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NUMERICAL LIMIT AND ITS APPLICATION TO A BLOW-UP PROBLEM RELATED TO DEFAULT RISK

Yuya Sasaki
Komi High School,
1006-2 Chiyosato, Komi, Minamisaku, Nagano, 384-1105, Japan
(sasaki-y@nagano-c.ed.jp)

Hiroko Soutome
Japan Woman's College of Physical Education,
8-19-1 Kitakarasuyama, Setagaya-ku, Tokyo, 157-8565, Japan
(sotome.hiroko@jwcpe.ac.jp)

Hitoshi Imai
Faculty of Science and Engineering, Doshisha University,
1-3 Tatara Miyakodani, Kyotanabe, Kyoto, 610-0394, Japan
(himai@mail.doshisha.ac.jp)

and

Naoyuki Ishimura
Faculty of Commerce, Chuo University,
Hachioji, Tokyo, 192-0393, Japan
(naoyuki@tamacc.chuo-u.ac.jp)

Abstract. In the paper, we propose the method of numerical limits, which consist of the extrapolation and the bounding transformation. We apply these numerical limits to the computation of the blow-up time for a model ODE system of interacting firms. Numerical implementation shows satisfactory results.

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1 Introduction

In the paper, we are concerned with a numerical treatment of the limiting behavior of blowing-up solutions. In particular, we apply our method to the following system of ordinary differential equations (ODEs), which is introduced in Ishimura and Nakamura [5][6] as a simple model dynamics of comercial firms:

$$\begin{cases} u'(t) = -\,\alpha\, u(t) + \alpha\, v(t)\, w(t) \\ v'(t) = -\,\beta\, v(t) + \beta\, w(t)\, u(t) \\ w'(t) = -\,\gamma\, w(t) + \gamma\, u(t)\, v(t) \end{cases} \tag{1}$$

where $\alpha\beta\gamma$ are positive constants.

As is well known, blow-up phenomena for differential equations have been investigated extensively so far. We recall the seminal work of Fujita [2] for the semilinear heat equation. Here we mean that a function $u(t) = u(t, \cdot)$ blows-up if $\lim_{t \uparrow T} \|u(t)\| = \infty$ with $\|\cdot\|$ being a suitable norm. $T(< \infty)$ is called a blow-up time of a function $u(t)$. It is shown that the solution to (1) may blow-up or breakdown within finite time [6], that is, according to initial conditions there corresponds $0 < T < \infty$ such that $|u(t)| + |v(t)| + |w(t)| \to \infty$ as $t \uparrow T$. In [6], the blow-up time is interpreted as a default time of firms.

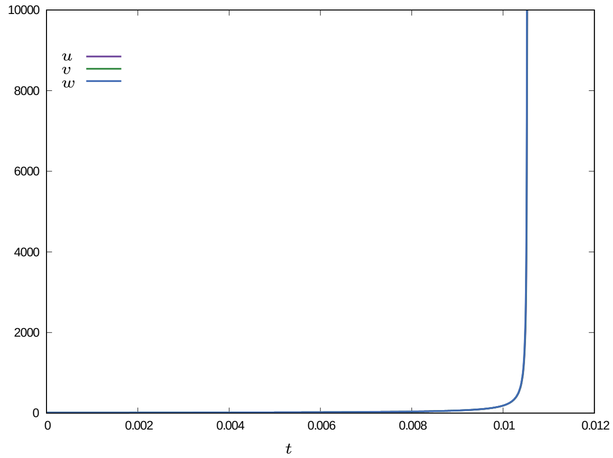
To illustrate the novelty of our nethod, we consider the following two cases of (1) as shown in Table 1.

Table 1. Typical two cases of (1).

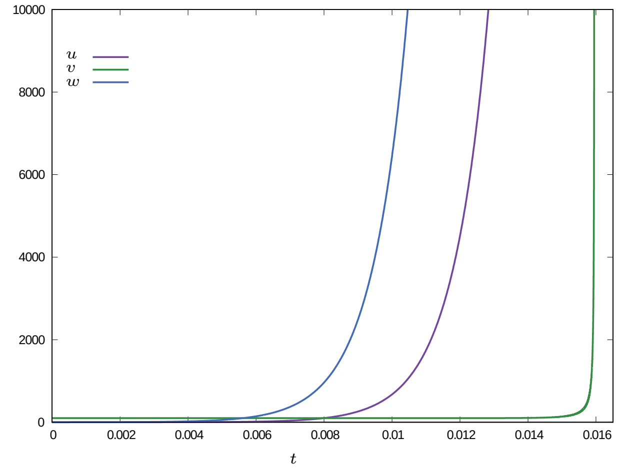
	$(\alpha,\ \beta,\ \gamma)$	$(u(0),\ v(0),\ w(0))$
Case 1	$(10,\ 10,\ 10)$	$(10,\ 10,\ 10)$
Case 2	$(1,\ 10^{-6},\ 100)$	$(10^{-6},\ 100,\ 1)$

It is noted that Case 2 treats extremely different values of the coefficients and the initial data.

Numerical solutions of (1) are depicted in Fig.1, where the Euler method with the variable temporal mesh size [7] $\Delta t = \min\left\{ \dfrac{1}{\max\{|u|, |v|, |w|\}},\ \Delta t_0 \right\}$ is used. Here Δt_0 denotes some fixed constant specified in each implementation. Numerical computation is performed in double precision. Obviously, in Case 1 $u(t) = v(t) = w(t)$ holds theoretically and numerically; indeed, three profiles of u, v and w in Fig. 1 (1) (Case 1) are seen to ovrelap each other. Fig. 1 (1) (Case 1) suggests that the blow-up times for u, v, w are the same. On the other hand, Fig.1 (2) (Case 2) indicates that the blow-up times may be different, while this is not true according to the theorem in [6]; for this system, it is proved that blow-up times for u, v, w are the same.



(1) Case 1.



(2) Case 2.

Fig.1. Numerical solution profiles ($\Delta t_0 < 10^{-5}$).

Our numerical method gives the numerical blow-up time T_{nu} as shown in Table 2; here T_{nu} is regarded as the blow-up time when $\Delta t < 10^{-16}$.

Table 2. Numerical blow-up time T_{nu}.

Δt_0	Case 1	Case 2
10^{-5}	$0.0105434262\cdots$	$0.0159578799\cdots$
10^{-6}	$0.0105370056\cdots$	$0.0159632635\cdots$
10^{-7}	$0.0105361214\cdots$	$0.0159638484\cdots$

We should remark that this numerical method does not apparently guarantee that blow-up times for u, v, w are the same. On the other hand, our method yields the same blow-up time even numerically.

The numerical computation of blow-up problems is challenging, since the blow-up phenomena involve limit and infinity. To list up some literatures, we recall the work of Nakagawa [7], where an adaptive temporal mesh control as shown in the above is proposed to compute the blow-up time, and it is shown that the numerical blow-up time converges to the exact analytical one. Recently, Cho [1] proposes a nice scheme with the uniform temporal mesh size and shows the convergence property as well. In his scheme the difficulty of infinity is removed theoretically, and the difficulty of limit is partially removed by taking the temporal mesh size be smaller.

In the present article, we propose a numerical method which enables us to remove the difficulty both of limit and inifinity. The method consists of the extrapolation and the bounding transform [4]. It is applied to (1). Our interest is not only in the numerical computation of the blow-up time, but also in the possibility that numerical results may show that blow-up times for u, v, w are the same.

2 Our methods

To remove the difficulty that the valueof a function u becomes infinite, we consider the next bounding transformation [4] of u into $\tilde{u}$.

$$\tilde{u} = \frac{-1+\sqrt{1+4u^2}}{2u} \quad \left(u = \frac{\tilde{u}}{1-\tilde{u}^2} \right). \tag{2}$$

By this bounding transform, there follows the onto-mapping between the intervals of u and $\tilde{u}$.

$$\tilde{u} \in [0,1] \iff u \in [0,\infty]. \tag{3}$$

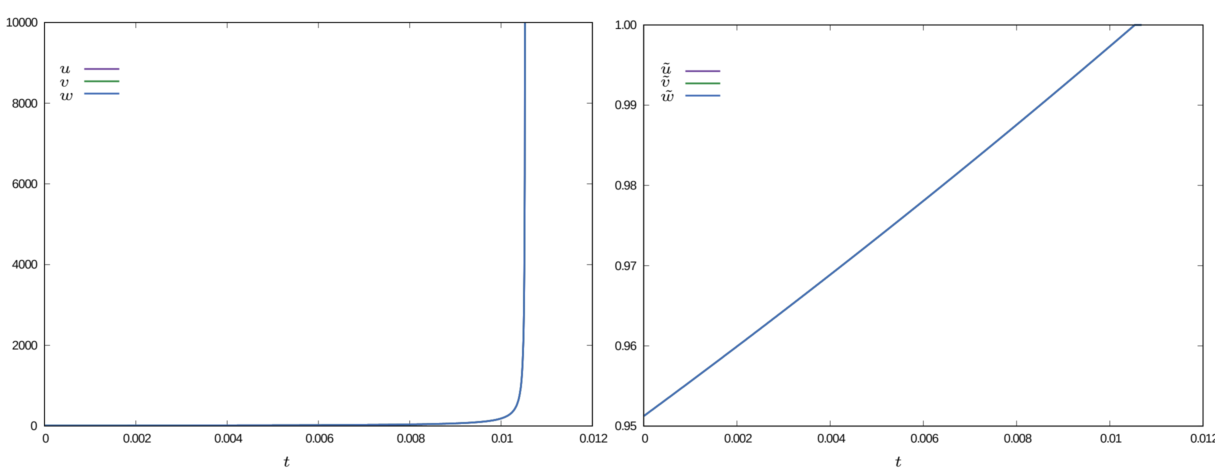
There are two ways for the application of the bounding transform in numerical computation. The first way is to numerically compute the transformed equations, which is known to become more complicated than the original equations. The second way is to perform the bounding transform to a numerical data, which are obtained from original equations. The second way is simple enough to use it here. We should remark that in the current numerical computation, a standard double precision does not work effectively. For example, due to the loss of trailing digits, $\tilde{u} = 1$ corresponds to $u \simeq 10^{16}$ in double precision. Therefore. multiple-precision arithmetic is necessary.

To remove the difficulty of limit we use the extrapolation. For example, we fix the temporal mesh size Δt and then the extrapolation is employed for numerical computation: $\lim_{\Delta t \to 0}$. The details of our procedure will be explained in the next section.

3 Numerical results

The forth order Runge-Kutta method is used for the discretization. The temporal mesh size Δt is fixed. Numerical computation is performed in multiple precision (100 digits) by using exflib [3] which is a compact and fast library.

Firstly, we show numerical results for Case 1.



(1) Original profiles. (2) Profiles of transformed solutions.

Fig.2. Numerical solution profiles in Case 1 ($\Delta t = 10^{-5}$).

In this case $u(t) = v(t) = w(t)$ holds theoretically and numerically. It is seen that three graphs in each of Figs. 2−5 overlap each other. The original profiles of u, v and w obtained by the forth order Runge-Kutta method with $\Delta t = 10^{-5}$ are shown in Fig.2 (1). Of course, profies are almost same as in Fig.1 (1). Then we apply the bounding transform to these numerical data of u, v, w. Profiles of transformed solutions $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are shown in Fig.2 (2). It is interesting that profiles of $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are almost straight lines. To get better straight lines, we consider a narrow interval $[0.0100,\ 0.0105]$ which is close to the blow-up time. Numerical data of $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ in this interval are plotted and regression lines are exhibited in Fig. 3. The $t-$coordinate at the intersection of two lines, namely the intersection of $\tilde{u} = 1$ with the regression line for $\tilde{u}$, means the numerical blow-up time of u for $\Delta t = 10^{-5}$. We denote this time as $T_u(\Delta t = 10^{-5})$. We obtain $T_v(\Delta t = 10^{-5})$ and $T_w(\Delta t = 10^{-5})$ similarly. We note once again that in Case 1 these three numerical blow-up times are the same.

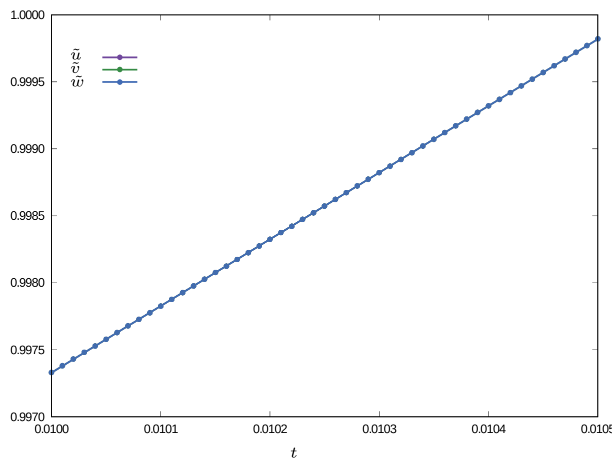


Fig. 3. Regression lines in Case 1 ($\Delta t = 10^{-5}$,$t \in [0.0100,\ 0.0105]$).

In Fig. 4, $T_u(\Delta t)$, $T_v(\Delta t)$ and $T_w(\Delta t)$ are plotted for various Δt, and regression lines are shown. The vertical intersection with the regression line for $\tilde{u}$ means $\lim_{\Delta t \to 0} T_u(\Delta t)(\equiv T_u)$. From numerical data in the interval $[0.0100,\ 0.0105]$ we obtain that $T_u = T_v = T_w = 0.01053650\cdots$.

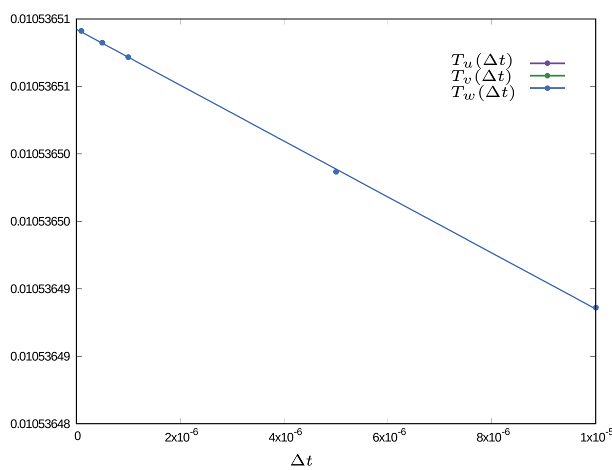


Fig. 4. Numerical blow-up times in Case 1 ($t \in [0.0100,\ 0.0105]$).

Similarly, from numerical data in the interval [0.01050, 0.01053] which is closer to the blow-up time, we obtain that $T_u = T_v = T_w = 0.01053605\cdots$ (Fig. 5). Please compare this with $T_{nu} = 0.0105361214\cdots$ in numerical computation with variable Δt (Table. 2).

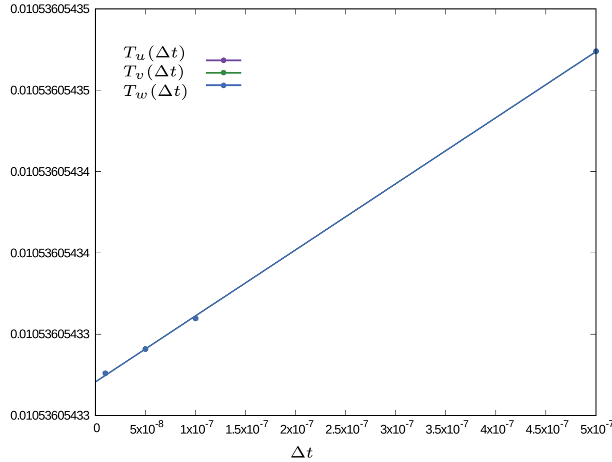
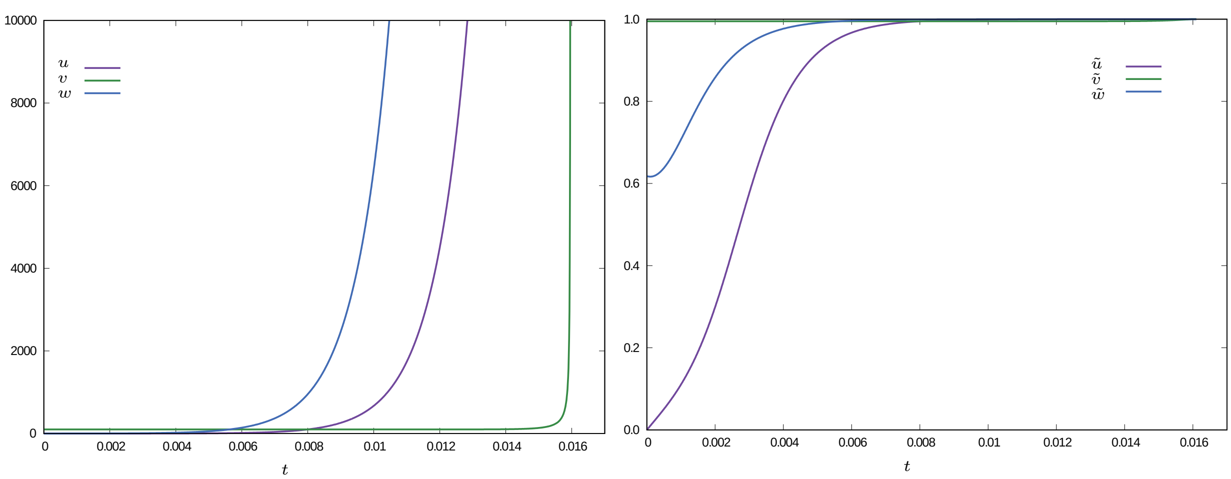


Fig. 5. Numerical blow-up times in Case 1 ($t \in [0.01050,\ 0.01053]$).

We remark that the exactness of our numerical results depends on the interval of t.

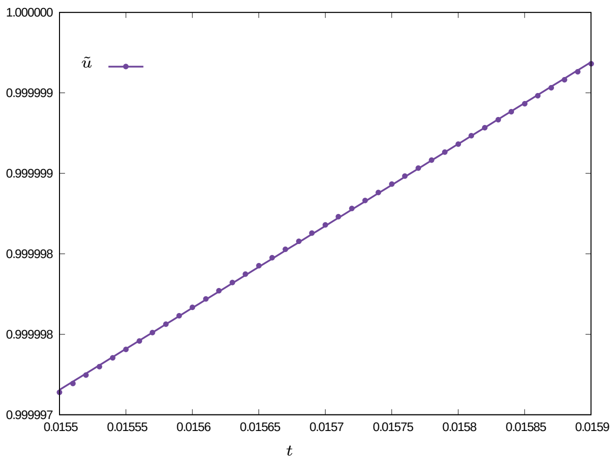
We apply the same proceedure to Case 2. The original profiles of u, v and w obtained by the forth order Runge-Kutta method with $\Delta t = 10^{-5}$ are shown in Fig. 6 (1). Of course, profies are almost same as in Fig. 1 (2). Then we apply the bounding transform to these numerical data of u, v, w. Profiles of transformed solutions $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are shown in Fig. 6 (2).



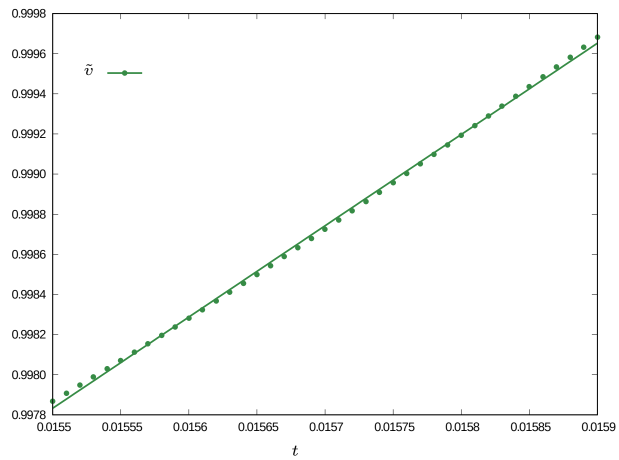
(1) Original profiles. (2) Profiles of transformed solutions.

Fig. 6. Numerical solution profiles in Case 2 ($\Delta t = 10^{-5}$).

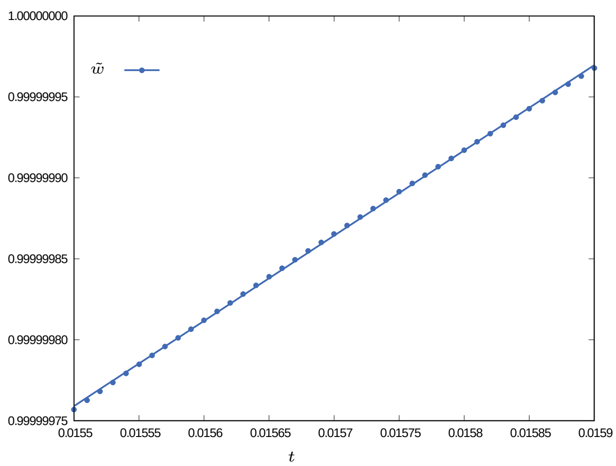
To get better straight lines we consider a narrow interval [0.0155, 0.0159] which is close to the blow-up time. Numerical data of $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ in this interval are plotted and regression lines are shown in Fig. 7.



(1) $\tilde{u}$.



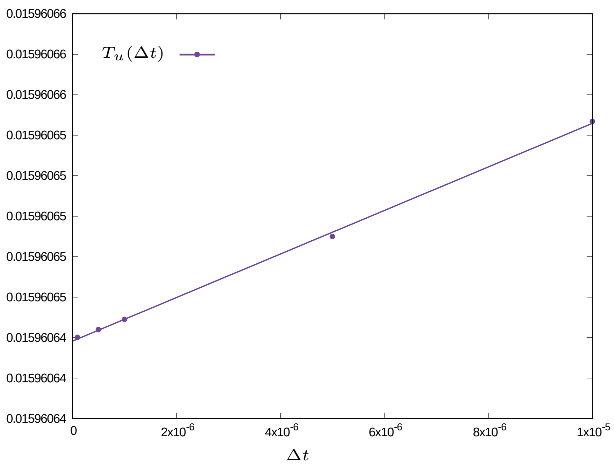
(2) $\tilde{v}$.



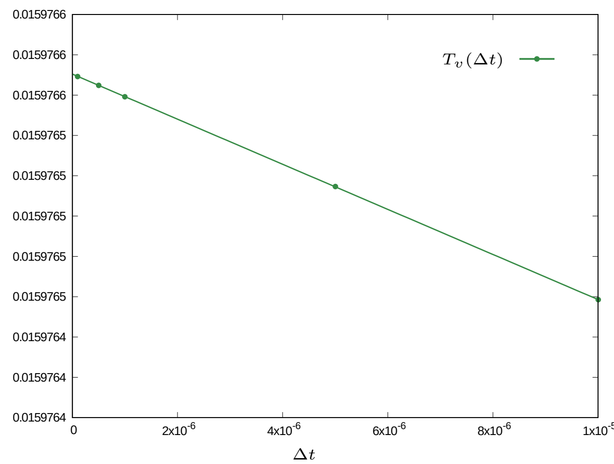
(3) $\tilde{w}$.

Fig. 7. Regression lines in Case 2 ($\Delta t = 10^{-5}$, $t \in [0.0155,\ 0.0159]$).

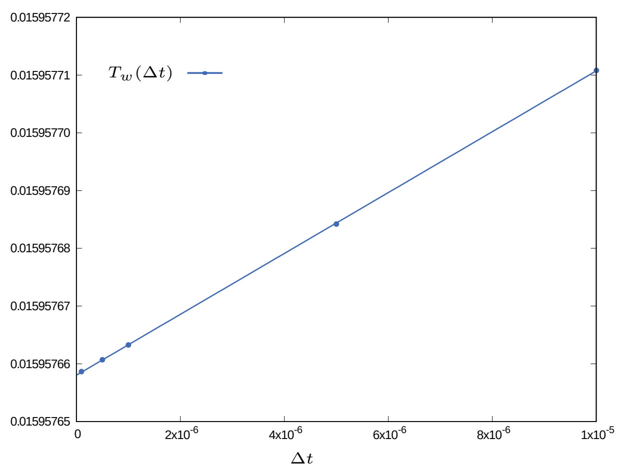
In Fig. 8, $T_u(\Delta t)$, $T_v(\Delta t)$ and $T_w(\Delta t)$ are plotted for various Δt, and regression lines are exhibited. From numerical data in the interval [0.0155, 0.0159] we obtain that $T_u = 0.01596064\cdots$, $T_v = 0.01597657\cdots$, $T_w = 0.01595765\cdots$. The first 3 digits are the same.



(1) $T_u(\Delta t)$.



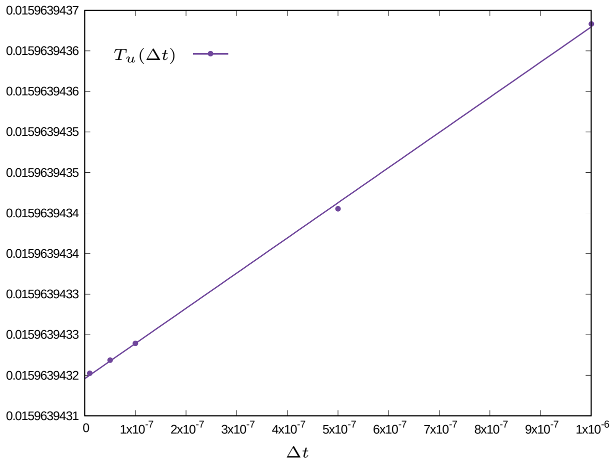
(2) $T_v(\Delta t)$.



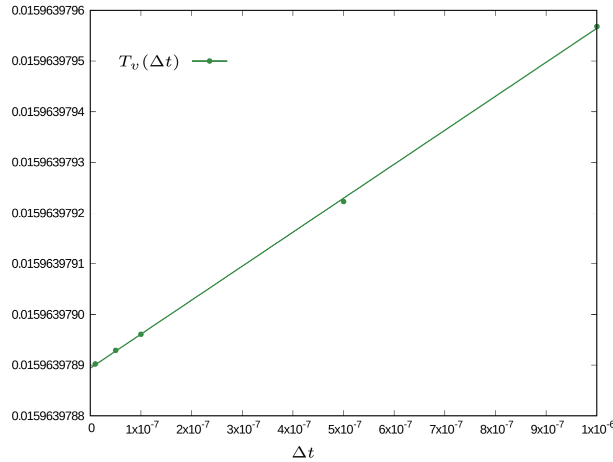
(3) $T_w(\Delta t)$.

Fig. 8. Numerical blow-up times in Case 2 ($t \in [0.0155,\ 0.0159]$).

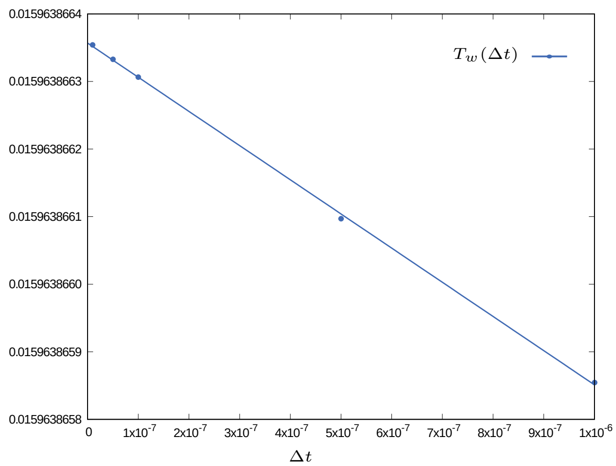
Similarly, from numerical data in the interval $[0.0159,\ 0.01596]$ which is closer to the blow-uptime we obtain that $T_u = 0.01596394\cdots$, $T_v = 0.01596397\cdots$, $T_w = 0.01596386\cdots$ (Fig. 9). The first 5 digits are the same. Please compare this with $T_{nu} = 0.0159638484\cdots$ in numerical computation with variable Δt(Table. 2). Numerical results suggest that solutions blow up at the same time.



(1) $T_u(\Delta t)$.



(2) $T_v(\Delta t)$.



(3) $T_w(\Delta t)$.

Fig. 9. Numerical blow-up times in Case 2 ($t \in [0.0159,\ 0.01596]$).

4 Conclusion

In the paper, a blow-up problem governed by the system of ordinary differential equations is analyzed numerically. The system is related to the default risk and various properties of solutions are theoretically clarified. The interesting property from a view point of numerical analysis is that solutions blow up at the same time. Traditional numerical methods are not suitable for ascertaining this property. We have here, on the other hand, proposed the method of numerical limits which consist of the extrapolation and the bounding transform. We apply this method to the problem of the numerical computation of blow-up phenomena. Numerical implementation is performed in multiple precision and it is realized that our procedure works efficiently. Numerical results successfully indicate that the blow-up time are the same.

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