon}-\vec{q_1})u_i\varphi\,dx\,dy\,dt\right| \\
&\to 0 \; as\; \varepsilon\to 0.
\end{aligned} \tag{4.3}$$

We have used the weak convergence of $\mathcal{T}^\varepsilon\vec{q^\varepsilon}$ to $\vec{q_1}$, strong convergence of $\mathcal{T}^\varepsilon u_i^\varepsilon$ to u_i and the norm preserving property of $\mathcal{T}^\varepsilon$, i.e. $||\mathcal{T}^\varepsilon\vec{q^\varepsilon}||_{L^2(S\times\Omega\times Y)} = ||\vec{q^\varepsilon}||_{L^2(S\times\Omega)}$. From (4.3) it follows that $\mathcal{T}^\varepsilon\vec{q^\varepsilon}\mathcal{T}^\varepsilon u_i^\varepsilon$ is weakly convergent to $\vec{q_1}u_i$. This gives the two-scale convergence of $\vec{q^\varepsilon}u_i^\varepsilon$ to $\vec{q_1}u_i$.

Next we focus on the nonlinear rate term on Γ^ε. We use the similar method as shown in [MP08, FM12]. Let $\mathcal{T}_b^\varepsilon : L^2(S\times\Gamma^\varepsilon) \to L^2(S\times\Omega\times\Gamma)$ be the boundary unfolding operator defined as

$$\mathcal{T}_b^\varepsilon w_k^\varepsilon(t,x,y) := w_k^\varepsilon(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y), \quad \text{for every } (t,x,y)\in S\times\Omega\times\Gamma.$$

We apply $\mathcal{T}_b^\varepsilon$ to (3.22f)-(3.22h) which gives

$$\begin{aligned}\mathcal{T}_b^\varepsilon\left(\frac{\partial w_k^\varepsilon}{\partial t}\right)(t,x,y) = \frac{\partial}{\partial t}(w_k^\varepsilon)(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y) &= \frac{\partial}{\partial t}\left(w_k^\varepsilon(t,\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y)\right)\\ &= \frac{\partial}{\partial t}\mathcal{T}_b^\varepsilon(w_k^\varepsilon(t,x,y))\end{aligned}$$

and,

$$\mathcal{T}_b^\varepsilon(R_p(u^\varepsilon)_k - k_k^d z_k^\varepsilon) = R_p(\mathcal{T}_b^\varepsilon u^\varepsilon)_k - k_k^d\mathcal{T}_b^\varepsilon z_k^\varepsilon \text{ and } \mathcal{T}_b^\varepsilon z_k^\varepsilon \in \psi(\mathcal{T}_b^\varepsilon w_k^\varepsilon).$$

By definition of ψ,

$$\psi(w_k^\varepsilon) \;=\; \begin{cases} 0 & \text{if } w_k^\varepsilon < 0,\\ 0\le\psi(w_k^\varepsilon)\le 1 & \text{if } w_k^\varepsilon = 0,\\ 1 & \text{if } w_k^\varepsilon > 0.\end{cases}$$

Via the substitution $x \to \varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y$, we obtain

$$\psi(\mathcal{T}_b^\varepsilon w_k^\varepsilon) = \mathcal{T}_b^\varepsilon\psi(w_k^\varepsilon) \;=\; \begin{cases} 0 & \text{if } \mathcal{T}_b^\varepsilon w_k^\varepsilon < 0,\\ 0\le\psi(\mathcal{T}_b^\varepsilon w_k^\varepsilon)\le 1 & \text{if } \mathcal{T}_b^\varepsilon w_k^\varepsilon = 0,\\ 1 & \text{if } \mathcal{T}_b^\varepsilon w_k^\varepsilon > 0,\end{cases}$$

i.e. $\mathcal{T}_b^\varepsilon z^\varepsilon \in \psi(\mathcal{T}_b^\varepsilon w_k^\varepsilon)$. Therefore the unfolded ODE system is

$$\frac{\partial}{\partial t}\mathcal{T}_b^\varepsilon w_k^\varepsilon = R_p(\mathcal{T}_b^\varepsilon u^\varepsilon)_k - k_k^d\mathcal{T}_b^\varepsilon z_k^\varepsilon \quad \text{in} \quad (0,T)\times\Omega\times\Gamma, \tag{4.4a}$$

$$\mathcal{T}_b^\varepsilon z^\varepsilon \in \psi(\mathcal{T}_b^\varepsilon w_k^\varepsilon) \quad \text{in} \quad (0,T)\times\Omega\times\Gamma, \tag{4.4b}$$

$$\mathcal{T}_b^\varepsilon(w_k^\varepsilon)(0,x,y) = w_{0_k}(x,y) \quad \text{on} \quad \Omega\times\Gamma. \tag{4.4c}$$

for $k=1,2,...,\bar{I}$. We show in lemma that $(\mathcal{T}_b^\varepsilon u_i^\varepsilon)$ is strongly convergent to u_i in $L^2(S\times\Omega\times\Gamma)$. This implies that $\mathcal{T}_b^\varepsilon u_i^\varepsilon$ is pointwise convergent to u_i in $S\times\Omega\times\Gamma$. Since $|\mathcal{T}_b^\varepsilon u_i^\varepsilon| < \frac{u_i}{2}$, $|\mathcal{T}_b^\varepsilon R_p(u^\varepsilon)_k| = |R_p(\mathcal{T}_b^\varepsilon u^\varepsilon)_k| = |\mathcal{T}_b^\varepsilon u_i^\varepsilon \mathcal{T}_b^\varepsilon u_j^\varepsilon| < \frac{|u_i||u_j|}{4}$. Then by *dominated convergence theorem for L^p spaces* it follows that $||\mathcal{T}_b^\varepsilon R_p(u^\varepsilon)_k - R_p(u)_k||_{L^2(S\times\Omega\times\Gamma)} \to 0$ as $\varepsilon\to 0$. This gives the weak convergence of $\mathcal{T}_b^\varepsilon R_p(u^\varepsilon)_k$ and hence, the two-scale convergence of $R_p(u^\varepsilon)_k$ to $R_p(u)_k$.

Lemma 4.7. *The sequence $(\mathcal{T}_b^\varepsilon u^\varepsilon)$ is strongly convergent in $L^2(S\times\Omega\times\Gamma)^I$.*

Proof. We use the arguments of Lemma 3.5 to obtain the strong convergence of u_i^ε to u_i in $L^2(S;H^{s-\frac{1}{2}}(\Gamma^\varepsilon))$ for $0<s<\frac{1}{2}$, i.e. in $L^2(S;L^2(\Gamma^\varepsilon))$ for $s=\frac{1}{2}$. Thus by norm preserving property

$$||\mathcal{T}_b^\varepsilon u^\varepsilon - u||^2_{L^2(S\times\Omega\times\Gamma)^I} = \sum_{i=1}^{I}||\mathcal{T}_b^\varepsilon u_i^\varepsilon - u_i||^2_{L^2(S\times\Omega\times\Gamma)} = \sum_{i=1}^{I}||u_i^\varepsilon - u_i||^2_{L^2(S\times\Gamma^\varepsilon)} \to 0 \; as \; \varepsilon\to 0,$$

i.e. $\mathcal{T}_b^\varepsilon u^\varepsilon$ is strongly convergent to u in $L^2(S\times\Omega\times\Gamma)^I$. □

4.2 Passage to the limit as $\varepsilon \to 0$

Lemma 4.8. *The limit function* $(w,z)\in H^{1,p}((0,T);L^p(\Omega\times\Gamma))^{\bar{I}}\times L^\infty((0,T)\times\Omega\times\Gamma)^{\bar{I}}$ *satisfies the homogenized problem*

$$\frac{\partial w_k}{\partial t} = R(u)_k - k_k^d z_k \qquad in \qquad (0,T)\times\Omega\times\Gamma, \tag{4.5a}$$

$$z_k \in \psi(w)_k \qquad in \qquad (0,T)\times\Omega\times\Gamma, \tag{4.5b}$$

$$w_k(0,x,y) = w_{0_k}(x,y) \qquad on \qquad \Omega\times\Gamma. \tag{4.5c}$$

for $k=1,2,...,\bar{I}$.

Proof. Passing the two-scale limit in (3.22f) is straightforward whereas (3.22g) will require special attention. Below we will show that (3.22g) reduces to a similar form as $\varepsilon\to 0$. The limit problem will be obtained by using lemmas 4.6 and 4.7. Testing (3.22f) by $\phi\in C_0^\infty((0,T)\times\Omega;C_{per}^\infty(Y))^{\bar{I}}$, we obtain

$$\sum_{k=1}^{\bar{I}}\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\frac{\partial w_k^\varepsilon(t,x)}{\partial t}\phi_k(t,x,\frac{x}{\varepsilon})\,d\sigma_x\,dt$$
$$= \sum_{k=1}^{\bar{I}}\left(\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}R_p(u^\varepsilon)_k\phi_k(t,x,\frac{x}{\varepsilon})\,d\sigma_x\,dt - k_k^d\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}z_k^\varepsilon\phi_k(t,x,\frac{x}{\varepsilon})\,d\sigma_x\,dt\right),$$

$$\sum_{k=1}^{\bar{I}}\int_0^T\int_\Omega\int_\Gamma\frac{\partial w_k^\varepsilon(t,x,y)}{\partial t}\phi_k(t,x,y)\,dx\,d\sigma_y\,dt$$
$$= \sum_{k=1}^{\bar{I}}\left(\int_0^T\int_\Omega\int_\Gamma R_p(u)_k\phi_k(t,x,y)\,dx\,d\sigma_y\,dt - k_k^d\int_0^T\int_\Omega\int_\Gamma z_k(t,x,y)\phi_k(t,x,y)\,dx\,d\sigma_y\,dt\right),$$

$$\int_0^T\int_\Omega\int_\Gamma\langle\frac{\partial w(t,x,y)}{\partial t},\phi(t,x,y)\rangle\,dx\,d\sigma_y\,dt = \int_0^T\int_\Omega\int_\Gamma\langle R(u,w),\phi(t,x,y)\rangle\,dx\,d\sigma_y\,dt. \tag{4.6}$$

Since $(\mathcal{T}_b^\varepsilon w^\varepsilon)$ is strongly convergent to w in $L^2(S\times\Omega\times\Gamma)^{\bar{I}}$, it is pointwise convergent to w almost everywhere in $S\times\Omega\times\Gamma$, i.e. $\lim_{\varepsilon\to 0}T_b^\varepsilon w^\varepsilon = w$ a.e. in $S\times\Omega\times\Gamma$. As $\mathcal{T}_b^\varepsilon w^\varepsilon \geq 0$, $w\geq 0$. We have two different cases.

Case 1: Let $w \geq 0$.
Then for any $\eta > 0$ there exists a $\delta > 0$ such that $|\varepsilon| < \delta \Rightarrow |\mathcal{T}_b^\varepsilon w^\varepsilon - w| < \eta$. We choose $\eta = \frac{|w|}{2} = \frac{w}{2}$ then $|\varepsilon| < \delta \Rightarrow \mathcal{T}_b^\varepsilon w^\varepsilon > \frac{w}{2}$. By definition of ψ, we have $\mathcal{T}_b^\varepsilon z^\varepsilon = 1$ which implies $z = 1$ due to weak convergence.
Case 2: Let $w = 0$.
Note that $\frac{\partial w_k^\varepsilon}{\partial t} \in L^p(S \times \Gamma^\varepsilon)^{\bar{I}}$. For a test function $\varphi \in C_0^\infty(S \times \Omega)$, we obtain

$$\int_0^T \int_{\Gamma^\varepsilon} \frac{\partial w_k^\varepsilon}{\partial t} \varphi \, d\sigma_x \, dt = -\int_0^T \int_{\Gamma^\varepsilon} w_k^\varepsilon \frac{\partial \varphi}{\partial t} \, d\sigma_x \, dt \to 0 \text{ as } \varepsilon \to 0, \text{ since the limit } w_k = 0.$$

Since φ is arbitrary, $\frac{\partial w_k^\varepsilon}{\partial t} \overset{2}{\rightharpoonup} 0$, i.e. $\frac{\partial}{\partial t}(\mathcal{T}_b^\varepsilon w_k^\varepsilon) \overset{w}{\rightharpoonup} 0$ in $L^2(S \times \Omega \times \Gamma)$. Then

$$\int_0^T \int_\Omega \int_\Gamma \frac{\partial \mathcal{T}_b^\varepsilon w_k^\varepsilon}{\partial t} \varphi \, dx \, d\sigma_y \, dt = \int_0^T \int_\Omega \int_\Gamma (\mathcal{T}_b^\varepsilon R_p(u^\varepsilon)_k - k_k^d \mathcal{T}_b^\varepsilon z_k^\varepsilon) \varphi \, dx \, d\sigma_y \, dt.$$

We make $\varepsilon \to 0$,

$$\int_0^T \int_\Omega \int_\Gamma 0.\varphi \, dx \, d\sigma_y \, dt = \int_0^T \int_\Omega \int_\Gamma (R_p(u)_k - k_k^d z_k) \varphi \, dx \, d\sigma_y \, dt, \tag{4.7}$$

which leads to $R(u) = z$ and $0 \leq z \leq 1$ when $w = 0$ on Γ. Since $w_{0_k} \in L^p(\Omega)$, $T_b^\varepsilon w_{0_k}$ is strongly convergent to some w_{0_k} (using the same notation) in $L^p(\Omega \times \Gamma)$. Therefore for $k = 1, 2, ..., \bar{I}$, the limit problem is

$$\frac{\partial w_k}{\partial t} = R_p(u)_k - k_k^d z_k \quad \text{in} \quad (0,T) \times \Omega \times \Gamma, \tag{4.8a}$$
$$z_k \in \psi(w)_k \quad \text{in} \quad (0,T) \times \Omega \times \Gamma, \tag{4.8b}$$
$$w_k(0,x,y) = w_{0_k}(x,y) \quad \text{on} \quad \Omega \times \Gamma. \tag{4.8c}$$

□

We use following lemma from [MZ11] for the homogenization of (1.14a)-(1.14e):

Lemma 4.9. *Let $a_j(y)$ for $j = 1, 2, ..., n$ be the $Y-$ periodic solution of the integral identity*

$$\int_0^T \int_\Omega \int_{Y^p} (e_j + \nabla_y a_j(y)) \cdot \nabla_y \phi_{1_i} \, dx \, dy \, dt = 0, \tag{4.9}$$

and $a_0(t,x,y)$ be the solution to the integral identity

$$\int_0^T \int_\Omega \int_{Y^p} (\vec{q}_1 + \nabla_y a_0) \cdot \nabla_y \phi_{1_i} \, dx \, dy \, dt = 0, \tag{4.10}$$

for any $Y-$ periodic smooth function ϕ_1. Then the function

$$u_i^1(x,y,t) = \sum_{j=1}^n \frac{\partial u_i(t,x)}{\partial x_j} a_j(y) + a_0(x,y,t) u_i(t,x) + c_i(x)$$

satisfies

$$\int_0^T \int_\Omega \int_{Y^p} \left(D_{diff} \left(\nabla u_i + \nabla_y u_i^1 \right) - u_i \vec{q}_1 \right) \cdot \nabla_y \phi_{1_i} \, dx \, dy \, dt = 0. \tag{4.11}$$

Note that $c_i(.)$ in the expression for u_i^1 is a function of x only, however, without loss of generality we can assume that $c_i(x)=0$ for each $i=1,2,...,I$. We now let $\varepsilon\to 0$ in (3.22a). Let us choose the functions $\phi_0\in C_0^\infty((0,T)\times\Omega)^I$ and $\phi_1\in C_0^\infty((0,T)\times\Omega;C_{per}^\infty(Y))^I$. Set $\phi(t,x,\frac{x}{\varepsilon})=\phi_0(t,x)+\varepsilon\phi_1(t,x,\frac{x}{\varepsilon})\in C_0^\infty((0,T)\times\Omega;C_{per}^\infty(Y))^I$. Using ϕ as test function in the weak formulation of (3.22a), we get

$$\sum_{i=1}^{I}\int_0^T\left\langle\frac{\partial u_i^\varepsilon}{\partial t},\phi_i\right\rangle dt+\sum_{i=1}^{I}\int_0^T\int_{\Omega_\varepsilon^p}\left(D_{diff}\nabla u_i^\varepsilon-\vec{q}^\varepsilon u_i^\varepsilon\right)\nabla\phi_i\,dx\,dt$$
$$+\sum_{k=1}^{\bar{I}}\varepsilon\sigma_{ik}\int_0^T\int_{\Gamma^\varepsilon}\frac{\partial w_k^\varepsilon}{\partial t}\phi_i\,d\sigma_x\,dt=\sum_{i=1}^{I}\int_0^T\langle\mathcal{S}R(u^\varepsilon)_i,\phi_i\rangle\,dt,$$

i.e.,

$$I_{time}+I_{diff}+I_{bound}=I_{reac}, \tag{4.12}$$

where

$$I_{time}=\sum_{i=1}^{I}\int_0^T\left\langle\frac{\partial u_i^\varepsilon}{\partial t},\phi_i\right\rangle dt, \tag{4.13a}$$
$$I_{diff}=\sum_{i=1}^{I}\int_0^T\int_{\Omega_\varepsilon^p}\left(D_{diff}\nabla u_i^\varepsilon-\vec{q}^\varepsilon u_i^\varepsilon\right)\nabla\phi_i\,dx\,dt, \tag{4.13b}$$
$$I_{bound}=\sum_{k=1}^{\bar{I}}\varepsilon\sigma_{ik}\int_0^T\int_{\Gamma^\varepsilon}\frac{\partial w_k^\varepsilon}{\partial t}\phi_i\,d\sigma_x\,dt, \tag{4.13c}$$
$$I_{reac}=\sum_{i=1}^{I}\int_0^T\langle\mathcal{S}R(u^\varepsilon)_i,\phi_i\rangle\,dt. \tag{4.13d}$$

Now we pass to the *two-scale* limit in each term separately. We have

$$\lim_{\varepsilon\to 0}I_{time}=-\lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_\Omega\chi^\varepsilon(x)\left\langle u_i^\varepsilon,\frac{\partial}{\partial t}\phi_i\right\rangle dx\,dt=-\sum_{i=1}^{I}\int_0^T\int_\Omega\int_Y\chi(y)\left\langle u_i,\frac{\partial}{\partial t}\phi_i\right\rangle dx\,dy\,dt \tag{4.14}$$

and by (4.13d)

$$\lim_{\varepsilon\to 0}I_{reac}=\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\chi^\varepsilon(x)\,\langle\mathcal{S}R(u^\varepsilon),\phi_0\rangle\,dx\,dt=\int_0^T\int_\Omega\int_Y\chi(y)\,\langle\mathcal{S}R(u),\phi_0\rangle\,dx\,dy\,dt. \tag{4.15}$$

Next,

$$\begin{aligned}\lim_{\varepsilon\to 0} I_{diff} =&\lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_{\Omega_\varepsilon^p}(D\nabla u_i^\varepsilon-\vec{q}^\varepsilon u_i^\varepsilon)(\nabla\phi_{0_i}+\nabla_y\phi_{1_i}+\varepsilon\nabla\phi_{1_i})\,dx\,dt\\ =&\lim_{\varepsilon\to 0}\sum_{i=1}^{I}\int_0^T\int_\Omega \chi\left(\frac{x}{\varepsilon}\right)(D\nabla u_i^\varepsilon-\vec{q}^\varepsilon u_i^\varepsilon)(\nabla\phi_{0_i}+\nabla_y\phi_{1_i})\,dx\,dt\\ &+\underbrace{\lim_{\varepsilon\to 0}\varepsilon\sum_{i=1}^{I}\int_0^T\int_\Omega \chi\left(\frac{x}{\varepsilon}\right)(D\nabla u_i^\varepsilon-\vec{q}^\varepsilon u_i^\varepsilon)\nabla\phi_{1_i}\,dx\,dt}_{=0}\\ =&\sum_{i=1}^{I}\int_0^T\int_\Omega\int_Y \chi(y)\left(D\left(\nabla u_i+\nabla_y u_i^1\right)-u_i\vec{q}_1\right)(\nabla\phi_{0_i}+\nabla_y\phi_{1_i})\,dx\,dy\,dt \qquad (4.16)\end{aligned}$$

Again,

$$\begin{aligned}\lim_{\varepsilon\to 0} I_{bound} &= \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}}\sigma_{ik}\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\frac{\partial w_k^\varepsilon}{\partial t}\phi_i\,d\sigma_x\,dt\\ &= \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}}\sigma_{ik}\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\frac{\partial w_k^\varepsilon}{\partial t}\phi_{0_i}\,d\sigma_x\,dt+\underbrace{\sum_{i=1}^{I}\sum_{k=1}^{\bar{I}}\sigma_{ik}\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\frac{\partial w_k^\varepsilon}{\partial t}\phi_{1_i}\,d\sigma_x\,dt}_{=0}\\ &= \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}}\sigma_{ik}\int_0^T\int_\Omega\int_\Gamma\frac{\partial w_k}{\partial t}\phi_{0_i}\,dx\,d\sigma_y\,dt\\ &= \int_0^T\int_\Omega\int_\Gamma\left\langle\sigma\frac{\partial w}{\partial t},\phi_0\right\rangle dx\,d\sigma_y\,dt. \qquad (4.17)\end{aligned}$$

Combining (4.12), (4.14), (4.15), (4.16) and (4.17), we obtain

$$\begin{aligned}&|Y^p|\int_0^T\left\langle\frac{\partial u}{\partial t},\phi_0\right\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}dt\\ &+\sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p}\left(D\left(\nabla u_ß+\nabla_y u_i^1\right)-u_i\vec{q}_1\right)(\nabla\phi_{0_i}+\nabla_y\phi_{1_i})\,dx\,dy\,dt\\ &+\int_0^T\int_\Omega\int_\Gamma\left\langle\sigma\frac{\partial w}{\partial t},\phi_0\right\rangle dx\,d\sigma_y\,dt=|Y^p|\int_0^T\langle\mathcal{S}R(u),\phi_0\rangle_{[H^{1,2}(\Omega)^*]^I\times[H^{1,2}(\Omega)]^I}\,dt. \qquad (4.18)\end{aligned}$$

We decouple the equation (4.18) to achieve the homogenized equation and the *Cell-Problem*. Setting $\phi_0\equiv 0$, the equation (4.18) reduces to

$$\sum_{i=1}^{I}\int_0^T\int_\Omega\int_{Y^p}\left(D\left(\nabla u_i+\nabla_y u_i^1\right)-u_i\vec{q}_1\right)\cdot\nabla_y\phi_{1_i}\,dx\,dy\,dt=0, \qquad (4.19)$$

We set $\vec{q}_0 = D\int_{Y^p} \nabla_y a_0 \, dy$. Now setting $\phi_1 \equiv 0$, then (4.18) reduces to

$$\begin{aligned}
&|Y^p| \int_0^T \left\langle \frac{\partial u}{\partial t}, \phi_0 \right\rangle dt + \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} \left(D\left(\nabla u_i + \nabla_y u_i^1\right) - u_i \vec{q}_1 \right) \nabla \phi_{0_i} \, dx \, dy \, dt \\
&\quad + \int_0^T \int_\Omega \int_\Gamma \left\langle \sigma \frac{\partial w}{\partial t}, \phi_0 \right\rangle dx \, d\sigma_y \, dt = |Y^p| \int_0^T \langle \mathcal{S}R(u), \phi_0 \rangle \, dt, \\
&\int_0^T \left\langle \frac{\partial u}{\partial t}, \phi_0 \right\rangle dt + \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} \frac{D}{|Y^p|} \left(\nabla u_i + \nabla_y u_i^1\right) \nabla \phi_{0_i} \, dx \, dy \, dt \\
&\quad - \frac{1}{|Y^p|} \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} u_i \vec{q}_1 \nabla \phi_{0_i} \, dx \, dy \, dt \\
&\quad = \int_0^T \langle \mathcal{S}R(u), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \sigma \frac{\partial w}{\partial t}, \phi_0 \right\rangle dx \, d\sigma_y \, dt. \qquad (4.20)
\end{aligned}$$

Substituting $u_i^1(x,y,t) = \sum_{j=1}^{n} \frac{\partial u_i(t,x)}{\partial x_j} a_j(y) + a_0(x,y,t) u_i(t,x)$, for $i = 1, 2, ..., I$, in (4.20) leaves

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial u}{\partial t}, \phi_0 \right\rangle dt + \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} \frac{D}{|Y^p|} \left(\nabla u_i + \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} \nabla_y a_j + \nabla_y a_0 u_i \right) \nabla \phi_{0_i} \, dx \, dy \, dt \\
&\quad - \frac{1}{|Y^p|} \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} u_i \vec{q}_1 \nabla \phi_{0_i} \, dx \, dy \, dt \\
&\quad = \int_0^T \langle \mathcal{S}R(u), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \sigma \frac{\partial w}{\partial t}, \phi_0 \right\rangle dx \, d\sigma_y \, dt, \\
&\int_0^T \left\langle \frac{\partial u}{\partial t}, \phi_0 \right\rangle dt + \sum_{i=1}^{I} \int_0^T \int_\Omega \sum_{l,j=1}^{n} \left(\frac{D}{|Y^p|} \int_{Y^p} \left(\delta_{jl} + \frac{\partial a_j}{\partial y_l} \right) dy \right) \frac{\partial u_i}{\partial x_j} \frac{\partial \phi_{0_i}}{\partial x_l} \, dx \, dt \\
&\quad - \frac{1}{|Y^p|} \sum_{i=1}^{I} \int_0^T \int_\Omega \int_{Y^p} (\vec{q}_1 - D \nabla_y a_0) u_i \nabla \phi_{0_i} \, dx \, dy \, dt \\
&\quad = \int_0^T \langle \mathcal{S}R(u), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \sigma \frac{\partial w_\delta}{\partial t}, \phi_0 \right\rangle dx \, d\sigma_y \, dt, \\
&\int_0^T \left\langle \frac{\partial u}{\partial t}, \phi_0 \right\rangle dt + \sum_{i=1}^{I} \int_0^T \int_\Omega P \nabla u_i \cdot \nabla \phi_{0_i} \, dx \, dt - \frac{1}{|Y^p|} \sum_{i=1}^{I} \int_0^T \int_\Omega (\vec{q} - \vec{q}_0) u_i \nabla \phi_{0_i} \, dx \, dt \\
&\quad = \int_0^T \langle \mathcal{S}R(u), \phi_0 \rangle \, dt - \frac{1}{|Y^p|} \int_0^T \int_\Omega \int_\Gamma \left\langle \sigma \frac{\partial w}{\partial t}, \phi_0 \right\rangle dx \, d\sigma_y \, dt, \qquad (4.21)
\end{aligned}$$

where $P = \left(p_{jl}\right)_{\substack{1 \le j \le n \\ 1 \le l \le n}}$ is a positive definite second order symmetric tensor whose components are given by

$$p_{jl} = \int_{Y^p} \frac{D}{|Y^p|} \left(\delta_{jl} + \frac{\partial a_j}{\partial y_l} \right) dy \quad \text{for } j,l = 1,2,...,n. \tag{4.22}$$

Therefore the strong form of the homogenized equation is

$$\frac{\partial u}{\partial t} - \nabla \left(P\nabla u - \frac{1}{|Y^p|}(\vec{q} - \vec{q}_0)u \right) = \mathcal{S}R(u) - \frac{1}{|Y^p|}\int_\Gamma \sigma \frac{\partial w}{\partial t}\, d\sigma_y \quad \text{in} \quad (0,T)\times\Omega, \tag{4.23a}$$

$$-\left(P\nabla u - \frac{1}{|Y^p|}(\vec{q} - \vec{q}_0)u \right) \cdot \vec{n} = d \quad \text{on} \quad (0,T)\times\Gamma_{in}, \tag{4.23b}$$

$$-\left(P\nabla u + \frac{1}{|Y^p|}\vec{q}_0 u \right) \cdot \vec{n} = 0 \quad \text{on} \quad (0,T)\times\Gamma_{out}, \tag{4.23c}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \Omega, \tag{4.23d}$$

$$\frac{\partial w}{\partial t} = R(u) - k^d z \quad \text{in} \quad (0,T)\times\Omega\times\Gamma, \tag{4.23e}$$

$$z \in \psi(w) \quad \text{in} \quad (0,T)\times\Omega\times\Gamma, \tag{4.23f}$$

$$w(0,x,y) = w_0(x,y) \quad \text{on} \quad \Omega\times\Gamma. \tag{4.23g}$$

Lemma 4.10. *There exists a unique positive global weak solution* $(u,w,z) \in \mathcal{F}_p^I(\Omega) \cap L^r(S\times\Omega)^I \times \mathcal{M}_p^{\bar{I}}(\Gamma) \times \mathcal{N}_\infty^{\bar{I}}(\Gamma)$ *of the problem (4.23a)-(4.23g).*

Proof. We have shown in lemma 4.8 that $(w,z) \in \mathcal{M}_p^{\bar{I}}(\Gamma) \times \mathcal{N}_\infty^{\bar{I}}(\Gamma)$ solves (4.23e)-(4.23g). Again to show that $u \in \mathcal{F}_p^I(\Omega) \cap L^r(S\times\Omega)$ solves (4.23a)-(4.23d) for $(w,z) \in \mathcal{M}_p^{\bar{I}}(\Gamma) \times \mathcal{N}_\infty^{\bar{I}}(\Gamma)$, we take the abstract formulation of (4.23a)-(4.23g) and proceed as as in step 2 of lemma 3.5. The uniqueness can be shown by the same arguments as in lemma 3.5. This terminates the proof. □

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5 Appendix

Lemma 5.1 (Schaefer's fixed point theorem, cf. theorem 9.2.2.4 in [Eva98])**.** *Let X be Banach space. Assume that $\mathcal{Z}: X \to X$ is a continuous and compact mapping and the set*

$$\{u \in X | \exists \lambda \in [0,1] : u = \lambda \mathcal{Z}(u)\}$$

is bounded. Then $\mathcal{Z}$ has a fixed point.

Lemma 5.2 (Extension theorem, cf. [MB13b])**.** *Let $1 \leq p,q \leq \infty$. There exists a bounded linear (extension) operator $E^\varepsilon : L^q((0,T);H^{1,p}(\Omega_p^\varepsilon)) \to L^q((0,T);H^{1,p}(\Omega))$ such that for all $u^\varepsilon \in L^q((0,T);H^{1,p}(\Omega_p^\varepsilon))$ following estimate holds:*

$$||E^\varepsilon u^\varepsilon||_{L^q((0,T);H^{1,p}(\Omega))} \leq C||u^\varepsilon||_{L^q((0,T);H^{1,p}(\Omega_p^\varepsilon))}, \tag{5.1}$$

where C is independent of ε and u^ε.

Lemma 5.3 (Trace theorem on Γ^ε, cf. lemma 2.7.2 in [NR92])**.** *Let* $1 \le p < \infty$. *There exists a bounded linear operator* $T^\varepsilon : H^{1,p}(\Omega_p^\varepsilon) \to L^p(\Gamma^\varepsilon)$ *such that*

$$(i)\; T^\varepsilon u^\varepsilon := u^\varepsilon|_{\Gamma^\varepsilon} \text{ for } u^\varepsilon \in H^{1,p}(\Omega_p^\varepsilon) \cap C(\bar{\Omega}_p^\varepsilon), \tag{5.2}$$

$$(ii)\; \varepsilon \int_{\Gamma^\varepsilon} |T^\varepsilon u^\varepsilon|^p \, d\sigma_x \le C \left(\int_{\Omega_p^\varepsilon} |u^\varepsilon|^p \, dx + \varepsilon^p \int_{\Omega_p^\varepsilon} |\nabla_x u|^p \, dx \right), \tag{5.3}$$

where C *is independent of* ε *and* u^ε.

Lemma 5.4 (Boundary trace inequality for Ω_p^ε)**.** *Let* Ω, Ω_p^ε *and* $\partial\Omega$ *(*C^1*-boundary) be defined as in section 1.1 and* p, r, q *and* n *be chosen as in theorem 6.3 in [Auc14]. For* $u^\varepsilon \in H^{1,r}(\Omega_p^\varepsilon)$, *there exists a constant* C *independent of* ε *and* u^ε *such that*

$$\int_{\partial\Omega} |u^\varepsilon|^p \, ds \le C_1 \int_{\Omega_p^\varepsilon} |u^\varepsilon|^p \, dx + p\, C_2\, ||u^\varepsilon||^{p-1}_{L^q(\Omega_p^\varepsilon)} ||\nabla u^\varepsilon||_{L^r(\Omega_p^\varepsilon)}, \tag{5.4}$$

where C_1, C_2 *are the constants independent of* ε.

Proof. By extension theorem there exists an operator $P : H^{1,r}(\Omega_p^\varepsilon) \to H^{1,r}(\Omega)$ such that $Pu^\varepsilon|_{\Omega_p^\varepsilon} = u^\varepsilon$; $||P\nabla u^\varepsilon||_{L^r(\Omega)} \le C\, ||\nabla u^\varepsilon||_{L^r(\Omega_p^\varepsilon)}$ and $||Pu^\varepsilon||_{L^r(\Omega)} \le C\, ||u^\varepsilon||_{L^r(\Omega_p^\varepsilon)}$, where C is independent of ε. We still denote the extension Pu^ε by u^ε we then have by theorem 6.3 in [Auc14]

$$\begin{aligned}\int_{\partial\Omega} |u^\varepsilon|^p \, ds &\le C_1 \int_{\Omega} |u^\varepsilon|^p \, dx + p\, C_2\, ||u^\varepsilon||^{p-1}_{L^q(\Omega)} ||\nabla u^\varepsilon||_{L^r(\Omega)} \\ &\le C \left(\int_{\Omega_p^\varepsilon} |u^\varepsilon|^p \, dx + p\, ||u^\varepsilon||^{p-1}_{L^q(\Omega_p^\varepsilon)} ||\nabla u^\varepsilon||_{L^r(\Omega_p^\varepsilon)} \right).\end{aligned}$$

□

Similarly by theorem 6.3 in [Auc14] and a scaling argument it can be shown that

$$\varepsilon \int_{\Gamma^\varepsilon} |u^\varepsilon|^4 \, d\sigma_x \le C \int_{\Omega_p^\varepsilon} |u^\varepsilon|^4 \, dx + 4\, C_{Y_p} \left(\lambda\, \varepsilon^2 \int_{\Omega_p^\varepsilon} |\nabla u^\varepsilon|^2 \, dx + \Lambda_\lambda \int_{\Omega_p^\varepsilon} |u^\varepsilon|^6 \, dx \right), \tag{5.5}$$

where C, C_{Y_p} are finite and independent of ε and u^ε.

Definition 5.1. *(Maximal* L^p*-regularity) Let* $1 < p < \infty$, X *be a Banach space and* $A : D(A) \overset{d}{\hookrightarrow} X \to X$ *be a closed, not necessarily bounded, operator. The operator* A *is said to have the maximal (parabolic)* L^p*-regularity property if for every* $f \in L^p((0,T);X)$, *there exists a unique solution* $u \in L^p((0,T);D(A)) \cap H^{1,p}((0,T);X)$ *of*

$$\partial_t u + Au = f \text{ for a.e. } t, \quad u(0) = 0, \tag{5.6}$$

which satisfies

$$||u_t||_{L^p((0,T);X)} + ||u||_{L^p((0,T);D(A))} \;\le\; C\, ||f||_{L^p((0,T);X)}, \tag{5.7}$$

where $C > 0$ *is a constant.*

Let $A = (A_1, A_2, ..., A_I) : D(A)^I \to X^I$, then A is said to have the maximal regularity on X^I if each A_i has maximal regularity on X, where $i = 1, 2, ..., I$. For a detailed overview on maximal regularity, we refer the interested readers to [ACFP07], [Mon09], [Prü02], [RDR09], [KW04] and references therein.

5.1 Two-scale convergence

Definition 5.2. *A sequence of functions* (u^ε) *in* $L^p((0,T)\times\Omega)$ *is said to be two-scale convergent to a limit* $u \in L^p((0,T)\times\Omega\times Y)$ *if*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega u^\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon})\,dx\,dt = \int_0^T\int_\Omega\int_Y u(t,x,y)\,\phi(t,x,y)\,dx\,dt\,dy \tag{5.8}$$

for all $\phi \in L^q((0,T)\times\Omega; C_{per}(Y))$.

Definition 5.3 ([ADH96, NR96])**.** *Let* $1 \le p < \infty$. *A sequence* (u^ε) *in* $L^p((0,T)\times\Gamma^\varepsilon)$ *is said to be two-scale convergent to a limit* $u \in L^p((0,T)\times\Omega\times\Gamma)$ *if*

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon} u^\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon})\,d\sigma_x\,dt = \int_0^T\int_\Omega\int_\Gamma u(t,x,y)\phi(t,x,y)\,dx\,dy\,dt \tag{5.9}$$

for all $\phi \in C([0,T]\times\bar\Omega; C_{per}(Y))$.

Lemma 5.5. *For every bounded sequence* (u^ε) *in* $L^p(S\times\Omega)$ *there exists a subsequence* (u^ε) *(still denoted by same symbol) and a* $u \in L^p((0,T)\times\Omega\times Y)$ *such that* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$.

Lemma 5.6. *Let* (u^ε) *be strongly convergent to* $u \in L^p((0,T)\times\Omega)$, *then* $u^\varepsilon \stackrel{2}{\rightharpoonup} u_1$, *where* $u_1(t,x,y) = u(t,x)$.

Lemma 5.7. *Let* (u^ε) *be a sequence in* $L^p((0,T);H^{1,p}(\Omega))$ *such that* $u^\varepsilon \stackrel{w}{\rightharpoonup} u$ *in* $L^p((0,T);H^{1,p}(\Omega))$. *Then* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$ *and there exists a subsequence* (u^ε), *still denoted by same symbol, and an* $u_1 \in L^p((0,T)\times\Omega; H^{1,p}_{per}(Y))$ *such that* $\nabla_x u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla_x u + \nabla_y u_1$.

Lemma 5.8. *Let* (u^ε) *be a sequence in* $L^p((0,T)\times\Gamma^\varepsilon)$ *such that*

$$\varepsilon\int_0^T\int_{\Gamma^\varepsilon}|u^\varepsilon(t,x)|^p\,d\sigma_x\,dt \le C, \tag{5.10}$$

where C *is independent of* ε. *Then there exists a subsequence* (u^ε) *(still denoted by same symbol) and a two-scale limit* $u \in L^p((0,T)\times\Omega\times\Gamma)$ *such that* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$ *in the sense of (5.9).*

5.2 Periodic unfolding

Let $u^\varepsilon \in L^r(\Omega)$, $1 \le r \le \infty$. We define the unfolding operator $\mathcal{T}^\varepsilon : L^r(\Omega) \to L^r(\Omega\times Y)$ as

$$\mathcal{T}^\varepsilon u^\varepsilon(x,y) = u^\varepsilon(t^\varepsilon(x,y)) \qquad \text{for} \quad x \in Y_s^k \subset \Omega \tag{5.11a}$$

$$\mathcal{T}^\varepsilon u^\varepsilon(x,y) = u^\varepsilon(x) \qquad \text{for} \quad \bar{Y}_s^k \cap \partial\Omega \neq \emptyset. \tag{5.11b}$$

where $t^\varepsilon(x,y)=\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y$, $[s]$ being the lower integer part of s.

We note that the unfolding operator $\mathcal{T}^\varepsilon$ transforms a single variable function u on Ω into a two-variable function $\mathcal{T}^\varepsilon u^\varepsilon$ on $\Omega\times Y$, s.t. $u^\varepsilon(x)=\mathcal{T}^\varepsilon u^\varepsilon(x,x-\varepsilon\left[\frac{x}{\varepsilon}\right])$, cf. [CDG02, CDZ06].

Lemma 5.9. *Let (u^ε) be a bounded sequence in $L^p((0,T)\times\Omega)$. Then the following statements are equivalent:*

(a) $(\mathcal{T}^\varepsilon u^\varepsilon)$ weakly converges to u in $L^p((0,T)\times\Omega\times Y)$.

(b) (u^ε) two-scale converges to u in the sense of (5.9).

In the similar way we can define the boundary unfolding operator $\mathcal{T}_b^\varepsilon: L^r((0,T)\times\Gamma^\varepsilon)\to L^r((0,T)\times\Omega\times\Gamma)$, cf. [CDG08, CDZ06]. Some basic properties of periodic unfolding and its relation to the two-scale convergence are summarized, e.g., in Lemma 5.1 and Theorem 5.3 in [FS12]. Further information about unfolding operators and applications to homogenization can be found in [FS12, Fra10, CDD+12]. Now we are going to give the proofs of theorem 3.1, lemma 3.3 and corollary 3.1. We first begin with the proof of lemma 3.3.

5.3 Proof of lemma 3.3:

For $p>n+2$, $u^\varepsilon_{\delta,\tau}\in L^\infty(S\times\Omega^\varepsilon_p)^I$ (by theorem 2.2 in [MB13a]) and $\partial f_r(u^\varepsilon_{\delta,\tau})\in L^q(S;H^{1,q}(\Omega^\varepsilon_p))^I$ (see remark 3.1). Using $\partial f_r(u^\varepsilon_{\delta,\tau})$ in the weak formulation of (3.2a), we get

$$\int_0^t\langle\frac{\partial u^\varepsilon_\delta}{\partial\theta},\partial f_r(u^\varepsilon_{\delta,\tau})\rangle\,d\theta:=I^{(t)}_{diff}+I^{(t)}_{advec}+I^{(t)}_{bound}+I^{(t)}_{reac}\qquad\text{for a.e. } t\in S,\tag{5.12}$$

where

$$I^{(t)}_{diff}:=-\sum_{l=1}^n\int_0^t\int_{\Omega^\varepsilon_p}D\nabla_x u^\varepsilon_{\delta_i}\nabla_x(\partial f_r(u^\varepsilon_{\delta,\tau}))\,dx\,d\theta=-\sum_{i=1}^I\sum_{l=1}^n\int_0^t\int_{\Omega^\varepsilon_p}D\frac{\partial u^\varepsilon_{\delta_i}}{\partial x_l}\frac{\partial}{\partial x_l}(\partial f_r(u^\varepsilon_{\delta,\tau}))_i\,dx\,d\theta,\tag{5.13a}$$

$$I^{(t)}_{bound}:=\sum_{i=1}^I\int_0^t\int_{\Gamma_{in}}(-d_i+\vec{q}^\varepsilon\cdot\vec{n}u^\varepsilon_{\delta_i})\partial f_r(u^\varepsilon_{\delta,\tau})_i\,ds\,d\theta-\varepsilon\sum_{i=1}^I\sum_{k=1}^{\bar I}\int_0^t\int_{\Gamma^\varepsilon}\sigma_{ik}\frac{\partial w^\varepsilon_{\delta_k}}{\partial\theta}\partial f_r(u^\varepsilon_{\delta,\tau})_i\,d\sigma_x\,d\theta,\tag{5.13b}$$

$$I^{(t)}_{advec}:=-\sum_{i=1}^I\int_0^t\int_{\Omega^\varepsilon_p}\vec{q^\varepsilon}\cdot\nabla u^\varepsilon_{\delta_i}\,\partial f_r(u^\varepsilon_{\delta,\tau})_i\,dx\,d\theta,\tag{5.13c}$$

$$I^{(t)}_{reac}:=\int_0^t\langle S\bar R(u^\varepsilon_\delta),\partial f_r(u^\varepsilon_{\delta,\tau})\rangle\,dx\,d\theta.\tag{5.13d}$$

Now we estimate the r.h.s. of (5.12). The idea to estimate the terms $I^{(t)}_{diff}$, $I^{(t)}_{bound}$, $I^{(t)}_{advec}$ and $I^{(t)}_{reac}$ is similar to the one shown in [MB13a]. To begin with

$$I^{(t)}_{reac}\le r\,C\sum_{i=1}^I\int_0^t\int_{\Omega^\varepsilon_p}\tau\left[\left|\mu^0_i\right|+T\,|\Omega|\,|\log\tau|+u^\varepsilon_{\delta_i,\tau}\right]\,dx\,d\theta=:h(t,\tau,u^\varepsilon_{\delta,\tau})\qquad\text{for a.e. } t\in S,$$

where C is independent of ε, δ, τ, $u^\varepsilon_{\delta,\tau}$, $w^\varepsilon_{\delta,\tau}$ and all the other terms of $h(t,\tau,u^\varepsilon_{\delta,\tau})$ are bounded and tend to zero as $\tau\to 0$ for a.e. t, i.e.

$$I^{(t)}_{reac} \le h(t,\tau,u^\varepsilon_{\delta,\tau}) \to 0 \text{ as } \tau\to 0 \quad \text{for a.e. } t\in S. \tag{5.14}$$

$$\begin{aligned}
I^{(t)}_{advec} &= -\sum_{i=1}^{I}\int_0^t\int_{\Omega^\varepsilon_p} \vec{q^\varepsilon}\cdot\nabla u^\varepsilon_{\delta_i}\left(\partial f_r(u^\varepsilon_{\delta,\tau})\right)_i dx\,d\theta\\
&= -\int_0^t\int_{\Omega^\varepsilon_p}\nabla_x f_r(u^\varepsilon_{\delta,\tau})\cdot\vec{q^\varepsilon}\,dx\,d\theta\\
&\le -\int_0^t\int_{\Gamma_{in}} f_r(u^\varepsilon_{\delta,\tau})\vec{q^\varepsilon}\cdot\vec{n}\,ds\,d\theta,\ \text{ by (1.3)} - \text{(1.4) and } f_r(u^\varepsilon_{\delta,\tau})\ge 0.\\
&\le \left\|\vec{q^\varepsilon}\cdot\vec{n}\right\|_{L^\infty(S\times\Gamma_{in})}\int_0^t\int_{\Gamma_{in}} f_r(u^\varepsilon_{\delta,\tau})\,ds\,d\theta \le C_1\int_0^t\int_{\partial\Omega} f_r(u^\varepsilon_{\delta,\tau})\,ds\,d\theta \qquad \text{for a.e. } t\in S,
\end{aligned} \tag{5.15}$$

where $f_r\ge 0$ and $C_1 := ||\vec{q}\cdot\vec{n}||_{L^\infty(S\times\Gamma_{in})}$ which is independent of ε, δ, τ and t.

$$\begin{aligned}
I^{(t)}_{diff} &= -D\sum_{i=1}^{I}\sum_{l=1}^{n}\int_0^t\int_{\Omega^\varepsilon_p} r(r-1)f_{r-2}(u^\varepsilon_{\delta,\tau})\sum_{\upsilon=1}^{I}\left(\mu^0_\upsilon+\log u^\varepsilon_{\delta_\upsilon,\tau}\right)\left(\mu^0_i+\log u^\varepsilon_{\delta_i,\tau}\right)\frac{\partial u^\varepsilon_{\delta_\upsilon,\tau}}{\partial x_l}\frac{\partial u^\varepsilon_{\delta_i}}{\partial x_l}\,dx\,d\theta\\
&\qquad\underbrace{-D\sum_{i=1}^{I}\sum_{l=1}^{n}\int_0^t\int_{\Omega^\varepsilon_p} r f_{r-1}(u^\varepsilon_{\delta,\tau})\frac{1}{u^\varepsilon_{\delta_i,\tau}}\frac{\partial u^\varepsilon_{\delta_i}}{\partial x_l}\frac{\partial u^\varepsilon_{\delta_i}}{\partial x_l}\,dx\,d\theta}_{\le 0}\\
&\le -D\sum_{l=1}^{n}\int_0^t\int_{\Omega^\varepsilon_p} r(r-1)f_{r-2}(u^\varepsilon_{\delta,\tau})\left(\sum_{i=1}^{I}\left(\mu^0_i+\log u^\varepsilon_{\delta_i,\tau}\right)\frac{\partial u^\varepsilon_{\delta_i}}{\partial x_l}\right)^2 dx\,d\theta \qquad \text{for a.e. } t\in S.
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
I^{(t)}_{bound} &= \sum_{i=1}^{I}\int_0^t\int_{\Gamma_{in}}(-d_i+\vec{q^\varepsilon}\cdot\vec{n}u^\varepsilon_{\delta_i})\partial f_r(u^\varepsilon_{\delta,\tau})_i\,ds\,d\theta - \varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\int_0^t\int_{\Gamma^\varepsilon}\sigma_{ik}\frac{\partial w^\varepsilon_{\delta_k}}{\partial\theta}\partial f_r(u^\varepsilon_{\delta,\tau})_i\,d\sigma_x\,d\theta,\\
&= \sum_{i=1}^{I}\int_0^t\int_{\Gamma_{in}} r\ f_{r-1}(u^\varepsilon_{\delta,\tau})\Big((-d_i+|\vec{q^\varepsilon}\cdot\vec{n}|\tau)-|\vec{q^\varepsilon}\cdot\vec{n}|u^\varepsilon_{\delta_i,\tau}\Big)(\mu^0_i+\log u^\varepsilon_{\delta_i,\tau})\,ds\,d\theta\\
&\quad -\varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\int_0^t\int_{\Gamma^\varepsilon} r\ f_{r-1}(u^\varepsilon_{\delta,\tau})\sigma_{ik}\left(k^p_k\prod_{m=1}^{I}(u^{\varepsilon^+}_{\delta_m})^{\sigma_{mk}}\right)(\mu^0_i+\log u^\varepsilon_{\delta_i,\tau})\,d\sigma_x\,d\theta\\
&\qquad +\varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\int_0^t\int_{\Gamma^\varepsilon} r\ f_{r-1}(u^\varepsilon_{\delta,\tau})\sigma_{ik}k^d_k\psi_\delta(w^\varepsilon_\delta)_k(\mu^0_i+\log u^\varepsilon_{\delta_i,\tau})\,d\sigma_x\,d\theta.
\end{aligned} \tag{5.17}$$

We now estimate I_{bound} term by term. Set $\Gamma^+_{in} := \{x\in\Gamma_{in} : \mu^0_i+\log u^\varepsilon_{\delta_i,\tau}\ge 0\}$ and $\Gamma^-_{in} := \{x\in\Gamma_{in} : \mu^0_i+\log u^\varepsilon_{\delta_i,\tau}\le 0\}$. Note that $g(u^\varepsilon_{\delta,\tau})\ge g_i(u^\varepsilon_{\delta_i,\tau})\ge (e-1)e^{-\mu^0_i} > 0$ and $\log u^\varepsilon_{\delta_i,\tau}\le$

$u^{\varepsilon}_{\delta_i,\tau} \le g_i(u^{\varepsilon}_{\delta_i,\tau})$. Therefore $\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau} \le |\mu_i^0| + g_i(u^{\varepsilon}_{\delta_i,\tau}) \le \left(1 + \max_{1\le i\le I} |\mu_i^0| e^{\mu_i^0}(e-1)^{-1}\right) g_i(u^{\varepsilon}_{\delta_i,\tau})$ and $f_r(u^{\varepsilon}_{\delta,\tau}) = f_{r-1}(u^{\varepsilon}_{\delta,\tau}) g(u^{\varepsilon}_{\delta,\tau})$. This gives

$$\begin{aligned}
&\sum_{i=1}^{I} \int_0^t \int_{\Gamma_{in}} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau})(-d_i + |\vec{q^{\varepsilon}} \cdot \vec{n}|\tau)(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, ds\, d\theta \\
&= \sum_{i=1}^{I} \int_0^t \Bigg[\int_{\Gamma_{in}^+} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau})(-d_i + |\vec{q^{\varepsilon}} \cdot \vec{n}|\tau)(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, ds \\
&\qquad + \int_{\Gamma_{in}^-} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau})(-d_i + |\vec{q^{\varepsilon}} \cdot \vec{n}|\tau)(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, ds \Bigg] d\theta \qquad (5.18)
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{i=1}^{I} \int_0^t \int_{\Gamma_{in}^+} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau})(-d_i + |\vec{q^{\varepsilon}} \cdot \vec{n}|\tau)(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, ds\, d\theta \\
&\le C_2\, r \int_0^t \int_{\Gamma_{in}} f_{r-1}(u^{\varepsilon}_{\delta,\tau}) g(u^{\varepsilon}_{\delta,\tau})\, ds\, d\theta \\
&\le C_2\, r \int_0^t \int_{\partial\Omega} f_r(u^{\varepsilon}_{\delta,\tau})\, ds\, d\theta, \text{ since } \Gamma_{in} \subset \partial\Omega, \qquad (5.19)
\end{aligned}$$

where $C_2 = I \left(1 + \max_{1\le i\le I} |\mu_i^0| e^{\mu_i^0}(e-1)^{-1}\right) \left(\max_{1\le i\le I} ||d_i||_{L^\infty(S\times\Gamma_{in})} + \tau\, ||\vec{q}^{\varepsilon} \cdot \vec{n}||_{L^\infty(S\times\Gamma_{in})}\right)$. We know $-u^{\varepsilon}_{\delta_i,\tau}(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau}) \le e^{-(1+\mu_i^0)}$ and $1 \le e^{\mu_i^0}(e-1)^{-1} g(u^{\varepsilon}_{\delta,\tau})$ for all $i = 1, 2, ..., I$.

$$\begin{aligned}
&-\sum_{i=1}^{I} \int_0^t \int_{\Gamma^{in}} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau}) |\vec{q^{\varepsilon}} \cdot \vec{n}| u^{\varepsilon}_{\delta_i,\tau}(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, ds\, d\theta \\
&\le\ r\ \sum_{i=1}^{I} e^{-(1+\mu_i^0)}\, ||\vec{q^{\varepsilon}} \cdot \vec{n}||_{L^\infty(S\times\Gamma_{in})} \int_0^t \int_{\Gamma_{in}} f_{r-1}(u^{\varepsilon}_{\delta,\tau})\, ds\, d\theta \\
&\le\ r\ ||\vec{q^{\varepsilon}} \cdot \vec{n}||_{L^\infty(S\times\Gamma_{in})} \sum_{i=1}^{I} e^{-(1+\mu_i^0)} \int_0^t \int_{\Gamma_{in}} f_{r-1}(u^{\varepsilon}_{\delta,\tau}) e^{\mu_i^0}(e-1)^{-1} g(u^{\varepsilon}_{\delta,\tau})\, ds\, d\theta \\
&\le C_3\, r \int_0^t \int_{\partial\Omega} f_r(u^{\varepsilon}_{\delta,\tau})\, ds\, d\theta, \qquad (5.20)
\end{aligned}$$

where $C_3 = I\, ||\vec{q^{\varepsilon}} \cdot \vec{n}||_{L^\infty(S\times\Gamma_{in})}[e(e-1)]^{-1}$. Again $\Gamma^{\varepsilon}_{+} := \{x \in \Gamma^{\varepsilon} : \mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau} \ge 0\}$ and $\Gamma^{\varepsilon}_{-} := \{x \in \Gamma^{\varepsilon} : \mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau} \le 0\}$. Since the solution u^{ε}_{δ} of (3.2a) is positive, we have $u^{\varepsilon^+}_{\delta} = u^{\varepsilon}_{\delta} \le u^{\varepsilon}_{\delta,\tau}$. Then as in (5.19)

$$\begin{aligned}
&-\varepsilon \sum_{i=1}^{I} \sum_{k=1}^{\bar{I}} \int_0^t \int_{\Gamma^{\varepsilon}} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau}) \sigma_{ik} \left(k_k^p \prod_{m=1}^{I} (u^{\varepsilon^+}_{\delta_m})^{\sigma_{mk}} \right) (\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, d\sigma_x\, d\theta \\
&\le -\varepsilon \sum_{i=1}^{I} \sum_{k=1}^{\bar{I}} \int_0^t \int_{\Gamma^{\varepsilon}_{-}} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau}) \sigma_{ik} \left(k_k^p \prod_{m=1}^{I} (u^{\varepsilon}_{\delta_m,\tau})^{\sigma_{mk}} \right) (\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, d\sigma_x\, d\theta. \qquad (5.21)
\end{aligned}$$

By (1.10) we know that $\sigma_{mk} \in \{0,1\}$. Note that for each $k = 1,2,...,\bar{I}$, $\prod_{m=1}^{I}(u^{\varepsilon}_{\delta_m,\tau})^{\sigma_{mk}}$ contains the product of concentrations of only two species with exponents as 1. In this product one of them is $u^{\varepsilon}_{\delta_i,\tau}$ and the other one is $u^{\varepsilon}_{\delta_j,\tau}$ for $i,\, j = 1,2,...,I$ and $i \neq j$. Also $-u^{\varepsilon}_{\delta_i,\tau}\left(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau}\right) \le e^{-(1+\mu_i^0)}$ and $u^{\varepsilon}_{\delta_j,\tau} \le g(u^{\varepsilon}_{\delta_j,\tau}) \le g(u^{\varepsilon}_{\delta,\tau})$ for $i,j = 1,2,...,I$ and $i \neq j$. Therefore from (5.21) we have

$$\begin{aligned}
&-\varepsilon \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}} \int_0^t \int_{\Gamma^\varepsilon} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau})\sigma_{ik}\left(k_k^p \prod_{m=1}^{I} (u^{\varepsilon^+}_{\delta_m})^{\sigma_{mk}}\right)(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, d\sigma_x\, d\theta \\
&\le \varepsilon \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}} r e^{-(1+\mu_i^0)}\sigma_{ik}\, k_k^p \int_0^t \int_{\Gamma^\varepsilon} f_{r-1}(u^{\varepsilon}_{\delta,\tau}) g(u^{\varepsilon}_{\delta,\tau})\, d\sigma_x\, d\theta \\
&\le \varepsilon\, C_4\, r \int_0^t \int_{\Gamma^\varepsilon} f_r(u^{\varepsilon}_{\delta,\tau})\, d\sigma_x\, d\theta, \qquad (5.22)
\end{aligned}$$

where $C_4 := \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}} e^{-(1+\mu_i^0)}\sigma_{ik}\, k_k^p$. Again following the steps of (5.19) we obtain

$$\begin{aligned}
&\varepsilon \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}} \int_0^t \int_{\Gamma^\varepsilon} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau})\sigma_{ik} k_k^d \psi_\delta(w^{\varepsilon}_\delta)_k (\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, d\sigma_x\, d\theta \\
&\le \varepsilon \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}} \int_0^t \int_{\Gamma^\varepsilon_+} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau})\sigma_{ik} k_k^d (\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau})\, d\sigma_x\, d\theta, \text{ since } |\psi_\delta(w^{\varepsilon}_\delta)_k| \le 1 \\
&\le \varepsilon \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}} \int_0^t \int_{\Gamma^\varepsilon_+} r\, f_{r-1}(u^{\varepsilon}_{\delta,\tau})\sigma_{ik} k_k^d \left(1 + \max_{1\le i\le I}|\mu_i^0| e^{\mu_i^0}(e-1)^{-1}\right) g_i(u^{\varepsilon}_{\delta_i,\tau})\, d\sigma_x\, d\theta \\
&\le \varepsilon\, C_5\, r \int_0^t \int_{\Gamma^\varepsilon} f_r(u^{\varepsilon}_{\delta,\tau})\, d\sigma_x\, d\theta, \text{ since } g_i \le g \text{ and } f_r = f_{r-1} g, \qquad (5.23)
\end{aligned}$$

where $C_5 := \sum_{i=1}^{I}\sum_{k=1}^{\bar{I}} \sigma_{ik} k_k^d \left(1 + \max_{1\le i\le I}|\mu_i^0| e^{\mu_i^0}(e-1)^{-1}\right)$. With the help of (5.19)-(5.23), (5.17) reduces to

$$I^{(t)}_{bound} \le C_6\, r \int_0^t \int_{\partial\Omega} f_r(u^{\varepsilon}_{\delta,\tau})\, ds\, d\theta + \varepsilon\, C_7\, r \int_0^t \int_{\Gamma^\varepsilon} f_r(u^{\varepsilon}_{\delta,\tau})\, d\sigma_x\, d\theta, \qquad (5.24)$$

where $C_6 := (C_2 + C_3)$ and $C_7 := (C_4 + C_5)$. Note that both the constants C_5 and C_6 are independent of ε, δ, τ, t, $u^{\varepsilon}_{\delta,\tau}$ and $w^{\varepsilon}_{\delta,\tau}$. Now combining (5.12), (5.14), (5.15), (5.16) and (5.24), we obtain

$$\begin{aligned}
&\int_0^t \left\langle \partial_\theta u^{\varepsilon}_\delta, \partial f_r(u^{\varepsilon}_{\delta,\tau})\right\rangle d\theta \\
&\le -Dr(r-1)\sum_{l=1}^{n} \int_0^t \int_{\Omega^\varepsilon_p} f_{r-2}(u^{\varepsilon}_{\delta,\tau}) \left(\sum_{i=1}^{I}\left(\mu_i^0 + \log u^{\varepsilon}_{\delta_i,\tau}\right)\frac{\partial u^{\varepsilon}_{\delta_i,\tau}}{\partial x_l}\right)^2 dx\, d\theta + C_1 \int_0^t \int_{\partial\Omega} f_r(u^{\varepsilon}_{\delta,\tau})\, dx\, d\theta \\
&\quad + C_6\, r \int_0^t \int_{\partial\Omega} f_r(u^{\varepsilon}_{\delta,\tau})\, dx\, d\theta + \varepsilon\, C_7\, r \int_0^t \int_{\Gamma^\varepsilon} f_r(u^{\varepsilon}_{\delta,\tau})\, d\sigma_x\, d\theta + h(t,\tau,u^{\varepsilon}_{\delta,\tau}). \qquad (5.25)
\end{aligned}$$

We note that $f_r = f_{\frac{r}{2}}^2$ and $|\nabla_x f_{\frac{r}{2}}(u^\varepsilon_{\delta,\tau})|_n^2 = \frac{r^2}{4} f_{r-2}(u^\varepsilon_{\delta,\tau}) \sum_{l=1}^n \left(\sum_{i=1}^I \left(\mu_i^0 + \log u^\varepsilon_{\delta_i,\tau} \right) \frac{\partial u^\varepsilon_{\delta_i,\tau}}{\partial x_l} \right)^2$. cf. [Krä08]. Therefore applying (5.3) and (5.4), (5.25) reduces to

$$
\begin{aligned}
&\int_0^t \left\langle \partial_\theta u^\varepsilon_\delta, \partial f_r(u^\varepsilon_{\delta,\tau}) \right\rangle d\theta \\
&\leq -Dr(r-1) \sum_{l=1}^n \int_0^t \int_{\Omega_p^\varepsilon} f_{r-2}(u^\varepsilon_{\delta,\tau}) \left(\sum_{i=1}^I \left(\mu_i^0 + \log u^\varepsilon_{\delta_i,\tau} \right) \frac{\partial u^\varepsilon_{\delta_i,\tau}}{\partial x_l} \right)^2 dx\, d\theta + h(t,\tau,u^\varepsilon_{\delta,\tau}) \\
&\quad + C_1 \int_0^t \int_{\Omega_p^\varepsilon} \left[(1+\Lambda_{\lambda_1}) |f_{\frac{r}{2}}(u^\varepsilon_{\delta,\tau})|^2 + \lambda_1 \frac{r^2}{4} f_{r-2}(u^\varepsilon_{\delta,\tau}) \left(\sum_{i=1}^I \left(\mu_i^0 + \log u^\varepsilon_{\delta_i,\tau} \right) \frac{\partial u^\varepsilon_{\delta_i,\tau}}{\partial x_l} \right)^2 \right] dx\, d\theta \\
&\quad + r\, C_6 \int_0^t \int_{\Omega_p^\varepsilon} \left[(1+\Lambda_{\lambda_2}) |f_{\frac{r}{2}}(u^\varepsilon_{\delta,\tau})|^2 + \lambda_2 \frac{r^2}{4} f_{r-2}(u^\varepsilon_{\delta,\tau}) \left(\sum_{i=1}^I \left(\mu_i^0 + \log u^\varepsilon_{\delta_i,\tau} \right) \frac{\partial u^\varepsilon_{\delta_i,\tau}}{\partial x_l} \right)^2 \right] dx\, d\theta \\
&\quad + r\, C_7 \int_0^t \int_{\Omega_p^\varepsilon} \left[(1+\Lambda_{\lambda_3}) |f_{\frac{r}{2}}(u^\varepsilon_{\delta,\tau})|^2 + \varepsilon^2 \lambda_3 \frac{r^2}{4} f_{r-2}(u^\varepsilon_{\delta,\tau}) \left(\sum_{i=1}^I \left(\mu_i^0 + \log u^\varepsilon_{\delta_i,\tau} \right) \frac{\partial u^\varepsilon_{\delta_i,\tau}}{\partial x_l} \right)^2 \right] dx\, d\theta.
\end{aligned}
\tag{5.26}
$$

We choose $\lambda_1 = \frac{(r-1)D}{C_1 r}$, $\lambda_2 = \frac{(r-1)D}{r^2 C_6}$ and $\lambda_3 = \frac{(r-1)D}{r^2\varepsilon^2 C_7}$ which implies $\Lambda_{\lambda_1} = \frac{C_1 r}{4D(r-1)}$, $\Lambda_{\lambda_2} = \frac{C_6 r^2}{4D(r-1)}$ and $\Lambda_{\lambda_3} = \frac{r^2\varepsilon^2 C_7}{4D(r-1)}$. Note that Λ_{λ_3} involves ε^2 as a factor but since $\varepsilon \ll 1$, we have $\Lambda_{\lambda_3} = \frac{r^2\varepsilon^2 C_7}{4D(r-1)} < \frac{r^2 C_7}{4D(r-1)}$. We set $C := C_1 \left(1 + \frac{C_1 r}{4D(r-1)}\right) + r\, C_6 \left(1 + \frac{C_6 r^2}{4D(r-1)}\right) + r\, C_7 \left(1 + \frac{C_7 r^2}{4D(r-1)}\right)$. Therefore (5.26) gives

$$
\begin{aligned}
&\int_0^t \left\langle \partial_\theta u^\varepsilon_\delta, \partial f_r(u^\varepsilon_{\delta,\tau}) \right\rangle d\theta \\
&\leq h(t,\tau,u^\varepsilon_{\delta,\tau}) + C \int_0^t \int_{\Omega_p^\varepsilon} f_{\frac{r}{2}}^2(u^\varepsilon_{\delta,\tau})\, dx\, d\theta \\
&\leq h(t,\tau,u^\varepsilon_{\delta,\tau}) + C \int_0^t F_r(u^\varepsilon_{\delta,\tau})\, d\theta, \quad \text{since } f_r = f_{\frac{r}{2}}^2 \text{ and } F_r(u^\varepsilon_{\delta,\tau}) = \int_{\Omega_p^\varepsilon} f_r(u^\varepsilon_{\delta,\tau})\, dx,
\end{aligned}
$$

where $h(t,\tau,u^\varepsilon_{\delta,\tau}) \to 0$ as $\tau \to 0$ for a.e. t and C is independent of ε, δ, τ, t, $u^\varepsilon_{\delta,\tau}$ and $w^\varepsilon_{\delta,\tau}$ but it depends on r. □

5.4 Proof of theorem 3.1

Let $(u^\varepsilon_\delta, w^\varepsilon_\delta)$ be a solution of the problem $(P_\delta^{\varepsilon^+})$. Since we only the know the nonnegativity of u^ε_δ, let $u^\varepsilon_{\delta,\tau} := u^\varepsilon_\delta + \tau$ for $\tau > 0$. Clearly $u^\varepsilon_{\delta,\tau} \in \mathcal{F}_p^I(\Omega_p^\varepsilon)$. Replicating the steps of theorem 3.3 in [Krä08] (see also theorem 4.1.1.3 in [MB13a]), we obtain

$$
F_r(u^\varepsilon_\delta(t)) \leq F_r(u^\varepsilon_\delta(0)) + C \int_0^t F_r(u^\varepsilon_\delta)\, d\theta \qquad \text{for a.e. } t \in S. \tag{5.27}
$$

Here we have used the properties that $u^\varepsilon_{\delta,\tau} \to u^\varepsilon_\delta$, $h(t,\tau,u^\varepsilon_{\delta,\tau}) \to 0$ as $\tau \to 0$ for a.e. t and $F_r(u^\varepsilon_{\delta,\tau})$ is continuous (cf. remark 3.1). Application of Gronwall's inequality to (5.27) gives

$$F_r(u^\varepsilon_\delta(t)) \le e^{Ct} F_r(u^\varepsilon_\delta(0)) \qquad \text{for a.e. } t \in S,$$

where C is independent of ε, δ, τ, t, $u^\varepsilon_{\delta,\tau}$ and $w^\varepsilon_{\delta,\tau}$ but it depends on r. This establishes the inequality (3.13). □.

5.5 Proof of corollary 3.1

The proof is divided in several steps.

(I) The estimate $\sup_{\varepsilon,\delta>0} |||u^\varepsilon_\delta(t)|||_{L^r(\Omega^\varepsilon_p)^I} < \infty$ for a.e. t follows like the one for corollary 3.3.4 in [MB14]. This gives $\sup_{\varepsilon,\delta>0} |||u^\varepsilon_\delta|||_{L^\infty(S;L^r(\Omega^\varepsilon_p))^I} = \sup_{\varepsilon,\delta>0} \operatorname{ess\,sup}_{t\in S} |||u^\varepsilon_\delta(t)|||_{L^r(\Omega^\varepsilon_p)^I} \le C < \infty$. This implies $|||u^\varepsilon_\delta|||_{L^r(S\times\Omega^\varepsilon_p)^I} \le C$.

(II) Testing (3.2a) by u^ε_δ, then

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{I}\left[||u^\varepsilon_{\delta_i}(T)||^2_{L^2(\Omega^\varepsilon_p)} - ||u_i(0)||^2_{L^2(\Omega^\varepsilon_p)}\right] + \sum_{i=1}^{I}\int_0^T\int_{\Omega^\varepsilon_p} D\,|\nabla u^\varepsilon_{\delta_i}|^2\,dx\,dt = \sum_{i=1}^{I}\int_0^T\int_{\Omega^\varepsilon_p} S\bar{R}(u^\varepsilon_\delta)_i u^\varepsilon_{\delta_i}\,dx\,dt\\
&\quad -\sum_{i=1}^{I}\int_0^T\int_{\Omega^\varepsilon_p} \vec{q}^\varepsilon\cdot\nabla u^\varepsilon_{\delta_i}\; u^\varepsilon_{\delta_i}\,dx\,dt - \sum_{i=1}^{I}\int_0^T\int_{\Gamma_{in}} d_i\, u^\varepsilon_{\delta_i}\,ds\,dt + \sum_{i=1}^{I}\int_0^T\int_{\Gamma_{in}} \vec{q}^\varepsilon\cdot\vec{n}\,|u^\varepsilon_{\delta_i}|^2\,ds\,dt\\
&\quad -\varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar{I}}\sigma_{ik}\int_0^T\int_{\Gamma^\varepsilon}\left(k^p_k\prod_{i=1}^{I}(u^{\varepsilon^+}_{\delta_i})^{\sigma_{ik}} - k^d_k\psi_\delta(w^\varepsilon_\delta)_k\right)u^\varepsilon_{\delta_i}\,d\sigma_x\,dt
\end{aligned} \tag{5.28}$$

Now we estimate each term in (5.28) using step (I), the boundary inequality (5.4) and Young's inequality.

$$\sum_{i=1}^{I}\int_0^T\int_{\Omega^\varepsilon_p}\mathcal{S}\bar{R}(u^\varepsilon_\delta)_i u^\varepsilon_{\delta_i}\,dx\,dt \le \frac{1}{2}\sum_{i=1}^{I}\left[\int_0^T\int_{\Omega^\varepsilon_p}|\mathcal{S}\bar{R}(u^\varepsilon_\delta)_i|^2\,dx\,dt + \int_0^T\int_{\Omega^\varepsilon_p}|u^\varepsilon_{\delta_i}|^2\,dx\,dt\right], \tag{5.29}$$

$$\sum_{i=1}^{I}\int_0^T\int_{\Omega^\varepsilon_p}\vec{q^\varepsilon}\cdot\nabla u^\varepsilon_{\delta_i}u^\varepsilon_{\delta_i}\,dx\,dt \le \sum_{i=1}^{I}\left[\frac{Q_1^2}{2}\lambda_1\int_0^T\int_{\Omega^\varepsilon_p}|\nabla u^\varepsilon_{\delta_i}|^2\,dx\,dt + \Lambda_{\lambda_1}\int_0^T\int_{\Omega^\varepsilon_p}|u^\varepsilon_{\delta_i}|^2\,dx\,dt\right] \tag{5.30}$$

$$\begin{aligned}
&\sum_{i=1}^{I}\int_0^T\int_{\Gamma_{in}}(-d_i)u^\varepsilon_{\delta_i}\,ds\,dt\\
\le&\frac{1}{2}\sum_{i=1}^{I}\left[\int_0^T\int_{\Gamma_{in}}||d_i||^2_{L^\infty(S\times\Gamma_{in})}\,ds\,dt + \int_0^T\int_{\partial\Omega}|u^\varepsilon_{\delta_i}|^2\,ds\,dt\right]\\
\le&\frac{1}{2}\sum_{i=1}^{I}\left[T|\Gamma_{in}|||d_i||^2_{L^\infty(S\times\Gamma_{in})} + C\int_0^T\left(||\nabla u^\varepsilon_{\delta_i}||_{L^2(S\times\Omega^\varepsilon_p)}\,||u^\varepsilon_{\delta_i}||_{L^2(S\times\Omega^\varepsilon_p)} + ||u^\varepsilon_{\delta_i}||^2_{L^2(S\times\Omega^\varepsilon_p)}\right)dt\right]\\
\le&\frac{1}{2}\sum_{i=1}^{I}\left[T|\Gamma_{in}|||d_i||^2_{L^\infty(S\times\Gamma_{in})} + C\left(\lambda_2||\nabla u^\varepsilon_{\delta_i}||^2_{L^2(S\times\Omega^\varepsilon_p)} + (1+\Lambda_{\lambda_2})||u^\varepsilon_{\delta_i}||^2\right)\right]
\end{aligned} \tag{5.31}$$

$$
\begin{aligned}
&-\varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\sigma_{ik}\int_0^T\int_{\Gamma^\varepsilon}\left(k_k^p\prod_{m=1}^{I}(u^{\varepsilon^+}_{\delta_m})^{\sigma_{mk}}-k_k^d\psi_\delta(w^\varepsilon_\delta)_k\right)\,u^\varepsilon_{\delta_i}\,d\sigma_x\,dt\\
&=-\varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\sigma_{ik}\int_0^T\int_{\Gamma^\varepsilon}k_k^p\prod_{m=1}^{I}(u^{\varepsilon^+}_{\delta_m})^{\sigma_{mk}}\,u^\varepsilon_{\delta_i}\,d\sigma_x\,dt+\varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\sigma_{ik}\int_0^T\int_{\Gamma^\varepsilon}k_k^d\psi_\delta(w^\varepsilon_\delta)_k\,u^\varepsilon_{\delta_i}\,d\sigma_x\,dt.
\end{aligned}
\tag{5.32}
$$

We estimate both the terms on the r.h.s. of (5.32). By (5.3) the second term in (5.32) can be estimated as

$$
\begin{aligned}
&\varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\sigma_{ik}\int_0^T\int_{\Gamma^\varepsilon}k_k^d\psi_\delta(w^\varepsilon_\delta)_k\,u^\varepsilon_{\delta_i}\,d\sigma_x\,dt\\
&\le\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\left[\varepsilon\int_{\Gamma^\varepsilon}|k_k^d|^2\,d\sigma_x\,dt+C\left\{(1+\Lambda_{\lambda_3})\int_{\Omega^\varepsilon_p}|u^\varepsilon_{\delta_i}|^2\,dx\,dt+\lambda_3\varepsilon^2\int_{\Omega^\varepsilon_p}|\nabla u^\varepsilon_{\delta_i}|^2\,dx\,dt\right\}\right].
\end{aligned}
\tag{5.33}
$$

To estimate the first term in (5.32) we recall that $u^{\varepsilon^+}_{\delta_i}\ge u^\varepsilon_{\delta_i}$, $|||u^\varepsilon_{\delta_i}|||_{L^\infty(S;L^r(\Omega^\varepsilon_p))^I}\le C<\infty$, $|\Omega^\varepsilon_p|\le|\Omega|$, $\sigma_{ik}\in\{0,1\}$ and $\prod_{m=1}^{I}(u^{\varepsilon^+}_{\delta_m})^{\sigma_{mk}}$ is the product of concentrations of only two species with exponent as 1 and on expansion the factor $u^\varepsilon_{\delta_i}\prod_{m=1}^{I}(u^{\varepsilon^+}_{\delta_m})^{\sigma_{mk}}$ gives $u^{\varepsilon^+}_{\delta_i}\,u^\varepsilon_{\delta_i}\,u^{\varepsilon^+}_{\delta_j}$, i.e. $-u^{\varepsilon^+}_{\delta_i}\,u^\varepsilon_{\delta_i}\,u^{\varepsilon^+}_{\delta_j}\le|u^\varepsilon_{\delta_i}|^2\,|u^\varepsilon_{\delta_j}|$ for $i,j=1,2,...,I$, $i\ne j$. Then

$$
\begin{aligned}
&-\varepsilon\sum_{i=1}^{I}\sum_{k=1}^{\bar I}\sigma_{ik}\int_0^T\int_{\Gamma^\varepsilon}k_k^p\prod_{m=1}^{I}(u^{\varepsilon^+}_{\delta_m})^{\sigma_{mk}}\,u^\varepsilon_{\delta_i}\,d\sigma_x\,dt\\
&\le\varepsilon\sum_{\substack{i,j=1\\ i\ne j}}^{I}\sum_{k=1}^{\bar I}\sigma_{ik}k_k^p\int_0^T\int_{\Gamma^\varepsilon}|u^\varepsilon_{\delta_i}|^2|u^\varepsilon_{\delta_j}|\,d\sigma_x\,dt\le\frac{\varepsilon}{2}\sum_{\substack{i,j=1\\ i\ne j}}^{I}\sum_{k=1}^{\bar I}\sigma_{ik}\,k_k^p\int_0^T\int_{\Gamma^\varepsilon}\left[|u^\varepsilon_{\delta_i}|^4+|u^\varepsilon_{\delta_j}|^2\right]d\sigma_x\,dt.
\end{aligned}
\tag{5.34}
$$

Now we employ the inequality (5.4), (5.5) to estimate (5.34) as follows:

$$
\varepsilon\int_0^T\int_{\Gamma^\varepsilon}|u^\varepsilon_{\delta_i}|^4\,d\sigma_x\,dt\le C\left[\int_0^T\int_{\Omega^\varepsilon_p}|u^\varepsilon_{\delta_i}|^4\,dx\,dt+\lambda_4\varepsilon^2\int_0^T\int_{\Omega^\varepsilon_p}|\nabla u^\varepsilon_{\delta_i}|^6\,dx\,dt+\Lambda_{\lambda_4}\int_0^T\int_{\Omega^\varepsilon_p}|u^\varepsilon_{\delta_i}|^6\,dx\,dt\right],
\tag{5.35}
$$

$$
\varepsilon\int_0^T\int_{\Gamma^\varepsilon}|u^\varepsilon_{\delta_i}|^2\,d\sigma_x\,dt\le C\left[\int_0^T\int_{\Omega^\varepsilon_p}|u^\varepsilon_{\delta_i}|^2\,dx\,dt+\frac{\varepsilon^2}{4}\int_0^T\int_{\Omega^\varepsilon_p}|\nabla u^\varepsilon_{\delta_i}|^2\,dx\,dt+\varepsilon^2\int_0^T\int_{\Omega^\varepsilon_p}|u^\varepsilon_{\delta_i}|^2\,dx\,dt\right].
\tag{5.36}
$$

We combine (5.35) and (5.36) with (5.28)-(5.34), choose $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ sufficiently small and r sufficiently large in step (I) yields an estimate for $\nabla u^\varepsilon_\delta$ independent of ε, δ, t, u^ε_δ and w^ε_δ.

(III) Testing (3.2a) with $\phi \in L^2((0,T);H^{1,2}(\Omega_p^\varepsilon))^I$ and proceeding in the similar way as in step (II), we obtain

$$\sup_{\substack{\phi\in L^2((0,T);H^{1,2}(\Omega_p^\varepsilon))^I \\ |||\phi|||_{L^2((0,T);H^{1,2}(\Omega_p^\varepsilon))^I}\leq 1}} \left|\left\langle \frac{\partial u_\delta^\varepsilon}{\partial t},\phi\right\rangle\right| \leq C \Longrightarrow \left|\left|\left|\frac{\partial u_\delta^\varepsilon}{\partial t}\right|\right|\right|_{L^2((0,T);H^{1,2}(\Omega_p^\varepsilon)^*)^I} \leq C, \tag{5.37}$$

for instance see theorem 2.19 in [vDP04].

(IV) Testing the *k-th* equation of (3.2f) with $\left|\frac{\partial w_{\delta_k}^\varepsilon}{\partial t}\right|^{p-2}\frac{\partial w_{\delta_k}^\varepsilon}{\partial t}$, we then obtain

$$\begin{aligned}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\left|\frac{\partial w_{\delta_k}^\varepsilon}{\partial t}\right|^p d\sigma_x\,dt \leq k_k^p\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\left[\frac{1}{4k_k^p}\left|\frac{\partial w_{\delta_k}^\varepsilon}{\partial t}\right|^p+\frac{1}{p}\left(\frac{p-1}{4pk_k^p}\right)^{-(p-1)}\left(\prod_{m=1}^I|u_{\delta_m}^\varepsilon|^{\sigma_{mk}}\right)^p\right]d\sigma_x\,dt\\ +k_k^d\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\left[\frac{1}{4k_k^d}\left|\frac{\partial w_{\delta_k}^\varepsilon}{\partial t}\right|^p+\frac{1}{p}\left(\frac{p-1}{4k_k^dp}\right)^{-(p-1)}\right]d\sigma_x\,dt.\end{aligned} \tag{5.38}$$

To estimate $\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\left(\prod_{m=1}^I|u_{\delta_m}^\varepsilon|^{\sigma_{mk}}\right)^p d\sigma_x\,dt$, we notice that the term $\left(\prod_{m=1}^I|u_{\delta_m}^\varepsilon|^{\sigma_{mk}}\right)^p$ contains the product of concentrations of only two species with exponent p, i.e.

$$\begin{aligned}&\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\left(\prod_{m=1}^I|u_{\delta_m}^\varepsilon|^{\sigma_{mk}}\right)^p d\sigma_x\,dt\\ &=\varepsilon\int_0^T\int_{\Gamma^\varepsilon}|u_{\delta_i}^\varepsilon|^p|u_{\delta_j}^\varepsilon|^p\,d\sigma_x\,dt \quad \text{for } i,j=1,2,...,I; i\neq j\\ &\leq\frac{1}{2}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\left[|u_{\delta_i}^\varepsilon|^{2p}+|u_{\delta_j}^\varepsilon|^{2p}\right]d\sigma_x\,dt.\\ &\leq\frac{C}{2}\left[\int_0^T\int_{\Omega_p^\varepsilon}|u_{\delta_i}^\varepsilon|^{2p}\,dx\,dt+2p\varepsilon^{2p}\int_0^T\int_{\Omega_p^\varepsilon}|u_{\delta_i}^\varepsilon|^{2p-1}|\nabla u_{\delta_i}^\varepsilon|\,dx\,dt+\int_0^T\int_{\Omega_p^\varepsilon}|u_{\delta_j}^\varepsilon|^{2p}\,dx\,dt\right.\\ &\qquad\left.+2p\varepsilon^{2p}\int_0^T\int_{\Omega_p^\varepsilon}|u_{\delta_j}^\varepsilon|^{2p-1}|\nabla u_{\delta_j}^\varepsilon|\,dx\,dt\right] \text{ by lemma 5.4}\\ &\leq\frac{C}{2}\left[\int_0^T\int_{\Omega_p^\varepsilon}|u_{\delta_i}^\varepsilon|^{2p}\,dx\,dt+p\varepsilon^{2p}\int_0^T\int_{\Omega_p^\varepsilon}|\nabla u_{\delta_i}^\varepsilon|^2\,dx\,dt+p\varepsilon^{2p}\int_0^T\int_{\Omega_p^\varepsilon}|u_{\delta_i}^\varepsilon|^{4p-2}\,dx\,dt\right.\\ &\qquad\left.+\int_0^T\int_{\Omega_p^\varepsilon}|u_{\delta_j}^\varepsilon|^{2p}\,dx\,dt+p\varepsilon^{2p}\int_0^T\int_{\Omega_p^\varepsilon}|\nabla u_{\delta_j}^\varepsilon|^2\,dx\,dt+p\varepsilon^{2p}\int_0^T\int_{\Omega_p^\varepsilon}|u_{\delta_j}^\varepsilon|^{4p-2}\,dx\,dt\right],\end{aligned} \tag{5.39}$$

where $\varepsilon \ll 1$ and rest of the terms are bounded courtsey of steps (I) and (II). By (5.38) and (5.39), it follows that $\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\left|\frac{\partial w_{\delta_k}^\varepsilon}{\partial t}\right|^p d\sigma_x\,dt$ is bounded by a positive, ε and δ independent, constant C. We obtain the estimate for $\varepsilon\int_0^T\int_{\Gamma^\varepsilon}|w_{\delta_k}^\varepsilon|^p\,d\sigma_x\,dt$ by proceeding in the similar fashion as the one for $\varepsilon \int_0^T \int_{\Gamma^\varepsilon} \left| \frac{\partial w^\varepsilon_{\delta_k}}{\partial t} \right|^p d\sigma_x \, dt$ and a straightforward application of *Gronwall's inequality*. Consequently, $\sup_{\varepsilon>0} \sum_{k=1}^{\bar{I}} \varepsilon \int_0^T \int_{\Gamma^\varepsilon} |w^\varepsilon_{\delta_k}|^p \, d\sigma_x \, dt < \infty$ and $\sup_{\varepsilon>0} \sum_{k=1}^{\bar{I}} \varepsilon \int_0^T \int_{\Gamma^\varepsilon} \left| \frac{\partial w^\varepsilon_{\delta_k}}{\partial t} \right|^p d\sigma_x \, dt < \infty$. Adding the steps (I) - (IV), we obtain (3.15). Hence, the corollary is proved. □