Advances in Mathematical
Sciences and Applications
Vol. 26, No. 1 (2017), pp. 273-311

GAKKOTOSHO
TOKYO JAPAN

LOWER BOUND OF THE LIFESPAN OF THE SOLUTION TO SYSTEMS OF QUASI-LINEAR WAVE EQUATIONS WITH MULTIPLE PROPAGATION SPEEDS

Akira HOSHIGA
Department of Applied Mathematics
Shizuoka University
Johoku 3-5-1, Hamamatsu, Japan
(E-mail : hoshiga.akira@shizuoka.ac.jp)

Abstract. We consider the Cauchy problem of systems of quasilinear wave equations in 2-dimensional space. We assume that the propagation speeds are distinct and that the nonlinearities contain quadratic and cubic terms of the first and second order derivatives of the solution. We know that if the all quadratic and cubic terms of nonlinearities satisfy *Strong Null-condition*, then there exists a global solution for sufficiently small initial data. In this paper, we study about the lifespan of the smooth solution, when the cubic terms in the quasi-linear nonlinearities do not satisfy the Strong null-condition. In the proof of our claim, we use the *ghost weight* energy method and the L^∞-L^∞ estimates of the solution, which is slightly improved.

Communicated by T. Suzuki; Received October 4, 2017
AMS Subject Classification: 35L05, 35L15, 35L51, 35L70.
Keywords: Lifespan, Wave Eqation, Null-condition.

1 Intrduction

In this paper, we study the Cauchy problem;

$$\Box_i u^i = \partial_0^2 u^i - c_i^2 \triangle u^i = F^i(\partial u, \partial^2 u) \qquad (x,t) \in \mathbb{R}^2 \times (0,\infty), \tag{1.1}$$

$$u^i(x,0) = \varepsilon f^i(x), \quad \partial_t u^i(x,0) = \varepsilon g^i(x) \qquad x \in \mathbb{R}^2, \tag{1.2}$$

where $i = 1, 2, \cdots, m$ and $u(x,t) = (u^1(x,t), u^2(x,t), \cdots, u^m(x,t))$. We denote $\partial u = (\partial_\alpha u)_{\alpha=0,1,2}$ with $\partial_0 = \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1, 2)$ and $\partial^2 u = (\partial_\alpha \partial_\beta u)_{\alpha,\beta=0,1,2}$. Let $\varepsilon > 0$ is a small parameter and assume $f^i(x), g^i(x) \in C_0^\infty(\mathbb{R}^2)$ and $\mathrm{supp}\{f^i\}, \mathrm{supp}\{g^i\} \subset \{x \in \mathbb{R}^2 \ : \ |x| \le M\}$ for some positive constant M. We also assume that the propagation speeds of (1.1) are distinct constants, namely we assume

$$0 < c_1 < c_2 < \cdots < c_m. \tag{1.3}$$

Each nonlinearity $F^i(v,w)$ is smooth near the origin and is expressed as

$$F^i(\partial u, \partial^2 u) = \sum_{\ell=1}^{m} \sum_{\alpha,\beta=0}^{2} A_\ell^{i,\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^\ell + B^i(\partial u), \tag{1.4}$$

where

$$A_\ell^{i,\alpha\beta}(\partial u) = \sum_{j=1}^{m} \sum_{\gamma=0}^{2} a_{\ell j}^{i,\alpha\beta\gamma} \partial_\gamma u^j + \sum_{j,k=1}^{m} \sum_{\gamma,\delta=0}^{2} c_{\ell jk}^{i,\alpha\beta\gamma\delta} \partial_\gamma u^j \partial_\delta u^k + O(|\partial u|^3) \tag{1.5}$$

and

$$B^i(\partial u) = \sum_{j,k=1}^{m} \sum_{\alpha,\beta=0}^{2} b_{jk}^{i,\alpha\beta} \partial_\alpha u^j \partial_\beta u^k + \sum_{j,k,\ell=1}^{m} \sum_{\alpha,\beta,\gamma=0}^{2} d_{jk\ell}^{i,\alpha\beta\gamma} \partial_\alpha u^j \partial_\beta u^k \partial_\gamma u^\ell + O(|\partial u|^4). \tag{1.6}$$

Here $a_{\ell j}^{i,\alpha\beta\gamma}, b_{jk}^{i,\alpha\beta}, c_{\ell jk}^{i,\alpha\beta\gamma\delta}, d_{jk\ell}^{i,\alpha\beta\gamma}$ are constants.
In order to derive energy estimate, we need to assume that for each $i, \ell = 1, 2, \cdots, m$ and $\alpha, \beta = 0, 1, 2$,

$$A_\ell^{i,\alpha\beta}(v) = A_i^{\ell,\alpha\beta}(v) = A_i^{\ell,\beta\alpha}(v) \tag{1.7}$$

and

$$|A_\ell^{i,\alpha\beta}(v)| < \frac{(\min\{1, c_1\})^2}{2m}. \tag{1.8}$$

The assumption (1.8) constitutes no additional restriction, since we will only deal with small solutions. Note that by (1.3) and (1.4), we have for any $i = 1, 2, \cdots, m$,

$$u^i(x,t) = 0 \qquad \text{for} \quad |x| \ge c_m t + M. \tag{1.9}$$

For the proof of (1.9), see Theorem 4a in F. John [10]. Furthermore, in order to derive the *ghost weight* energy method, we need to assume that

$$\begin{aligned} a_{\ell j}^{i,\alpha\beta\gamma} &= 0 \qquad \text{when} \qquad (j,\ell) \ne (i,i) \\ b_{jk}^{i,\alpha\beta} &= 0 \qquad \text{when} \qquad j \ne k \end{aligned} \tag{1.10}$$

for each $i = 1, 2, \cdots, m$ and $\alpha, \beta, \gamma = 0, 1, 2$. This assumption means that only the terms $\partial u^i \partial^2 u^i$ and $\partial u^j \partial u^j$ $(j = 1, \cdots, m)$ appear in the quadratic terms of F^i.

Our purpose of this paper is to show a precise estimate for the *lifespan* T_ε. Here, we define T_ε by the supremum of $T > 0$ for which there exists a solution u to the Cauchy problem (1.1) and (1.2) in $\left(C^\infty(\mathbb{R}^2 \times [0, T))\right)^m$. To state the known results and our our result, we introduce some notations. Firstly, for $X = (X_0, X_1, X_2) \in \mathbb{R}^3$, we define $\Phi(X) = (\Phi^i_\ell(X))_{i,\ell=1,2,\cdots,m}$, $\Psi(X) = (\Psi^i_\ell(X))_{i,\ell=1,2,\cdots,m}$, $\Theta(X) = (\Theta^i_\ell(X))_{i,\ell=1,2,\cdots,m}$ and $\Xi(X) = (\Xi^i_\ell(X))_{i,\ell=1,2,\cdots,m}$ by

$$\Phi^i_\ell(X) = \sum_{\alpha,\beta,\gamma=0}^{2} a^{i,\alpha\beta\gamma}_{\ell\ell} X_\alpha X_\beta X_\gamma, \tag{1.11}$$

$$\Psi^i_\ell(X) = \sum_{\alpha,\beta=0}^{2} b^{i,\alpha\beta}_{\ell\ell} X_\alpha X_\beta, \tag{1.12}$$

$$\Theta^i_\ell(X) = \sum_{\alpha,\beta,\gamma,\delta=0}^{2} c^{i,\alpha\beta\gamma\delta}_{\ell\ell\ell} X_\alpha X_\beta X_\gamma X_\delta, \tag{1.13}$$

$$\Xi^i_\ell(X) = \sum_{\alpha,\beta,\gamma=0}^{2} d^{i,\alpha\beta\gamma}_{\ell\ell\ell} X_\alpha X_\beta X_\gamma. \tag{1.14}$$

Moreover, let $\phi(X) = (\phi^i_\ell(X))_{i,\ell=1,2,\cdots,m}$ be a function of $X = (X_0, X_1, X_2)$. If

$$\phi^i_\ell(X) \equiv 0 \quad \text{for} \quad X_0^2 = c_\ell^2 (X_1^2 + X_2^2) \tag{1.15}$$

holds for each $i, \ell = 1, 2, \cdots, m$, then we denote $\phi \approx 0$ and we say that ϕ satisfies *Strong Null-condition*. On the other hand, if (1.15) holds when $\ell = i$ $(i = 1, 2, \cdots, m)$, then we denote $\phi \sim 0$ and we say that ϕ satisfies *Standard Null-condition*. In [6], the author showed that $\liminf_{\varepsilon \to +0} \varepsilon \sqrt{T_\varepsilon} \geq C$ holds for a certain constant $C > 0$, when $B^i(\partial u) \equiv 0$ and Φ does not satisfy Standard Null-condition. Moreover, the author showed in [5] that $\liminf_{\varepsilon \to +0} \varepsilon^2 \log(1 + T_\varepsilon) \geq C$ holds for a certain constant $C > 0$, when $B^i(\partial u) \equiv 0$ and $a^{i,\alpha\beta\gamma}_{\ell j} = 0$ hold for any $i, j, \ell = 1, 2, \cdots, m$ and $\alpha, \beta, \gamma = 0, 1, 2$ and Θ does not satisfy Standard Null-condition. On the other hand, the author also showed in [7] that $T_\varepsilon = \infty$ for sufficiently small $\varepsilon > 0$, when Φ, Ψ, Θ and Ξ satisfy Strong Null-condition and $A^{i,\alpha\beta}_\ell(\partial u) \equiv 0$ holds for $\ell \neq i$. Thus, it is natural that we are concerned with the case excluded from those previous results. In this paper, we consider the case that Φ and Ψ satisfy Strong Null-condition and Ξ satisfies Standard Null-condition. Namely, we assume $\Phi \approx 0$, $\Psi \approx 0$ and $\Xi \sim 0$. For example, we consider the nonlinearities like

$$\begin{aligned}
F^1 = {} & (\partial_0 u^1)^2 - c_1^2 |\nabla u^1|^2 + (\partial_0 u^2)^2 - c_2^2 |\nabla u^2|^2 + \partial\{(\partial_0 u^1)^2 - c_1^2 |\nabla u^1|^2\} \\
& + \partial u^1 \{(\partial_0 u^1)^2 - c_1^2 |\nabla u^1|^2\} + (\partial u^2)^3 + \partial u^1 \partial u^2 \partial^2 u^1 \\
& + (\partial u^1)^2 \partial^2 u^1, \\
F^2 = {} & (\partial_0 u^1)^2 - c_1^2 |\nabla u^1|^2 + (\partial_0 u^2)^2 - c_2^2 |\nabla u^2|^2 + \partial\{(\partial_0 u^2)^2 - c_2^2 |\nabla u^2|^2\} \\
& + \partial u^2 \{(\partial_0 u^2)^2 - c_2^2 |\nabla u^2|^2\} + \partial u^1 (\partial u^2)^2 + (\partial u^1)^2 \partial^2 u^2 \\
& + (\partial u^2)^2 \partial^2 u^2,
\end{aligned}$$

where ∂ and ∂^2 stand for any of ∂_α $(\alpha = 0, 1, 2)$ and $\partial_\alpha\partial_\beta$ $(\alpha, \beta = 0, 1, 2)$ respectively.

Secondly, we introduce the Friedlander radiation field $\mathcal{F}^i(\rho, \omega)$. Let $u_0^i(x, t)$ be the solution to the Cauchy problem of the homogeneous linear wave equation;

$$\Box_i u_0^i = 0 \qquad (x, t) \in \mathbb{R}^2 \times (0, \infty), \tag{1.16}$$

$$u_0^i(x, 0) = f^i(x), \quad \partial_0 u_0^i(x, 0) = g^i(x) \qquad x \in \mathbb{R}^2. \tag{1.17}$$

Then we define $\mathcal{F}^i$ by

$$\mathcal{F}^i(\rho, \omega) = \lim_{r\to\infty} r^{\frac{1}{2}} u_0^i(x, t) \tag{1.18}$$

with $x = r\omega$ $(\omega \in S^1)$ and $\rho = r - c_i t$. We know that $\mathcal{F}^i(\rho, \omega)$ is expressed by

$$\mathcal{F}^i(\rho, \omega) = \frac{1}{2\sqrt{2}} \int_\rho^\infty \frac{1}{\sqrt{s-\rho}} (R_{g^i}(s, \omega) - \partial_s R_{f^i}(s, \omega)) ds,$$

where $R_h(s, \omega)$ is the Radon transform of $h \in C_0^\infty(\mathbb{R}^2)$, *i.e.*,

$$R_h(s, \omega) = \int_{\omega\cdot y = s} h(y)\ dS_y$$

for $s \in \mathbb{R},\ \omega \in S^1$. Note that $\mathcal{F}^i(\rho, \omega)$ satisfies

$$\mathcal{F}^i(\rho, \omega) = 0 \qquad \text{for} \qquad \rho \geq M, \tag{1.19}$$

$$\left|\partial_\rho^\ell \mathcal{F}^i(\rho, \omega)\right| \leq \frac{C}{(1 + |\rho|)^{\frac{1}{2}+\ell}} \tag{1.20}$$

and

$$\left|r^{\frac{1}{2}} \partial_0^\ell u_0^i(r\omega, t) - (-c_i)^\ell \partial_\rho^\ell \mathcal{F}^i(r - c_i t, \omega)\right| \leq \frac{C(1 + |r - c_i t|)^{\frac{1}{2}}}{t} \tag{1.21}$$

for $t \geq r/(2c_i) \geq 1$ and $\ell = 1, 2$. For the details about (1.19), (1.20) and (1.21), see L. Hörmander [3].

Then we define a constant

$$H_i = \max_{\substack{\rho\in\mathbb{R} \\ \omega\in S^1}} \left\{ -\frac{1}{c_i^2} \Theta_i^i(-c_i, \omega) \partial_\rho \mathcal{F}^i(\rho, \omega) \partial_\rho^2 \mathcal{F}^i(\rho, \omega) \right\} \tag{1.22}$$

and set

$$H = \max\{H_1, H_2, \cdots, H_m\}. \tag{1.23}$$

Note that by (1.19) and (1.20), each H_i is well-defined and nonnegative.

Now, we state our main result.

Theorem 1.1 *Assume that (1.7), (1.8) and (1.10) hold for the Cauchy problem (1.1) and (1.2). Also assume* $\Phi \approx 0$, $\Psi \approx 0$ *and* $\Xi \sim 0$. *Then, if* $H > 0$, *we have*

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log(1 + T_\varepsilon) \geq \frac{1}{H}. \tag{1.24}$$

Note that the estimate (1.24) coincides with the estimate obtained in [5], which was obtained with assumptions stronger than the present result. Hence, we can say that our result (1.24) is a generalization of the result in [5]. Also note that we can not improve the estimate (1.24), in general, since the counter result has been shown when $m = 1$ and $B^1(\partial u) \equiv 0$ in [4].

In the following sections, we aim at showing (1.24). In section 2, we prepare some notations and state a lemma which implies (1.24). We also discuss the estimates of the null-form. In section 3, we will show the L^∞-L^∞ estimates of solutions to the wave equation. It is an improvement of the one showed in [9]. In section 4, we concentrate to show *a priori* estimates of the solution, by using the ghost energy inequality and the method of ordinary differential equation along the characteristic curves.

2 Preliminary for the proof of Theorem 1.1

Our main theorem is immediately derived from the following lemma.

Lemma 2.1 *Under the same situation as Theorem 1.1, choose a positive constant* B *to be* $B < 1/H$. *Then there exists a constnat* $\varepsilon_0(B) > 0$ *such that*

$$\varepsilon^2 \log(1 + T_\varepsilon) \geq B \tag{2.1}$$

holds for $0 < \varepsilon < \varepsilon_0(B)$.

In order to state another lemma which causes Lemma 2.1, we introduce some notations. At first, we introduce the following differential operators,

$$\Omega = x_1\partial_2 - x_2\partial_1, \qquad S = t\partial_0 + x_1\partial_1 + x_2\partial_2$$

and denote

$$\Gamma = (\Gamma_0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4) = (\partial_0,\ \partial_1,\ \partial_2,\ \Omega,\ S)$$

and

$$\Gamma^a = \Gamma_0^{a_0}\Gamma_1^{a_1}\Gamma_2^{a_2}\Gamma_3^{a_3}\Gamma_4^{a_4}$$

for a multi-index $a = (a_0, a_1, a_2, a_3, a_4)$. We can verify the following commutator relations;

$$[\Gamma_\alpha, \Box_i] = -2\delta_{\alpha 4}\Box_i \qquad (\alpha = 0, 1, 2, 3, 4,\ i = 1, 2, \cdots, m)$$

and

$$[\partial_\alpha, \partial_\beta] = 0 \quad (\alpha, \beta = 0, 1, 2), \qquad [S, \partial_\alpha] = -\partial_\alpha \quad (\alpha = 0, 1, 2)$$
$$[\Omega, \partial_1] = -\partial_2, \quad [\Omega, \partial_2] = \partial_1, \quad [\Omega, \partial_0] = 0, \quad [S, \Omega] = 0.$$

Here, $[A, B] = AB - BA$ and $\delta_{\alpha\beta}$ is the Kronecker delta.

Secondly, we define norms. Let $v(x,t) = (v^1(x,t), v^2(x,t), \cdots, v^m(x,t))$ be a vector valued function defined on $\mathbb{R}^2 \times [0,T)$, then we set

$$\begin{aligned}
|v(x,t)|_k &= \sum_{i=1}^m |v^i(x,t)|_k = \sum_{i=1}^m \sum_{|a|\le k} |\Gamma^a v^i(x,t)|,\\
|v(t)|_k &= \sup_{x\in\mathbb{R}^2} |v(x,t)|_k, \qquad |v|_{k,T} = \sup_{0\leqq t<T} |v(t)|_k,\\
[v(x,t)]_k &= \sum_{i=1}^m [v^i(x,t)]_k = \sum_{i=1}^m \sum_{|a|\le k} |(1+|x|)^{\frac{1}{2}}(1+||x|-c_i t|)^{\frac{15}{16}}|\Gamma^a v^i(x,t)|,\\
[v(t)]_k &= \sup_{x\in\mathbb{R}^2} [v(x,t)]_k, \qquad [v]_{k,T} = \sup_{0\leqq t<T} [v(t)]_k,\\
[[v(x,t)]]_k &= \sum_{i=1}^m [[v^i(x,t)]]_k = \sum_{i=1}^m \sum_{|a|\le k} |(1+|x|)^{\frac{1}{2}}(1+||x|-c_i t|)|\Gamma^a v^i(x,t)|,\\
[[v(t)]]_k &= \sup_{x\in\mathbb{R}^2} [[v(x,t)]]_k, \qquad [[v]]_{k,T} = \sup_{0\leqq t<T} [[v(t)]]_k,\\
\langle v(x,t)\rangle_k &= \sum_{i=1}^m \langle v^i(x,t)\rangle_k = \sum_{i=1}^m \sum_{|a|\le k} (1+|x|+t)^{\frac{7}{16}}|\Gamma^a v^i(x,t)|,\\
\langle v(t)\rangle_k &= \sup_{x\in\mathbb{R}^2} \langle (x,t)\rangle_k, \qquad \langle v\rangle_k = \sup_{0\le t<T} \langle v(t)\rangle_k,
\end{aligned}$$

$$\begin{aligned}
\langle\langle v(x,t)\rangle\rangle_k &= \sum_{i=1}^m \langle\langle v^i(x,t)\rangle\rangle_k = \sum_{i=1}^m \sum_{|a|\le k} (1+|x|+t)^{\frac{1}{2}}|\Gamma^a v^i(x,t)|,\\
\langle\langle v(t)\rangle\rangle_k &= \sup_{x\in\mathbb{R}^2} \langle\langle (x,t)\rangle\rangle_k, \qquad \langle\langle v\rangle\rangle_{k,T} = \sup_{0\le t<T} \langle\langle v(t)\rangle\rangle_k,\\
||v(t)||_k &= \sum_{i=1}^m \sum_{|a|\le k} \left(\int_{\mathbb{R}^2} |\Gamma^a v^i(x,t)|^2 \, dx \right)^{\frac{1}{2}}, \qquad ||v||_{k,T} = \sup_{0\leqq t<T} ||v(t)||_k,
\end{aligned}$$

where k is a nonnegative integer and $|a| = a_0 + a_1 + \cdots + a_4$ for a multi-index $a = (a_0, a_1, a_2, a_3, a_4)$.
Then, we find that the following lemma implies Lemma 2.1.

Lemma 2.2 *Let $u(x,t) = (u^1(x,t), u^2(x,t), \cdots, u^m(x,t)) \in (C^\infty(\mathbb{R}^2 \times [0,T)))^m$ be a solution to (1.1) and (1.2). Choose an integer k so that $k \ge 21$. Let $B > 0$ be a constant so that $B < 1/H$ and also let $J > 0$ be a constant. Then, there exist constants $K = K(B) > 0$ and $\varepsilon_0 = \varepsilon_0(J,B) > 0$ such that, if*

$$[\partial u]_{k,T} + \langle u\rangle_{k+1,T} \le J\varepsilon \tag{2.2}$$

holds for $0 < \varepsilon < \varepsilon_0$, then

$$[\partial u]_{k,T_B} + \langle u\rangle_{k+1,T_B} \le K\varepsilon \tag{2.3}$$

holds for the same ε. Here, we have set $T_B = \min\{T, t_B\}$ and $t_B = \exp(B/\varepsilon^2) - 1$.

Proof of Lemma 2.1 providing Lemma 2.2: We show that Lemma 2.2 implies Lemma 2.1 by contradiction. If the statement of Lemma 2.1 is incorrect, there exists a positive constant $B_0(< 1/H)$ such that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\delta^2 \log(1 + T_\delta) \le B_0 \quad \text{and} \quad 0 < \delta < \varepsilon. \tag{2.4}$$

On the other hand, by the local existence theorem which was shown in A. Majda [13], we find that there are positive constants ε_1 and t_ε such that there exists a smooth solution $u(x,t) \in C^\infty(\mathbb{R}^2 \times [0, t_\varepsilon))$ to (1.1) and (1.2) for $0 < \varepsilon < \varepsilon_1$. Let $L > 0$ be a constant satisfying $[\partial u(0)]_k + \langle u(0)\rangle_{k+1} \le L\varepsilon$ and set $J_0 = 2\max\{K(B_0), L\}$, where $K(B_0)$ is the constant determined in Lemma 2.2 with $B = B_0$. Then we can define a positive constant τ_ε by

$$\tau_\varepsilon = \sup\{t > 0 \ : \ t < T_\varepsilon \ \text{ and } \ [\partial u(t)]_k + \langle u(t)\rangle_{k+1} \le J_0\varepsilon\} \ (\le T_\varepsilon)$$

for each $\varepsilon \in (0, \varepsilon_*)$. Here we have set $\varepsilon_* = \min\{\varepsilon_0(J_0, B_0), \varepsilon_1\}$. Note that (2.2) holds for $\varepsilon \in (0, \varepsilon_*)$ with $J = J_0$ and $T = \tau_\varepsilon$. Moreover, by using (1.7) and (1.8), we can show $\tau_\varepsilon < T_\varepsilon$ for each $\varepsilon \in (0, \varepsilon_*)$. (For the detail, see the proof of Lemma 2.1 in [6].) This means that

$$[\partial u]_{k,\tau_\varepsilon} + \langle u\rangle_{k+1,\tau_\varepsilon} = J_0\varepsilon \tag{2.5}$$

holds for $\varepsilon \in (0, \varepsilon_*)$. However, as mentioned above, there exists a constant $\delta = \delta(\varepsilon_*)$ such that (2.4) holds. In that case, we find that $T_{B_0} = \min\{\tau_\delta, t_{B_0}\} = \tau_\delta$ and hence that Lemma 2.2 implies

$$[\partial u]_{k,\tau_\delta} + \langle u\rangle_{k+1,\tau_\delta} \le K(B_0)\delta \le \frac{J_0}{2}\delta. \tag{2.6}$$

This contradicts to (2.5) and therefore we find that the claim of Lemma 2.1 is correct.

In the rest of this paper, we aim at showing Lemma 2.2. For this purpose, we prepare a proposition with respect to the *null-form*. Set $c_* = \min_{1\le i\le m}\{c_i - c_{i-1}\}/3$ with $c_0 = 0$. We see $c_* > 0$ from (1.3). Also we set

$$\Lambda_i(T) = \{(x,t) \in \mathbb{R}^2 \times [0,T) \ : \ ||x| - c_i t| \le c_* t\} \tag{2.7}$$

and

$$\Lambda_0(T) = \mathbb{R}^2 \times [0,T) \setminus \bigcup_{i=1}^m \Lambda_i(T). \tag{2.8}$$

We find that $\Lambda_i(T) \cap \Lambda_j(T) = \emptyset$ holds for any $T > 0$, if $i \ne j$ and that there exists a constant $C_1 > 0$ such that

$$\frac{1}{C_1}(1 + |x| + t) \le 1 + ||x| - c_j t| \le C_1(1 + |x| + t) \qquad (x,t) \in \Lambda_i(T) \tag{2.9}$$

holds for any $T>0$, if $i\neq j$.

In order to derive a good decay property from the null-form in $\Lambda_i(T)$, we introduce the following operators;

$$Z=(Z_1^i,Z_2^i),\qquad Z_\alpha^i=c_i\partial_\alpha+\frac{x_\alpha}{|x|}\partial_0\qquad (i=1,2,\cdots,m,\ \alpha=1,2). \tag{2.10}$$

Then we find that

$$Z_1^i=\frac{c_it-|x|}{t}\partial_1+\frac{x_1S-x_2\Omega}{|x|t},\qquad Z_2^i=\frac{c_it-|x|}{t}\partial_2+\frac{x_2S+x_1\Omega}{|x|t} \tag{2.11}$$

and hence that

$$|Z^iv(x,t)|\le\frac{||x|-c_it|}{t}|\partial v(x,t)|_0+\frac{2}{t}|v(x,t)|_1. \tag{2.12}$$

Now we have the following.

Proposition 2.1 *Let $T>1$ be a constant and let k be a positive integer. Let $v(x,t)=(v^1(x,t),v^2(x,t),\cdots,v^m(x,t))$ and $w(x,t)=(w^1(x,t),w^2(x,t),\cdots,w^m(x,t))$ be functions belonging to $(C^\infty(\mathbb{R}^2\times[0,T)))^m$. Assume that $\Phi\approx 0$, $\Psi\approx 0$, $\Xi\sim 0$ and (1.10) hold. Then, there exists a positive constant C_2 independent of T such that*

$$\begin{aligned}&\left|\sum_{\alpha,\beta,\gamma=0}^{2}a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma v^i(x,t)\partial_\alpha\partial_\beta w^i(x,t)\right|_k\\ &\le C_2\sum_{|b+c|\le k+1}(|Z^i\Gamma^bv^i(x,t)||\Gamma^c\partial^2w^i(x,t)|+|\Gamma^b\partial v^i(x,t)||Z^i\Gamma^c\partial w^i(x,t)|),\end{aligned} \tag{2.13}$$

$$\left|\sum_{\alpha,\beta=0}^{2}b_{jj}^{i,\alpha\beta}\partial_\alpha v^j(x,t)\partial_\beta v^j(x,t)\right|_k\le C_2\sum_{|b+c|\le k}|Z^j\Gamma^bv^j(x,t)||\Gamma^c\partial v^j(x,t)|, \tag{2.14}$$

$$\begin{aligned}&\left|\sum_{\alpha,\beta,\gamma=0}^{2}d_{iii}^{i,\alpha\beta\gamma}\partial_\alpha v^i(x,t)\partial_\beta v^i(x,t)\partial_\gamma v^i(x,t)\right|_k\\ &\le C_2\sum_{|b+c+d|\le k}|Z^i\Gamma^bv^i(x,t)||\Gamma^c\partial v^i(x,t)||\Gamma^d\partial v^i(x,t)|\end{aligned} \tag{2.15}$$

and especially

$$\begin{aligned}&\sum_{|a|\le k}\left|\sum_{\alpha,\beta,\gamma=0}^{2}\left\{\Gamma^a(a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma v^i(x,t)\partial_\alpha\partial_\beta v^i(x,t))-a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma v^i(x,t)\partial_\alpha\partial_\beta\Gamma^av^i(x,t)\right\}\right|\\ &\le C_2\sum_{\substack{|b+c|\le k+1\\ |b|,|c|\le k}}|Z^i\Gamma^bv^i(x,t)||\Gamma^c\partial v^i(x,t)|\end{aligned} \tag{2.16}$$

hold for $i, j = 1, 2, \cdots, m$. *Moreover, we find that*

$$\begin{aligned}&\left|\sum_{\alpha,\beta=0}^{2} b_{jj}^{i,\alpha\beta}\partial_\alpha v^j(x,t)\partial_\beta v^j(x,t)\right|_k \\ &\le C_2\Big(\frac{||x|-c_jt||\partial v^j(x,t)|_{[\frac{k}{2}]}|\partial v^j(x,t)|_k}{1+|x|+t}+\frac{P_k(v^j)(x,t)}{1+|x|+t}\Big)\end{aligned} \tag{2.17}$$

holds for $(x,t) \in \Lambda_j(T) \cap \{(y,s) : s \ge 1\}$, $i, j = 1, 2, \cdots, m$, *and that*

$$\begin{aligned}&\left|\sum_{\alpha,\beta,\gamma=0}^{2} a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma v^i(x,t)\partial_\alpha\partial_\beta w^i(x,t)\right|_k \\ &\le C_2\Big(\frac{||x|-c_it|(|\partial v^i(x,t)|_{[\frac{k}{2}]}|\partial w^i(x,t)|_{k+1}+|\partial w^i(x,t)|_{[\frac{k+1}{2}]}|\partial v^i(x,t)|_k)}{1+|x|+t}+ \\ &\qquad + \frac{P_k(v^i,w^i)(x,t)}{1+|x|+t}\Big),\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\left|\sum_{\alpha,\beta,\gamma=0}^{2} d_{iii}^{i,\alpha\beta\gamma}\partial_\alpha v^i(x,t)\partial_\beta v^i(x,t)\partial_\gamma v^i(x,t)\right|_k \\ &\le C_2\Big(\frac{||x|-c_it||\partial v^i(x,t)|^2_{[\frac{k}{2}]}|\partial v^i(x,t)|_k}{1+|x|+t}+\frac{|\partial v^i(x,t)|_{[\frac{k}{2}]}P_k(v^i)(x,t)}{1+|x|+t}\Big),\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\sum_{|a|\le k}\left|\sum_{\alpha,\beta,\gamma=0}^{2}\left\{\Gamma^a(a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma v^i(x,t)\partial_\alpha\partial_\beta v^i(x,t)) - a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma v^i(x,t)\partial_\alpha\partial_\beta\Gamma^a v^i(x,t)\right\}\right| \\ &\le C_2\Big(\frac{||x|-c_it||\partial v^i(x,t)|_{[\frac{k+1}{2}]}|\partial v^i(x,t)|_k}{1+|x|+t}+\frac{Q_k(v^i)(x,t)}{1+|x|+t}\Big)\end{aligned} \tag{2.20}$$

hold for $(x,t) \in \Lambda_i(T) \cap \{(y,s) : s \ge 1\}$, $i = 1, 2, \cdots, m$. *Here we have set*

$$\begin{aligned}P_k(v)(x,t) &= |v(x,t)|_{[\frac{k}{2}]+1}|\partial v(x,t)|_k + |\partial v(x,t)|_{[\frac{k}{2}]}|v(x,t)|_{k+1}, \\ P_k(v,w)(x,t) &= |v(x,t)|_{[\frac{k}{2}]+1}|\partial w(x,t)|_{k+1} + |\partial v(x,t)|_{[\frac{k}{2}]}|\partial w(x,t)|_{k+1} + \\ &\quad + |\partial w(x,t)|_{[\frac{k}{2}]+1}|\partial v(x,t)|_k + |\partial w(x,t)|_{[\frac{k}{2}]+1}|v(x,t)|_{k+1}, \\ Q_k(v)(x,t) &= |v(x,t)|_{[\frac{k}{2}]+1}|\partial v(x,t)|_k + |\partial v(x,t)|_{[\frac{k}{2}]}|\partial v(x,t)|_k + \\ &\quad + |\partial v(x,t)|_{[\frac{k}{2}]+1}|v(x,t)|_{k+1}.\end{aligned}$$

We can show the statements of Proposition 2.1 by the same manner with the proof of Proposition 2.1 in [7], except for (2.15). Since we assume that Ξ satisfies the Standard Null-condition instead of the Strong Null-condition, we have to review the proof of (2.15) carefully. According to the proof of Lemma 2.3, which is the key of the proof of

Proposition 2.1 in [7], we find that the each argument is closed with respect to the index $j = 1, 2, \cdots, m$, respectively. Hence, as well as Lemma 2.3 in [7], we can show that for any $\Gamma \in (\Gamma_0, \Gamma_1, \cdots, \Gamma_4)$, there exist constants $\hat{d}_{iii}^{i,\alpha\beta\gamma}$ such that $\hat{\Xi} = \left(\sum_{\alpha,\beta,\gamma=0}^{2} \hat{d}_{iii}^{i,\alpha\beta\gamma} X_\alpha X_\beta X_\gamma \right)$ satisfies the Standard Null-condition and

$$
\begin{aligned}
&\Gamma\left(\sum_{\alpha,\beta,\gamma=0}^{2} d_{iii}^{i,\alpha\beta\gamma} \partial_\alpha u^i \partial_\beta u^i \partial_\gamma u^i \right) \\
= &\sum_{\alpha,\beta,\gamma=0}^{2} d_{iii}^{i,\alpha\beta\gamma} (\Gamma\partial_\alpha u^i) \partial_\beta u^i \partial_\gamma u^i + \sum_{\alpha,\beta,\gamma=0}^{2} d_{iii}^{i,\alpha\beta\gamma} \partial_\alpha u^i (\Gamma\partial_\beta u^i) \partial_\gamma u^i \\
&+ \sum_{\alpha,\beta,\gamma=0}^{2} d_{iii}^{i,\alpha\beta\gamma} \partial_\alpha u^i \partial_\beta u^i (\Gamma\partial_\gamma u^i) + \sum_{\alpha,\beta,\gamma=0}^{2} \hat{d}_{iii}^{i,\alpha\beta\gamma} \partial_\alpha u^i \partial_\beta u^i \partial_\gamma u^i
\end{aligned}
$$

holds, provided Ξ satisfies Standard Null-condition. This leads to (2.15). For the detail of the proof, see Proposition 2.1 and Lemmas 2.2 and 2.3 in [7].

3 L^∞-L^∞ estimate

In this section, we will show a weighted L^∞-L^∞ estimate of solutions to inhomogeneous wave equations. It is an improvement of the estimate in Proposition 4.2 in [9]. Let c and T be positive constants and F be a function in $C^\infty(\mathbb{R}^2 \times [0, T))$. Then we introduce an operator $L_c(F)$;

$$
L_c(F)(x,t) = \frac{1}{2\pi c} \int_0^t \left(\int_{|x-y|<c(t-s)} \frac{F(y,s)}{\sqrt{c^2(t-s)^2 - |x-y|^2}} dy \right) ds \tag{3.1}
$$

for $(x,t) \in \mathbb{R}^2 \times [0,T)$. We know that $L_c(F)$ is the solution to the Cauchy problem;

$$
\begin{aligned}
&(\partial_0^2 - c^2\triangle) L_c(F) = F(x,t), \qquad && (x,t) \in \mathbb{R}^2 \times [0,T), \\
&L_c(x,0) = \partial_0 L_c(x,0) = 0, && x \in \mathbb{R}^2.
\end{aligned}
$$

Then we have the following.

Proposition 3.1 *Let c_i $(i = 1, 2, \cdots, m)$ be the propagation speeds defined in (1.3). Let $T > 0$ and $F, G, H \in C^\infty(\mathbb{R}^2 \times [0,T))$. Choose $\mu > 0$, $\nu > 0$ and $\rho > 0$ arbitrarily. Then, there exist positive constants $\tilde{C}_\mu$, $\hat{C}_\nu$ and $\dot{C}_\rho$ independent of T such that*

$$
|L_{c_i}(F)(x,t)|(1+|x|)^{\frac{1}{2}} \le \tilde{C}_\mu M_{\mu,0}^{(i)}(F)(x,t) \tag{3.2}
$$

and

$$
\begin{aligned}
&|\nabla L_{c_i}(G+H)(x,t)|(1+|x|)^{\frac{1}{2}}(1+||x|-c_i t|) \\
\le\ &\hat{C}_\nu M_{\nu,0}^{(i)}(G)(x,t) + \dot{C}_\rho\{1+\log(1+t+|x|)\} M_{0,\rho}^{(i)}(H)(x,t)
\end{aligned} \tag{3.3}
$$

hold for $(x,t)\in \mathbf{R}^2\times[0,T)$. *Here* $\nabla=(\partial_1,\partial_2)$ *and we have set*

$$M^{(i)}_{\mu,\nu}(F)(x,t)=\sum_{j=0}^{m}\sup_{\substack{(y,s)\in\\ \Lambda_j(T)\cap D^i(x,t)}}\{|y|^{\frac12}z^{(j)}_{\mu,\nu}(|y|,s)|F(y,s)|_1\},$$

$$z^{(j)}_{\mu,\nu}(\lambda,s)=(1+s+\lambda)^{1+\mu}(1+|\lambda-c_js|)^{1+\nu},$$

$$D^i(x,t)=\{(y,s)\ :\ |x-y|\le c_i(t-s)\ \}.$$

Proof of Proposition 3.1: By the same argument with the proof of Propositions 4.1 and 4.2 in [9], we obtain (3.2) and (3.3) when $H(x,t)\equiv 0$. Therefore, we have only to show (3.3) when $G(x,t)\equiv 0$. Without loss of generality, we may assume $c_i=1$ and for the sake of simplicity, we denote the constant depending on ρ by C which may change line by line, during this section. Set

$$\begin{aligned}
E_1 &= \{(y,s)\in\mathbf{R}^2\times[0,t)\ :\ |y|+s>t-r,\ |x-y|<t-s\}\\
E_2 &= \{(y,s)\in\mathbf{R}^2\times[0,t)\ :\ (t-r-1/2)_+<|y|+s<t-r\}\\
E_3 &= \{(y,s)\in\mathbf{R}^2\times[0,t)\ :\ |y|+s<(t-r-1/2)_+\}
\end{aligned}$$

with $r=|x|$ and define

$$P_j(H)(x,t)=\frac{1}{2\pi}\iint_{E_j}\frac{H(y,s)}{\sqrt{(t-s)^2-|x-y|^2}}\,dyds \qquad (j=1,2,3),$$

then we have

$$\partial_\ell L_{c_i}(H)(x,t)=\sum_{j=1}^{3}P_j(\partial_\ell H)(x,t) \qquad (\ell=1,2.)$$

Firstly, we deal with $P_1(\partial_\ell H)(x,t)$. Following the computation made in Section 4 of [9], we find

$$|P_1(\partial_\ell H)(x,t)|\le CM^{(i)}_{0,\rho}(H)\sum_{k=1}^{5}I_k,$$

where we have set

$$\begin{aligned}
I_1 &= \sum_{j=0}^{m}\iint_{D_1}\frac{\lambda^{\frac12}}{z^{(j)}_{0,\rho}(\lambda,s)}\left(\int_{-\varphi}^{\varphi}K_1(\lambda,\psi;r,t-s)\ d\psi\right)d\lambda ds,\\
I_2 &= \sum_{j=0}^{m}\int_{D_2'}\frac{\lambda^{\frac12}}{z^{(j)}_{0,\rho}(\lambda,s)}\left(\int_0^1 K_2(\lambda,\tau;r,t-s)\ d\tau\right)d\sigma,\\
I_3 &= \sum_{j=0}^{m}\iint_{D_2}\frac{1}{\lambda^{\frac12}z^{(j)}_{0,\rho}(\lambda,s)}\left(\int_0^1 K_2(\lambda,\tau;r,t-s)\ d\tau\right)d\lambda ds,\\
I_4 &= \sum_{j=0}^{m}\iint_{D_2}\frac{\lambda^{\frac12}}{z^{(j)}_{0,\rho}(\lambda,s)}\left(\int_0^1 |\partial_\lambda K_2(\lambda,\tau;r,t-s)|\ d\tau\right)d\lambda ds,\\
I_5 &= \sum_{j=0}^{m}\iint_{D_2}\frac{\lambda^{\frac12}}{z^{(j)}_{0,\rho}(\lambda,s)}\left(\int_0^1 |(\partial_\lambda\Psi\cdot K_2)(\lambda,\tau;r,t-s)|\ d\tau\right)d\lambda ds.
\end{aligned}$$

Here we have used the following notation:

$$
\begin{aligned}
K_1(\lambda,\varphi;r,t) &= \frac{1}{2\pi\sqrt{t^2-r^2-\lambda^2+2r\lambda\cos\psi}},\\
K_2(\lambda,\tau;r,t) &= \frac{1}{2\pi\sqrt{2r\lambda\tau(1-\tau)\{2-(1-\cos\varphi)\tau\}}},\\
\varphi(\lambda;r,t) &= \arccos\left(\frac{r^2+\lambda^2-t^2}{2r\lambda}\right),\\
\Psi(\lambda,\tau;r,t) &= \arccos\{1-(1-\cos\varphi)\tau\},\\
D_1 &= \{(\lambda,s)\in(0,\infty)\times(0,t)\ :\ \lambda_-<\lambda<\lambda_-+\delta \quad \text{or} \quad \lambda_+-\delta<\lambda<\lambda_+\},\\
D_2 &= \{(\lambda,s)\in(0,\infty)\times(0,t)\ :\ \lambda_-+\delta<\lambda<\lambda_+-\delta\},\\
D_2' &= \{(\lambda,s)\in(0,\infty)\times(0,t)\ :\ \lambda=\lambda_-+\delta \quad \text{or} \quad \lambda=\lambda_+-\delta\}
\end{aligned}
$$

with $\lambda_-=|t-s-r|$, $\lambda_+=t-s+r$ and $\delta=\min\{r,1/2\}$. Thus we aim to show

$$
I_k \le \frac{C\{1+\log(1+t+r)\}}{(1+r)^{\frac12}(1+|t-r|)} \qquad (k=1,2,3,4,5). \tag{3.4}
$$

In oder to show (3.4), we use the following estimates which are proved in Lemma 4.1 in [8].

Lemma 3.1 *Let $(\lambda,s)\in D_1\cup D_2$. then we have*

$$
\int_{-\varphi}^{\varphi} K_1\ d\psi = 2\int_0^1 K_2\ d\tau \le \frac{C}{(r\lambda)^{\frac12}}\log\left(2+\frac{r\lambda h(t-s-r)}{(\lambda-\lambda_-)(\lambda_+-\lambda)}\right), \tag{3.5}
$$

$$
\int_0^1 |\partial_\lambda K_2|\ d\tau \le \frac{C}{(r\lambda)^{\frac12}(\lambda+s+r-t)}, \tag{3.6}
$$

$$
\int_0^1 |\partial_\lambda \Psi\cdot K_2|\ d\tau \le \frac{C}{(r\lambda)^{\frac12}}\left(\frac{1}{(\lambda_+-\lambda)^{\frac12}(\lambda-\lambda_-)^{\frac12}}+\frac{1}{(\lambda^2-\lambda_-^2)^{\frac12}}\right), \tag{3.7}
$$

where, $h(p)=1$ for $p>0$ and $h(p)=0$ for $p\le 0$.

First we evaluate I_1. When $t-r-s>0$ and $\lambda>\lambda_+-\delta$, we have

$$
\log\left(2+\frac{r\lambda}{(\lambda-\lambda_-)(\lambda_++\lambda)}\right)\le \log 3,
$$

since $\lambda-\lambda_->r$. Moreover, we find that

$$
\begin{aligned}
z_{0,\rho}^{(j)}(\lambda,s) &\ge z_{0,\rho}^{(j)}(\lambda_+,s) && \text{for} \quad \lambda_+-\delta<\lambda<\lambda_+\\
z_{0,\rho}^{(j)}(\lambda,s) &\ge C z_{0,\rho}^{(j)}(\lambda_-,s) && \text{for} \quad \lambda_-<\lambda<\lambda_-+\delta.
\end{aligned}
$$

Hence, by (3.5), we get

$$
I_1 \le \frac{C}{r^{\frac12}}\sum_{j=0}^{m}(A_{1,j}+B_{1,j}+C_{1,j}), \tag{3.8}
$$

where we have set

$$
\begin{aligned}
A_{1,j} &= \int_0^t \left(\int_{\lambda_+ - \delta}^{\lambda_+} \frac{1}{z_{0,\rho}^{(j)}(\lambda_+, s)} \, d\lambda \right) ds, \\
B_{1,j} &= \int_0^{(t-r)_+} \left(\int_{\lambda_-}^{\lambda_- + \delta} \frac{1}{z_{0,\rho}^{(j)}(\lambda_-, s)} \log\left(2 + \frac{\lambda}{\lambda - \lambda_-}\right) d\lambda \right) ds, \\
C_{1,j} &= \int_{(t-r)_+}^t \left(\int_{\lambda_-}^{\lambda_- + \delta} \frac{1}{z_{0,\rho}^{(j)}(\lambda_-, s)} \, d\lambda \right) ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
A_{1,j} &= \int_0^t \left(\int_{\lambda_+ - \delta}^{\lambda_+} \frac{1}{(1+s+\lambda_+)(1+|\lambda_+ - c_j s|)^{1+\rho}} \, d\lambda \right) ds \\
&\leq \frac{C\delta}{1+t+r} \int_{-\infty}^{\infty} \frac{1}{(1+|(1+c_j)s - t - r|)^{1+\rho}} \, ds \\
&\leq \frac{C\delta}{1+|t-r|}.
\end{aligned} \tag{3.9}
$$

When we deal with $B_{1,j}$, we may assume $t > r$, since $B_{1,j} = 0$ if $t \leq r$. Integrating by parts, we find

$$
\begin{aligned}
\int_{\lambda_-}^{\lambda_- + \delta} \log\left(2 + \frac{\lambda}{\lambda - \lambda_-}\right) d\lambda &= \int_{\lambda_-}^{\lambda_- + \delta} \{\log(3\lambda - 2\lambda_-) - \log(\lambda - \lambda_-)\} \, d\lambda \\
&= \left[\frac{3\lambda - 2\lambda_-}{3} \log(3\lambda - 2\lambda_-) - (\lambda - \lambda_-)\log(\lambda - \lambda_-) \right]_{\lambda_-}^{\lambda_- + \delta} \\
&= \frac{\lambda_- + 3\delta}{3} \log(\lambda_- + 3\delta) - \delta \log \delta - \frac{\lambda_-}{3} \log \lambda_- \\
&= \frac{\lambda_-}{3} \log\left(1 + \frac{3\delta}{\lambda_-}\right) + \delta \log(\lambda_- + 3\delta) - \delta \log \delta \\
&\leq \delta + \delta \log(2 + |t-r|) + \delta^{\frac{1}{2}},
\end{aligned}
$$

where we have used $0 < \delta < 1/2$ and the facts

$$
\begin{aligned}
0 \leq \quad & \frac{\log(1+x)}{x} < 1 \qquad \text{for} \qquad x > 0, \\
0 \leq \quad & -\delta^{\frac{1}{2}} \log \delta < 1 \qquad \text{for} \qquad 0 < \delta < \frac{1}{2}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
B_{1,j} &\leq \frac{C\delta^{\frac{1}{2}} \log(2+|t-r|)}{1+|t-r|} \int_{-\infty}^{\infty} \frac{1}{(1+|(1+c_j)s - t - r|)^{1+\rho}} \, ds \\
&\leq \frac{C\delta^{\frac{1}{2}} \{1 + \log(1+t+r)\}}{1+|t-r|}.
\end{aligned} \tag{3.10}
$$

When $s > (t-r)_+$, we have

$$s + \lambda_- = 2s - t + r \geq |t-r|, \qquad s + \lambda_- \geq C|\lambda - c_j s| = C|(1-c_j)s - t + r|,$$

which imply

$$\begin{aligned} z_{0,\rho}^{(j)}(\lambda_-, s) &\geq C(1+|t-r|)(1+|(1-c_j)s-t+r|)^{1+\rho} \qquad \text{if} \quad j \neq i \\ z_{0,\rho}^{(i)}(\lambda_-, s) &\geq C(1+|t-r|)^{1+\rho}(1+2s-t+r). \end{aligned}$$

Therefore, we get

$$\begin{aligned} C_{1,j} &\leq \frac{C\delta}{1+|t-r|} \int_{-\infty}^{\infty} \frac{1}{(1+|(1-c_j)s-t+r|)^{1+\rho}}\, ds \\ &\leq \frac{C\delta}{1+|t-r|} \qquad\qquad \text{if} \qquad j \neq i \end{aligned} \tag{3.11}$$

$$\begin{aligned} C_{1,i} &\leq \frac{C\delta}{(1+|t-r|)^{1+\rho}} \int_{(t-r)_+}^{t} \frac{1}{1+2s+t-r}\, ds \\ &\leq \frac{C\delta\{1+\log(1+t+r)\}}{1+|t-r|}. \end{aligned} \tag{3.12}$$

Summing up (3.8), (3.9), (3.10), (3.11) and (3.12), we obtain (3.4) for $k=1$, since $\delta/r \leq 2/(1+r)$.

In the remainder of the proof of (3.4), we assume $r \geq 1/2$ so that $\delta = 1/2$, because D_2 is the empty set, if $0 < r < 1/2$.

Since $\lambda = \lambda_- + 1/2$ or $\lambda = \lambda_+ - 1/2$ for $(\lambda, s) \in D_2'$, we obtain (3.4) for $k=2$ analogously to the previous argument.

Next we evaluate I_3. Note that $\lambda > \lambda_- + 1/2$ for $(\lambda, s) \in D_2$ and that

$$\log\left(2 + \frac{r\lambda}{(\lambda-\lambda_-)(\lambda_+ + \lambda)}\right) \leq \log(2+2\lambda)$$

for $\lambda > \lambda_- + 1/2$. Therefore we get from (3.5)

$$\begin{aligned} r^{\frac{1}{2}} I_3 &\leq C\sum_{j=0}^{m} \iint_{D_2} \frac{\log(2+2\lambda)}{(1+\lambda) z_{0,\rho}^{(j)}(\lambda, s)} d\lambda ds \\ &\leq C\{1+\log(1+t+r)\} \sum_{j=0}^{m} A_{3,j}, \end{aligned}$$

where we have set

$$\begin{aligned} A_{3,j} &= \iint_{D_2} \frac{1}{(1+s+\lambda)^2(1+|\lambda - c_j s|)^{1+\rho}} d\lambda ds \qquad (1 \leq j \leq m), \\ A_{3,0} &= \iint_{D_2} \frac{1}{(1+s+\lambda)(1+\lambda)^{2+\rho}} d\lambda ds. \end{aligned}$$

When $1 \le j \le m$, changing variables by

$$\alpha = \lambda + s \qquad \text{and} \qquad \beta = \lambda - s, \tag{3.13}$$

we have

$$\begin{aligned} A_{3,j} &\le \frac{1}{2}\int_{|t-r|}^{t+r} \frac{1}{(1+\alpha)^2}\left(\int_{-|t-r|}^{\alpha} \frac{1}{(1+|\psi_j(\alpha,\beta)|)^{1+\rho}}\, d\beta\right) d\alpha \\ &\le \frac{C}{1+|t-r|}, \end{aligned}$$

where

$$2\psi_j(\alpha,\beta) = (c_j+1)\beta - (c_j-1)\alpha.$$

On the other hand, when $j = 0$, we have

$$A_{3,0} \le \frac{1}{1+|t-r|}\iint_{D_2} \frac{1}{(1+\lambda)^{2+\rho}}\, d\lambda ds \le \frac{C}{1+|t-r|}.$$

Therefore we have (3.4) for $k = 3$.

Next we evaluate I_4. Since $\lambda + s + r - t \ge 1/2$ for $\lambda \ge \lambda_- + 1/2$, we get from (3.6)

$$\begin{aligned} r^{\frac{1}{2}} I_4 &\le C\sum_{j=0}^{m}\iint_{D_2} \frac{1}{z_{0,\rho}^{(j)}(\lambda,s)(\lambda+s+r-t+1)}\, d\lambda ds \\ &\le \frac{C}{(1+|t-r|)}\int_{|t-r|}^{t+r} \frac{1}{\alpha+r-t+1}\left(\int_{-|r-t|}^{\alpha} \frac{1}{(1+|\psi_j(\alpha,\beta)|)^{1+\rho}}\, d\beta\right) d\alpha \\ &\le \frac{C\{1+\log(1+t+r)\}}{1+|t-r|}, \end{aligned}$$

which yields (3.4) for $k = 4$.

Next we evaluate I_5. It follows from $\lambda_- + 1/2 \le \lambda \le \lambda_+ - 1/2$ that

$$3(\lambda_+ - \lambda) \ge \lambda_+ - \lambda + 1, \quad 3(\lambda - \lambda_-) \ge \lambda - \lambda_- + 1, \quad 9(\lambda^2 - \lambda_-^2) \ge (\lambda+1)^2 - \lambda_-^2.$$

Hence we get from (3.7)

$$r^{\frac{1}{2}} I_5 \le C\sum_{j=0}^{m}(A_{5,j} + B_{5,j} + C_{5,j}),$$

where we have set

$$\begin{aligned} A_{5,j} &= \iint_{D_2\cap\{t-r\le s\}} \frac{1}{z_{0,\rho}^{(j)}(\lambda,s)(t+r-s-\lambda)^{\frac{1}{2}}(\lambda-t+s+r)^{\frac{1}{2}}}\, d\lambda ds, \\ B_{5,j} &= \iint_{D_2\cap\{t-r\ge s\}} \frac{1}{z_{0,\rho}^{(j)}(\lambda,s)(t+r-s-\lambda+1)^{\frac{1}{2}}(\lambda+t-s-r+1)^{\frac{1}{2}}}\, d\lambda ds, \\ C_{5,j} &= \iint_{D_2} \frac{1}{z_{0,\rho}^{(j)}(\lambda,s)(\lambda-t+s+r+1)^{\frac{1}{2}}(\lambda+t-s-r+1)^{\frac{1}{2}}}\, d\lambda ds. \end{aligned}$$

Changing variables by (3.13), we have

$$\begin{aligned}
A_{5,j} &\leq \frac{C}{1+|t-r|}\int_{|t-r|}^{t+r}\frac{1}{(t+r-\alpha)^{\frac{1}{2}}(\alpha-t+r)^{\frac{1}{2}}}\left(\int_{-|r-t|}^{\alpha}\frac{1}{(1+|\psi_j(\alpha,\beta)|)^{1+\rho}}\,d\beta\right)d\alpha\\
&\leq \frac{C}{1+|t-r|}\int_{t-r}^{t+r}\frac{1}{(t+r-\alpha)^{\frac{1}{2}}(\alpha-t+r)^{\frac{1}{2}}}\,d\alpha\\
&\leq \frac{C}{1+|t-r|}.
\end{aligned}$$

Changing variables by (3.13) and then by $\sigma=\psi_j(\alpha,\beta)$, we get

$$\begin{aligned}
B_{5,j} &\leq \frac{1}{2}\int_{|t-r|}^{t+r}\frac{1}{(1+\alpha)(t+r-\alpha+1)^{\frac{1}{2}}}\times\\
&\qquad\qquad\times\left(\int_{\gamma_j}^{\alpha}\frac{1}{(1+|\sigma|)^{1+\rho}\{1+\frac{2}{c_j+1}(\sigma-\gamma_j)\}^{\frac{1}{2}}}\,d\sigma\right)d\alpha,
\end{aligned}$$

where

$$2\gamma_j=(1-c_j)\alpha+(1+c_j)(r-t).$$

It has been shown in Lemma 3.13 in [12] that

$$\int_{\gamma_j}^{\alpha}\frac{1}{(1+|\sigma|)^{1+\rho}\{1+\frac{2}{c_j+1}(\sigma-\gamma_j)\}^{\frac{1}{2}}}\,d\sigma\leq\frac{C}{(1+|\gamma_j|)^{\frac{1}{2}}}.$$

Therefore, if $j\neq i$, we have

$$\begin{aligned}
B_{5,j} &\leq \frac{C}{(1+|t-r|)}\int_{|t-r|}^{t+r}\frac{1}{(t+r-\alpha+1)^{\frac{1}{2}}(1+|\gamma_j|)^{\frac{1}{2}}}\,d\alpha\\
&\leq \frac{C}{(1+|t-r|)}\int_{|t-r|}^{t+r}\left(\frac{1}{t+r-\alpha+1}+\frac{1}{1+|\gamma_j|}\right)d\alpha\\
&\leq \frac{C\{1+\log(1+t+r)\}}{1+|t-r|}.
\end{aligned}$$

On the other hand, if $j=i$, since $\gamma_i=r-t$, we have

$$\begin{aligned}
B_{5,i} &\leq \frac{C}{(1+|t-r|)}\int_{|t-r|}^{t+r}\frac{1}{(1+\alpha)^{\frac{1}{2}}(t+r-\alpha+1)^{\frac{1}{2}}}\,d\alpha\\
&\leq \frac{C}{(1+|t-r|)}\int_{|t-r|}^{t+r}\left(\frac{1}{1+\alpha}+\frac{1}{t+r-\alpha+1}\right)d\alpha\\
&\leq \frac{C\{1+\log(1+t+r)\}}{1+|t-r|}.
\end{aligned}$$

Since we can deal with $C_{5,j}$ similarly to $B_{5,j}$, we obtain (3.4) for $k=5$.

Secondly, we deal with $P_2(\partial_\ell H)$. We can assume $t>r$, since otherwise E_2 is empty. Switching to polar coordinates,

$$x=(r\cos\theta,r\sin\theta),\qquad y=\lambda\xi=(\lambda\cos(\theta+\psi),\lambda\sin(\theta+\psi)),\tag{3.14}$$

we get

$$P_2(\partial_\ell H)(x,t) = \int_0^{t-r} \left(\int_{(\lambda-\frac{1}{2})_+}^{\lambda_-} \lambda \partial_\ell H(\lambda\xi, s) \left(\int_{-\pi}^{\pi} K_1(\lambda,\psi;r,t-s)\, d\psi \right) d\lambda \right) ds$$

By Proposition 5.2 in [1], we have

$$\int_{-\pi}^{\pi} K_1(\lambda,\psi;r,t-s)\, d\psi \le \frac{C}{(\lambda+\lambda_-)^{\frac{1}{2}}(\lambda_+-\lambda)^{\frac{1}{2}}} \log\left(2+\frac{r\lambda}{(\lambda_- -\lambda)(\lambda_+ +\lambda)}\right) \quad (3.15)$$

for $0 < s < t-r$ and $0 < \lambda < \lambda_-$. It follows from the fact

$$\frac{1}{\lambda_+ - \lambda} \le \frac{2}{(r+1)(\lambda_- - \lambda)} \qquad \text{for} \qquad 0 < s < t-r, \quad \lambda_- - \frac{1}{2} \le \lambda < \lambda_- \quad (3.16)$$

that

$$(r+1)^{\frac{1}{2}} |P_2(\partial_\ell H)(x,t)| \le C M_{0,\rho}^{(i)}(F) \sum_{j=0}^{m} A_{6,j}, \quad (3.17)$$

where we have set

$$A_{6,j} = \int_0^{t-r} \left(\int_{(t-r-\frac{1}{2})_+}^{t-r} \frac{\log\left(2+\dfrac{\lambda+s}{\lambda_- -\lambda}\right)}{(1+s+\lambda)(1+|\lambda-c_j s|)^{1+\rho}(\lambda_- \lambda)^{\frac{1}{2}}}\, d\lambda \right) ds.$$

Changing variables by (3.13), we get

$$\begin{aligned} A_{6,j} &\le C \int_{t-r-\frac{1}{2}}^{t-r} \frac{\log\left(2+\dfrac{\alpha}{t-r+\alpha}\right)}{(1+\alpha)(t-r-\alpha)^{\frac{1}{2}}} \left(\int_{-\infty}^{\infty} \frac{1}{(1+|\psi_j|)^{1+\rho}}\, d\beta \right) d\alpha \\ &\le \frac{C}{t-r+1/2} \int_{t-r-\frac{1}{2}}^{t-r} \frac{\log(2(t-r)-\alpha) - \log(t-r-\alpha)}{(t-r-\alpha)^{\frac{1}{2}}}\, d\alpha. \end{aligned}$$

Here

$$\begin{aligned} &\int_{t-r-\frac{1}{2}}^{t-r} \frac{\log(2(t-r)-\alpha) - \log(t-r-\alpha)}{(t-r-\alpha)^{\frac{1}{2}}}\, d\alpha \\ = &\Big[-2(t-r-\alpha)(\log(2(t-r)-\alpha) - \log(t-r-\alpha)) \Big]_{t-r-\frac{1}{2}}^{t-r} \\ &\qquad -2\int_{t-r-\frac{1}{2}}^{t-r} \left(\frac{(t-r-\alpha)^{\frac{1}{2}}}{2(t-r)-\alpha} - \frac{(t-r-\alpha)^{\frac{1}{2}}}{t-r-\alpha} \right) d\alpha \\ = &\sqrt{2}\log\left(\frac{1}{2}+t-r\right) + \int_{t-r-\frac{1}{2}}^{t-r} \frac{t-r}{(2(t-r)-\alpha)(t-r-\alpha)^{\frac{1}{2}}}\, d\alpha \\ \le &C\log\left(\frac{3}{2}+2t\right) + \frac{t-r}{t-r+\frac{1}{2}} \int_{t-r-\frac{1}{2}}^{t-r} \frac{1}{(t-r-\alpha)^{\frac{1}{2}}}\, d\alpha \\ \le &C\left(1+\log\left(\frac{3}{2}+2t\right)\right) \end{aligned}$$

implies

$$A_{6,j} \leq \frac{C\{1+\log(1+t+r)\}}{1+|t-r|}. \tag{3.18}$$

Thus, combining (3.17) and (3.18), we obtain

$$(1+r)^{\frac{1}{2}}(1+|t-r|)|P_2(\partial_\ell H)(x,t)| \leq C\{1+\log(1+t+r)\}M_{0,\rho}^{(i)}(H).$$

Thirdly, we deal with $P_3(\partial_\ell H)$. We can assume $t > r+1/2$, since otherwise E_3 is empty. Integrating by parts in y and switching to polar coordinates as in (3.15), we get

$$\begin{aligned} P_3(\partial_\ell H)(x,t) &= \int_0^{t-r-\frac{1}{2}} \left(\int_0^{t-r-\frac{1}{2}} \left(\int_{-\pi}^{\pi} \lambda H(\lambda\xi, s) K_3(\lambda,\psi;x,t-s) d\psi \right) d\lambda \right) ds \\ &\quad + \int_0^{t-r-\frac{1}{2}} \left(\int_{-\pi}^{\pi} \lambda \xi_l H(\lambda\xi,s) K_1(\lambda,\psi;x,t-s) \Big|_{\lambda=t-s-r-\frac{1}{2}} d\psi \right) ds \\ &\equiv J_1 + J_2, \end{aligned}$$

where we have set

$$K_3(\lambda,\psi;x,t) = \frac{-(x_\ell - \lambda\xi_\ell)}{2\pi(t^2-r^2-\lambda^2+2r\lambda\cos\psi)^{\frac{3}{2}}}.$$

We see from (3.15) and (3.16) that

$$\begin{aligned} J_2 &\leq \frac{CM_{0,\rho}^{(i)}(H)\{1+\log(1+t+r)\}}{(r+1)^{\frac{1}{2}}(t-r+1/2)} \sum_{j=0}^{m} \int_0^{t-r-\frac{1}{2}} \frac{1}{(1+|t-r-(c_j+1)s-1/2|)^{1+\rho}} ds \\ &\leq \frac{C\{1+\log(1+t+r)\}M_{0,\rho}^{(i)}(H)}{(1+r)^{\frac{1}{2}}(1+|t-r|)}. \end{aligned}$$

As shown in the proof of Proposition 4.2 in [8], we have

$$\int_{-\pi}^{\pi} |K_3(\lambda,\psi;r,t-s)| d\psi \leq \frac{C}{(\lambda_- - \lambda)(\lambda_- + \lambda)^{\frac{1}{2}}(\lambda_+ - \lambda)^{\frac{1}{2}}}.$$

Therefore, since $r+1 \leq 2(\lambda_+ - \lambda)$ for $\lambda < t-r-s-1/2$, we get from (3.13)

$$\begin{aligned} & J_1 \\ \leq\ & \frac{CM_{0,\rho}^{(i)}(H)}{(1+r)^{\frac{1}{2}}} \sum_{j=0}^{m} \int_0^{t-r-\frac{1}{2}} \left(\int_0^{t-r-s-\frac{1}{2}} \frac{1}{(1+s+\lambda)(\lambda_- - \lambda)(1+|\lambda - c_j s|)^{1+\rho}} d\lambda \right) ds \\ \leq\ & \frac{CM_{0,\rho}^{(i)}(H)}{(1+r)^{\frac{1}{2}}} \sum_{j=0}^{m} \left\{ \int_{\frac{t-r}{2}}^{t-r-\frac{1}{2}} \left(\int_{-\alpha}^{\alpha} \frac{1}{(1+\alpha)(t-r-\alpha)(1+|\psi_j(\alpha,\beta)|)^{1+\rho}} d\beta \right) d\alpha + \right. \\ & \qquad \left. + \int_0^{\frac{t-r}{2}} \left(\int_{-\alpha}^{\alpha} \frac{1}{(1+\alpha)(t-r-\alpha)(1+|\psi_j(\alpha,\beta)|)^{1+\rho}} d\beta \right) d\alpha \right\} \\ \leq\ & \frac{C\{1+\log(1+t+r)\}M_{0,\rho}^{(i)}(H)}{(1+r)^{\frac{1}{2}}(1+|t-r|)}. \end{aligned}$$

This completes the proof of (3.3).

4 *a priori* estimates

In this section, we derive the *a priori* estimate (2.3) assuming (2.2). For this purpose, we introduce a notation. Let the assumptions of Lemma 2.1 be fulfilled and let $p(x,t;\varepsilon)$ and $q(x,t;\varepsilon)$ be functions defined on a set $D \subset \mathbb{R}^2 \times [0,T)$. Then we denote

$$p(x,t;\varepsilon) = O^*(q(x,t;\varepsilon)) \qquad \text{in} \quad D,$$

when there exist constants $K = K(B) > 0$ and $\varepsilon_0 = \varepsilon_0(J,B) > 0$ such that, if (2.2) holds for $0 < \varepsilon < \varepsilon_0$, then

$$|p(x,t;\varepsilon)| \le Kq(x,t;\varepsilon) \qquad \text{for} \quad (x,t) \in D$$

for the same ε. We can easily show that if $p_1(x,t;\varepsilon) = O^*(q(x,t;\varepsilon))$ and $p_2(x,t;\varepsilon) = O^*(q(x,t;\varepsilon))$, then $p_1(x,t;\varepsilon) + p_2(x,t;\varepsilon) = O^*(q(x,t;\varepsilon))$. Then our task to prove Lemma 2.1 is showing

$$[\partial u(x,t)]_k = O^*(\varepsilon) \quad \text{and} \quad \langle u(x,t)\rangle_{k+1} = O^*(\varepsilon) \qquad \text{in} \quad \mathbb{R}^2 \times [0,T_B). \tag{4.1}$$

Also we will express constants determined independently of J and T by K_n ($n \in \mathbb{N}$) in the following argument.

Now we aim to show (4.1). By (3.1), we can write

$$u^i(x,t) = u_0^i(x,t) + L_{c_i}(F^i)(x,t), \tag{4.2}$$

where $u_0^i(x,t)$ is the solution to (1.16), (1.17) and satisfies for any nonnegative integer p,

$$[[\partial u_0^i(t)]]_p + \langle\langle u_0^i(t)\rangle\rangle_{p+1} \le C_0\varepsilon \qquad \text{for} \quad 0 \le t < \infty, \tag{4.3}$$

with some constant $C_0 = C_0(f^i, g^i, p) > 0$. Then, we have for a multi-index $a = (a_0, a_1, \cdots, a_4)$,

$$\Gamma^a L_{c_i}(F^i) = v_a^i + \sum_{|b|\le|a|} C_{a,b} L_{c_i}(\Gamma^b F^i), \tag{4.4}$$

with some constants $C_{a,b}$. Here, $v_a^i = v_a^i(x,t)$ is the solution to the Cauchy problem;

$$\begin{aligned} \Box_i v_a^i &= 0 && \text{in} \quad \mathbb{R}^2 \times [0,\infty), \\ v_a^i(x,0) = \varepsilon^2 \phi_a^i(x), &\quad \partial_0 v_a^i(x,0) = \varepsilon^2 \psi_a^i(x) && \text{in} \quad \mathbb{R}^2, \end{aligned}$$

with functions $\phi_a^i,\ \psi_a^i \in C_0^\infty(\mathbb{R}^2)$ determined by $(f^j, g^j)_{j=1,2,\cdots,m}$ suitably. Indeed, by the commutation relations of Γ_α and $\Box_i$ and by the definition of $L_{c_i}(F^i)$, we have

$$\Box_i \Gamma_\alpha L_{c_i}(F^i) = \Gamma_\alpha \Box_i L_{c_i}(F^i) + 2\delta_{\alpha 4}\Box_i L_{c_i}(F^i) = \Gamma_\alpha F^i + 2\delta_{\alpha 4} F^i$$

and

$$\Gamma_\alpha L_{c_i}(F^i)(x,0)=0, \qquad \partial_0\Gamma_\alpha L_{c_i}(F^i)(x,0)=\delta_{\alpha 0}F^i(x,0). \tag{4.5}$$

Since F^i is quadratic, we can denote $F^i(x,0)=\varepsilon^2\psi^i(x)\in C_0^\infty(\mathbb{R}^2)$. Hence we have

$$\Gamma_\alpha L_{c_i}(F^i)=v^i+L_{c_i}(\Gamma_\alpha F^i+2\delta_{\alpha 4}F^i)=v^i+L_{c_i}(\Gamma_\alpha F^i)+2\delta_{\alpha 4}L_{c_i}(F^i) \tag{4.6}$$

where $v^i=v^i(x,t)$ is the solution to the Cauchy problem;

$$\begin{aligned} \Box_i v^i &= 0 \qquad\qquad \text{in} \quad \mathbb{R}^2\times[0,\infty),\\ v^i(x,0)=0, \quad \partial_0 v^i(x,0) &= \delta_{\alpha 0}\varepsilon^2\psi^i(x) \qquad \text{in} \quad \mathbb{R}^2. \end{aligned}$$

This implies (4.4) when $|a|=1$. Repeating the above argument, we can obtain (4.4) for any a.
Note that, as with (4.3), we have for any nonnegative integer p,

$$\sum_{|b|\le p}[[\partial v_b^i(t)]]_0+\sum_{|c|\le p+1}\langle\langle v_c^i(t)\rangle\rangle_0\le C_0'\varepsilon^2 \qquad \text{for} \quad 0\le t<\infty, \tag{4.7}$$

with some constant $C_0'=C_0'(f^1,\cdots,f^m,g^1,\cdots,g^m,p)>0$. It follows from (4.2) and (4.4) that

$$\Gamma^a u^i(x,t)=\Gamma^a u_0^i(x,t)+v_a^i(x,t)+\sum_{|b|\le|a|}C_{a,b}L_{c_i}(\Gamma^b F^i)(x,t) \tag{4.8}$$

Therefore, our task for the proof of (4.1) is to show

$$\sum_{|b|\le k}[\partial L_{c_i}(\Gamma^b F^i)(x,t)]_0+\sum_{|c|\le k+1}\langle L_{c_i}(\Gamma^c F^i)(x,t)\rangle_0=O^*(\varepsilon) \qquad \text{in} \quad \mathbb{R}^2\times[0,T_B). \tag{4.9}$$

We will show (4.9) by dividing the area into some parts.

Firstly, we assume $0\le t\le 1/\varepsilon$. In this region, we can show the sharper estimates;

$$\begin{aligned} &\sum_{|b|\le k}[[\partial L_{c_i}(\Gamma^b F^i)(x,t)]]_0+\sum_{|c|\le k+1}\langle\langle L_{c_i}(\Gamma^c F^i)(x,t)\rangle\rangle_0\\ &=O^*(\varepsilon^{\frac{5}{4}}) \qquad \text{in} \qquad \mathbb{R}^2\times\big[0,1/\varepsilon\big]. \end{aligned} \tag{4.10}$$

For this purpose, we prepare two propositions with respect to the energy.

Proposition 4.1 *Let $u(x,t)\in C^2(\mathbb{R}^2\times[0,T))$ be a function satisfying $||u||_{2,T}<\infty$. Then, there exists a constant $C_3>0$ such that*

$$|x|^{\frac{1}{2}}|u(x,t)|\le C_3||u(t)||_2 \tag{4.11}$$

holds for $(x,t)\in\mathbb{R}^2\times[0,T)$.

Proposition 4.2 *Let* $u(x,t)=(u^1(x,t),u^2(x,t),\cdots,u^m(x,t))\in(C^\infty(\mathbb{R}^2\times[0,T)))^m$ *be the solution to (1.1) and (1.2) and also let* ℓ *be a positive integer. Assume (1.7). Then there exist constants* $\delta>0$ *and* $C_4=C_4(\ell)>0$ *such that if* $|\partial u|_{[\frac{\ell+1}{2}],T}<\delta$ *holds, then*

$$||\partial u(t)||_\ell \le C_4||\partial u(0)||_\ell \exp\left(C_4\int_0^t |\partial u(s)|_{[\frac{\ell+1}{2}]}ds\right) \tag{4.12}$$

holds for $0\le t<T$.

We omit the proof of the propositions. For the details of Proposition 4.1, see [11]. On the other hand, we get Proposition 4.2 by the usual energy argument for the quasilinear wave equations with quadratic nonlinearities and by the Gronwall inequality.

By (2.2) and $k\ge 21$, we have $|\partial u|_{[\frac{k+5}{2}],\frac{1}{\varepsilon}}\le|\partial u|_{k,\frac{1}{\varepsilon}}\le J\varepsilon<\delta$ for $0<\varepsilon<\varepsilon_0$, if we take ε_0 to be $J\varepsilon_0<\delta$. Hence, by (2.2) and (4.12) with $\ell=k+4$ and $T=1/\varepsilon$, we have

$$\begin{aligned}
||\partial u(t)||_{k+4} &= O\Big(C_4||\partial u(0)||_{k+4}\exp\left(C_4\int_0^t|\partial u(s)|_{[\frac{k+5}{2}]}ds\right)\Big)\\
&= O\Big(C_4||\partial u(0)||_{k+4}\exp\left(C_4\int_0^t\frac{[\partial u(s)]_{[\frac{k+5}{2}]}}{(1+s)^{\frac12}}ds\right)\Big)\\
&= O^*\Big(\varepsilon\exp\left(\int_0^{\frac{1}{\varepsilon}}\frac{C_4J\varepsilon}{(1+s)^{\frac12}}ds\right)\Big)\\
&= O^*\left(\varepsilon\exp(4C_4J\varepsilon^{\frac12})\right)\\
&= O^*(\varepsilon) \qquad \text{in} \quad \big[0,\ 1/\varepsilon\big],
\end{aligned}\tag{4.13}$$

if we take ε_0 to be $\varepsilon_0\le 1$ and $J^2\varepsilon_0\le 1$. Therefore, by (1.9), (2.2), (3.2), (4.11) and (4.13), we have for $|c|\le k+1$

$$\begin{aligned}
&\sum_{|c|\le k+1}\langle\langle L_{c_i}(\Gamma^c F^i)(x,t)\rangle\rangle_0\\
&= O\left(\sum_{j=0}^m \sup_{\substack{(y,s)\in\\ \Lambda_j(\frac{1}{\varepsilon})\cap D^i(x,t)}}\{|y|^{\frac12}(1+s+|y|)^{1+\mu}(1+||y|-c_js|)|F^i(y,s)|_{k+1}\}\right)\\
&= O\Big(\sum_{j=0}^m \sup_{\substack{(y,s)\in\\ \Lambda_j(\frac{1}{\varepsilon})\cap D^i(x,t)}}\{|y|^{\frac12}(1+s+|y|)^{1+\mu}(1+||y|-c_js|)\times\\
&\qquad\qquad \times|\partial u(y,s)|_{[\frac{k+2}{2}]}|\partial u(y,s)|_{k+2}\}\Big)\\
&= O^*((1+t)^{\frac{9}{16}+\mu}[\partial u(t)]_{[\frac{k+2}{2}]}||\partial u(t)||_{k+4})\\
&= O^*(J\varepsilon^{\frac{23}{16}-\mu})\\
&= O^*(\varepsilon^{\frac54}) \qquad \text{in} \quad \mathbb{R}^2\times\big[0,\ 1/\varepsilon\big],
\end{aligned}\tag{4.14}$$

if we take μ and ε_0 to be $\mu < 1/16$ and $J^8\varepsilon_0 \le 1$. As for $\partial L_{c_i}(\Gamma^b F^i)$ with $|b| \le k$, when $0 \le t \le 1$, it follows from (1.9) that $1+|x|+t$ and $1+||x|-c_i t|$ are bounded in the support of the solution $u^i(x,t)$. Hence, by (4.4), (4.7) and (4.14), we find

$$\begin{aligned}
&\sum_{|b|\le k}[[\partial L_{c_i}(\Gamma^b F^i)(x,t)]]_0 \\
=\ & O\left(\sum_{|c|\le k+1}([[v_c^i(x,t)]]_0 + [[L_{c_i}(\Gamma^c F^i(x,t)]]_0)\right) \\
=\ & O\left(\sum_{|c|\le k+1}(\langle\langle v_c^i(x,t)\rangle\rangle_0 + \langle\langle L_{c_i}(\Gamma^c F^i(x,t)\rangle\rangle_0)\right) \\
=\ & O^*(\varepsilon^{\frac{5}{4}}) \qquad \text{in} \qquad \mathbb{R}^2\times[0,1].
\end{aligned}$$

On the other hand, by (1.9), (2.2), (3.3) with $G=0$, (4.11) and (4.13), we have

$$\begin{aligned}
&\sum_{|b|\le k}[[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0 \\
=\ & O\left(\sum_{j=0}^{m}\sup_{\substack{(y,s)\in\\ \Lambda_j\cap D^i(x,t)}}\{|y|^{\frac{1}{2}}(1+s+|y|)^{1+\nu}(1+||y|-c_j s|)|F^i(y,s)|_{k+1}\}\right) \\
=\ & O^*((1+t)^{\frac{9}{16}+\nu}[\partial u(t)]_{[\frac{k+2}{2}]}||\partial u(t)||_{k+2}) \qquad (4.15)\\
=\ & O^*(J\varepsilon^{\frac{23}{16}-\nu}) \\
=\ & O^*(\varepsilon^{\frac{5}{4}}) \qquad \text{in} \qquad \mathbb{R}^2\times\big[0,\ 1/\varepsilon\big],
\end{aligned}$$

if we take ν and ε_0 to be $\nu < 1/16$ and $J^8\varepsilon_0 < 1$. Therefore, when $1 \le t$, by (1.9), (4.4), (4.7) , (4.14), (4.15) and the identity

$$\partial_0 = -\frac{x_1}{t}\partial_1 - \frac{x_2}{t}\partial_2 + \frac{1}{t}S, \qquad (4.16)$$

we have

$$\begin{aligned}
&\sum_{|b|\le k}[[\partial_0 L_{c_i}(\Gamma^b F^i)(x,t)]]_0 \\
=\ & O\left(\sum_{|b|\le k}\left([[\nabla L_{c_i}(\Gamma^b F^i(x,t)]]_0 + \frac{1}{1+t}[[SL_{c_i}(\Gamma^b F^i)(x,t)]]_0\right)\right) \qquad (4.17)\\
=\ & O\left(\sum_{|b|\le k}[[\nabla L_{c_i}(\Gamma^b F^i(x,t)]]_0 + \sum_{|c|\le k+1}(\langle\langle v_c^i(x,t)\rangle\rangle_0 + \langle\langle L_{c_i}(\Gamma^c F^i(x,t)\rangle\rangle_0)\right) \\
=\ & O^*(\varepsilon^{\frac{5}{4}}) \qquad \text{in} \qquad \mathbb{R}^2\times[1,1/\varepsilon].
\end{aligned}$$

Therefore we obtain (4.10).

Secondly, we assume $1/\varepsilon \le t \le T_B$. In this region, we need more precise energy estimate:

Proposition 4.3 *Let* $u(x,t) = (u^1(x,t), u^2(x,t), \cdots, u^m(x,t)) \in (C^\infty(\mathbb{R}^2 \times [0,T))^m$ *be the solution to (1.1) and (1.2) under the same assumption in Theorem 1.1. Also let* ℓ *be a positive integer. Then there exist positive constants* C_5 *and* δ *such that if*

$$[[[\partial u]]]_{[\frac{\ell+1}{2}],T} + \langle u\rangle_{[\frac{\ell+1}{2}]+1,T} < \delta \tag{4.18}$$

holds, then

$$||\partial u(t)||_\ell \le C_5||\partial u(t_0)||_\ell \exp\left(C_5\int_{t_0}^t \frac{[[[\partial u(s)]]]_{[\frac{\ell+1}{2}]} + \langle u(s)\rangle_{[\frac{\ell+1}{2}]+1}}{1+s}\, ds\right) \tag{4.19}$$

holds for $0 \le t_0 \le t < T$. *Here, we have set*

$$[[[v]]]_{p,\tau} = \sup_{0\le t<\tau}[[[v(t)]]]_p, \quad [[[v(t)]]]_p = \sup_{x\in\mathbb{R}^2}[[[v(x,t)]]]_p,$$
$$[[[v(x,t)]]]_p = \begin{cases} [[v(x,t)]]_p & \textit{when} \ \ |x| \le t^{\frac{7}{8}}, \\ [v(x,t)]_p & \textit{when} \ \ |x| > t^{\frac{7}{8}}. \end{cases}$$

In order to prove (4.19), we use the ghost weight energy method, which was developed in S. Alinhac [2], like we did in the proof of Proposition 4.1 in [7]. In that argument, it was essential to show

$$\begin{aligned}&\int_{\mathbb{R}^2} e^{p_i(x,s)}\Gamma^a F^i(x,s)\Gamma^a\partial_0 u^i(x,s)\ dx \\ =\ & O\Big(\frac{[\partial u(s)]_{[\frac{|a|+1}{2}]} + \langle u(s)\rangle_{[\frac{|a|+1}{2}]+1}}{1+s}\int_{\mathbb{R}^2} e^{p_i(x,s)}|\Gamma^a\partial_0 u^i(x,s)|^2\ dx\Big)\end{aligned} \tag{4.20}$$

with a certain bounded function $p_i(x,s)$. We showed (4.20) by dividing the integration region $\mathbb{R}^2$ into Λ_i and Λ_i^c. For the case $(x,s) \in \Lambda_i$, we used the ghost weight method which is also applicable to the present situation. On the other hand, for the case $(x,s) \in \Lambda_i^c$, we extracted the decay $(1+s)^{-1}$ from the term $(1+||x|-c_is|)$ in $[[\partial u(x,s)]]_{[\frac{|a|+1}{2}]}$, which was the target to estimate in [7]. However, in our situation to estimate $[\partial u(x,s)]_{[\frac{|a|+1}{2}]}$, we can not earn the decay $(1+s)^{-1}$ from it in the region near $|x| = 0$. That is the reason why we introduced the norm $[[[\partial u(s)]]]_\ell$ and assumed (4.18) in Proposition 4.3. For the details of the ghost weight energy method, see the proof of Proposition 4.1 in [7].

In order to use (4.19) with $\ell = k+9$ and $T = T_B$, we also show that

$$[[[\partial u(x,t)]]]_{[\frac{k+10}{2}]} + \langle u(x,t)\rangle_{[\frac{k+10}{2}]+1} = O^*(\varepsilon^{\frac{1}{2}}) \qquad \text{in} \qquad \mathbb{R}^2 \times [0,\ T_B) \tag{4.21}$$

holds. By (2.2) and $k \ge 21$, we find that

$$[\partial u(x,t)]_{[\frac{k+10}{2}]} + \langle u(x,t)\rangle_{[\frac{k+10}{2}]+1} = O^*(J\varepsilon) = O^*(\varepsilon^{\frac{1}{2}}) \qquad \text{in} \qquad \mathbb{R}^2 \times [0, T_B) \tag{4.22}$$

if we take ε_0 to be $J^2\varepsilon_0 \le 1$. Furthermore, if we obtain

$$\sum_{|b|\le[\frac{k+10}{2}]}[[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0 = O^*(\varepsilon^{\frac{1}{2}}) \qquad \text{in} \qquad \mathbb{R}^2 \times \big[0,\ T_B\big), \tag{4.23}$$

then by (1.9), (2.2), (4.4), (4.16) and (4.23), we find that

$$
\begin{aligned}
&\sum_{|b|\le[\frac{k+10}{2}]}[[\partial_0 L_{c_i}(\Gamma^b F^i)(x,t)]]_0\\
&= O\Bigg(\sum_{|b|\le[\frac{k+10}{2}]}\Big(\frac{|x|}{t}[[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0+\frac{1}{t}[[SL_{c_i}(\Gamma^b F^i(x,t))]]_0\Big)\Bigg)\\
&= O\Bigg(\sum_{|b|\le[\frac{k+10}{2}]}[[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0+\\
&\qquad +\sum_{|c|\le[\frac{k+10}{2}]+1}\Big(\langle\langle v_c^i(x,t)\rangle\rangle_0+\frac{(1+|x|)^{\frac12}}{(1+|x|+t)^{\frac{7}{16}}}\langle L_{c_i}(\Gamma^c F^i(x,t))\rangle_0\Big)\Bigg) \qquad (4.24)\\
&= O\Bigg(\sum_{|b|\le[\frac{k+10}{2}]}[[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0+\\
&\qquad +\sum_{|c|\le[\frac{k+10}{2}]+1}(\langle\langle v_c^i(x,t)\rangle\rangle_0+\langle L_{c_i}(\Gamma^c F^i(x,t))\rangle_0)\Bigg)\\
&= O^*(\varepsilon^{\frac12}+J\varepsilon)\\
&= O^*(\varepsilon^{\frac12}) \qquad \text{in} \qquad \{(x,t)\ :\ |x|\le t^{\frac78},\ 1/\varepsilon\le t<T_B\},
\end{aligned}
$$

if we take ε_0 to be $J^2\varepsilon_0\le 1$. Hence, by (4.3), (4.7), (4.10), (4.23) and (4.24), we have (4.21).

In order to prove (4.9) and (4.23), we show that for any positive integer $\ell\le k+1$ and for any positive constant η

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{|c|\le\ell+1}\langle\langle L_{c_i}(\Gamma^c F^i)(x,t)\rangle\rangle_0\\
&= O^*\big(\varepsilon+J^2\varepsilon^2(1+t)^\eta\sup_{0\le s\le t}||\partial u(s)||_{\ell+8}\big) \qquad \text{in} \qquad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big) \qquad (4.25)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{|b|\le\ell}[[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0\\
&= O^*\big(\varepsilon+J^2\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}|y|^{\frac12}|\partial u(y,s)|_{\ell+6}\big) \qquad \text{in} \qquad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big). \qquad (4.26)
\end{aligned}
$$

hold. We will show (4.25) and (4.26) step by step.

At first, by (1.5), (1.6), (1.9), (2.17), (2.18), (3.2), (3.3), (4.11) and $\varepsilon^2\log(1+t)\le B$, we have for any $\mu_1 > 0$ and $\rho_1 > 0$,

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{|c|\le \ell+1}\langle\langle L_{c_i}(\Gamma^c F^i)(x,t)\rangle\rangle_0\\
=\ & O\Big(\sum_{i=1}^{m}\sum_{|c|\le \ell+1} M^{(i)}_{1+\mu_1,1}(\Gamma^c F^i)(x,t)\Big)\\
=\ & O\Big(\sum_{i=1}^{m}\sum_{j=0}^{m}\sup_{(y,s)\in D^i(x,t)\cap\Lambda_j}|y|^{\frac12} z^{(j)}_{1+\mu_1,1}(|y|,s)|F^i(y,s)|_{\ell+1}\Big)\\
=\ & O\Big(\sum_{j=0}^{m}\sup_{\substack{(y,s)\in\Lambda_j\\ 0\le s\le t}}\Big\{|y|^{\frac12} z^{(j)}_{\mu_1,2}(|y|,s)\sum_{h=1}^{m}|\partial u^h(y,s)|_{[\frac{\ell+2}{2}]}|\partial u^h(y,s)|_{\ell+2}+\\
&+|y|^{\frac12} z^{(j)}_{\mu_1,1}(|y|,s)\sum_{h=1}^{m}(|u^h(y,s)|_{[\frac{\ell+2}{2}]+1}|\partial u^h(y,s)|_{\ell+2}+|\partial u^h(y,s)|_{[\frac{\ell+2}{2}]}|u^h(y,s)|_{\ell+3})+\\
&+|y|^{\frac12} z^{(j)}_{1+\mu_1,1}(|y|,s)|\partial u(y,s)|^2_{[\frac{\ell+2}{2}]}|\partial u(y,s)|_{\ell+2}\Big\}\Big)\\
=\ & O\Big(([\partial u]_{[\frac{\ell+2}{2}],t}+\langle u\rangle_{[\frac{\ell+2}{2}]+1,t})\times\\
&\times\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}\{(1+s+|y|)^{-\frac{7}{16}+\mu_1}([[\partial u(y,s)]]_{\ell+2}+\langle\langle u(y,s)\rangle\rangle_{\ell+3})\}+\\
&+(1+t)^{\mu_1}[\partial u]^2_{[\frac{\ell+2}{2}],t}\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}|y|^{\frac12}|\partial u(y,s)|_{\ell+2}\Big)\\
=\ & O^*\Big(J\varepsilon\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}\{(1+s+|y|)^{-\frac{7}{16}+\mu_1}([[\partial u(y,s)]]_{\ell+2}+\langle\langle u(y,s)\rangle\rangle_{\ell+3})\}+\\
&+J^2\varepsilon^2(1+t)^{\mu_1}\sup_{0\le s\le t}||\partial u(s)||_{\ell+4}\Big)\qquad \text{in}\qquad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big)
\end{aligned}
\tag{4.27}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{|b|\le \ell}[[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0\\
=\ & O\Big(\sum_{i=1}^{m}\sum_{|c|\le \ell+1}\Big\{M^{(i)}_{1+\mu_1,1}(\Gamma^c N_2^i)+(1+\log(1+t))M^{(i)}_{1,1+\rho_1}(\Gamma^c(F^i-N_2^i))\Big\}\Big)\\
=\ & O\Big(\sum_{i=1}^{m}\sum_{j=0}^{m}\sup_{(y,s)\in D^i(x,t)\cap\Lambda_j}\{|y|^{\frac12} z^{(j)}_{1+\mu_1,1}(|y|,s)|N_2^i(y,s)|_{\ell+1}+\\
&\qquad +(1+\log(1+t))|y|^{\frac12} z^{(j)}_{1,1+\rho_1}(|y|,s)|(F^i-N_2^i)(y,s)|_{\ell+1}\}\Big)\\
=\ & O\Big(\sum_{j=0}^{m}\sup_{\substack{(y,s)\in\Lambda_j\\ 0\le s\le t}}\Big\{|y|^{\frac12} z^{(j)}_{\mu_1,2}(|y|,s)\sum_{h=1}^{m}|\partial u^h(y,s)|_{[\frac{\ell+2}{2}]}|\partial u^h(y,s)|_{\ell+2}+
\end{aligned}
$$

$$
\begin{aligned}
&+|y|^{\frac{1}{2}} z^{(j)}_{\mu_1,1}(|y|,s)\sum_{h=1}^{m}(|u^h(y,s)|_{[\frac{\ell+2}{2}]+1}|\partial u^h(y,s)|_{\ell+2}+|\partial u^h(y,s)|_{[\frac{\ell+2}{2}]}|u^h(y,s)|_{\ell+3})\Big\}+\\
&+(1+\log(1+t))\sum_{j=0}^{m}\sup_{\substack{(y,s)\in\Lambda_j\\0\le s\le t}}\{|y|^{\frac{1}{2}} z^{(j)}_{1,1+\rho_1}(|y|,s)|\partial u(y,s)|^2_{[\frac{\ell+2}{2}]}|\partial u(y,s)|_{\ell+2}\}\Big)\\
=\;& O\Big(([\partial u]_{[\frac{\ell+2}{2}],t}+\langle u\rangle_{[\frac{\ell+2}{2}]+1,t})\times \qquad (4.28)\\
&\times\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}\{(1+s+|y|)^{-\frac{7}{16}+\mu_1}([[\partial u(y,s)]]_{\ell+2}+\langle\langle u(y,s)\rangle\rangle_{\ell+3})\}+\\
&+(1+\log(1+t))[\partial u]^2_{[\frac{\ell+2}{2}],t}\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}|y|^{\frac{1}{2}}|\partial u(y,s)|_{\ell+2}\Big)\\
=\;& O^*\Big(J\varepsilon\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}\{(1+s+|y|)^{-\frac{7}{16}+\mu_1}([[\partial u(y,s)]]_{\ell+2}+\langle\langle u(y,s)\rangle\rangle_{\ell+3})\}+\\
&+J^2\varepsilon^2(1+\log(1+t))\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}|y|^{\frac{1}{2}}|\partial u(y,s)|_{\ell+2}\Big)\\
=\;& O^*\Big(J\varepsilon\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}\{(1+s+|y|)^{-\frac{7}{16}+\mu_1}([[\partial u(y,s)]]_{\ell+2}+\langle\langle u(y,s)\rangle\rangle_{\ell+3})\}+\\
&+J^2\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}|y|^{\frac{1}{2}}|\partial u(y,s)|_{\ell+2}\Big)\quad\text{in}\quad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big),
\end{aligned}
$$

where we have set

$$
N_2^i=\sum_{j,\ell=1}^{m}\sum_{\alpha,\beta=0}^{2}a^{i,\alpha\beta\gamma}_{\ell j}\partial_\gamma u^j\partial_\alpha\partial_\beta u^\ell+\sum_{j,k=1}^{m}\sum_{\alpha,\beta=0}^{2}b^{i,\alpha\beta}_{jk}\partial_\alpha u^j\partial_\beta u^k.
$$

Next, we estimate $(1+s+|y|)^{-\frac{7}{16}+\mu_1}([[\partial u(y,s)]]_{\ell+2}+\langle\langle u(y,s)\rangle\rangle_{\ell+3})$ for $(y,s)\in\mathbb{R}^2\times[0,t]$. By the same manner as (4.27), for any $\mu_2>0$, we obtain by (1.5), (1.6), (1.9), (2.17), (2.18), (3.2), (3.3), (4.3), (4.7), (4.8) and (4.17)

$$
\begin{aligned}
&(1+s+|y|)^{-\frac{7}{16}+\mu_1}([[\partial u(y,s)]]_{\ell+2}+\langle\langle u(y,s)\rangle\rangle_{\ell+3})\\
=\;& O\Big(\varepsilon+\sum_{i=1}^{m}\sum_{\substack{|b|\le\ell+2\\|c|\le\ell+3}}(1+s+|y|)^{-\frac{7}{16}+\mu_1}\times\\
&\times([[\nabla L_{c_i}(\Gamma^b F^i)(y,s)]]_0+\langle\langle L_{c_i}(\Gamma^c F^i)(y,s)\rangle\rangle_0)\Big)\\
=\;& O\Big(\varepsilon+\sum_{i=1}^{m}\sum_{|c|\le\ell+3}\sup_{y\in\mathbb{R}^2}\{(1+s+|y|)^{-\frac{7}{16}+\mu_1}M^{(i)}_{1+\mu_2,1}(\Gamma^c F^i)(y,s)\}\Big)\\
=\;& O\Big(\varepsilon+\sum_{i=1}^{m}\sum_{j=0}^{m}\sup_{y\in\mathbb{R}^2}\sup_{(\xi,\tau)\in D^i(y,s)\cap\Lambda_j}\{|\xi|^{\frac{1}{2}} z^{(j)}_{\frac{9}{16}+\mu_1+\mu_2,1}(|\xi|,\tau)|F^i(\xi,\tau)|_{\ell+3}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= O\biggl(\varepsilon + ([\partial u]_{[\frac{\ell+4}{2}],t} + \langle u\rangle_{[\frac{\ell+4}{2}]+1,t}) \times \\
&\qquad \times \sup_{(\xi,\tau)\in\mathbb{R}^2\times[0,s]} \{(1+\tau+|\xi|)^{-\frac{7}{8}+\mu_1+\mu_2}([[\partial u(\xi,\tau)]]_{\ell+4} + \langle\langle u(\xi,\tau)\rangle\rangle_{\ell+5})\} + \\
&\quad + [\partial u]^2_{[\frac{\ell+4}{2}],t} \sup_{(\xi,\tau)\in\mathbb{R}^2\times[0,s]} |\xi|^{\frac{1}{2}} |\partial u(\xi,\tau)|_{\ell+4}\biggr) \\
&= O^*\biggl(\varepsilon + J\varepsilon \sup_{(\xi,\tau)\in\mathbb{R}^2\times[0,s]} \{(1+\tau+|\xi|)^{-\frac{7}{8}+\mu_1+\mu_2}([[\partial u(\xi,\tau)]]_{\ell+4} + \langle\langle u(\xi,\tau)\rangle\rangle_{\ell+5})\} + \\
&\quad + J^2\varepsilon^2 \sup_{(\xi,\tau)\in\mathbb{R}^2\times[0,s]} |\xi|^{\frac{1}{2}} |\partial u(\xi,\tau)|_{\ell+4}\biggr) \quad \text{in} \quad \mathbb{R}^2 \times \bigl[1/\varepsilon,\ T_B\bigr).
\end{aligned}
\tag{4.29}
$$

Moreover, by the same manner as (4.29), for any $\mu_3 > 0$ we obtain (1.5), (1.6), (1.9), (3.2), (3.3), (4.3), (4.7), (4.8) and (4.17)

$$
\begin{aligned}
&(1+\tau+|\xi|)^{-\frac{7}{8}+\mu_1+\mu_2}([[\partial u(\xi,\tau)]]_{\ell+4} + \langle\langle u(\xi,\tau)\rangle\rangle_{\ell+5}) \\
&= O\biggl(\varepsilon + \sum_{i=1}^m \sum_{\substack{|b|\le \ell+4\\ |c|\le \ell+5}} (1+\tau+|\xi|)^{-\frac{7}{8}+\mu_1+\mu_2} \times \\
&\qquad \times ([[\nabla L_{c_i}(\Gamma^b F^i)(\xi,\tau)]]_0 + \langle\langle L_{c_i}(\Gamma^c F^i)(\xi,\tau)\rangle\rangle_0)\biggr) \\
&= O\biggl(\varepsilon + \sum_{i=1}^m \sum_{|c|\le \ell+5} \sup_{y\in\mathbb{R}^2} \{(1+\tau+|\xi|)^{-\frac{7}{8}+\mu_1+\mu_2} M^{(i)}_{1+\mu_3,1}(\Gamma^c F^i)(\xi,\tau)\}\biggr) \qquad (4.30) \\
&= O\biggl(\varepsilon + \sum_{i=1}^m \sum_{j=0}^m \sup_{y\in\mathbb{R}^2} \sup_{(\zeta,\theta)\in D^i(\xi,\tau)\cap\Lambda_j} \{|\zeta|^{\frac{1}{2}} z^{(j)}_{\frac{1}{8}+\mu_1+\mu_2+\mu_3,1}(|\zeta|,\theta) |F^i(\zeta,\theta)|_{\ell+5}\biggr) \\
&= O\biggl(\varepsilon + [\partial u]_{[\frac{\ell+6}{2}],t} \sup_{(\zeta,\theta)\in\mathbb{R}^2\times[0,\tau]} \{|\zeta|^{\frac{1}{2}} (1+\theta+|\zeta|)^{-\frac{5}{16}+\mu_1+\mu_2+\mu_3} |\partial u(\zeta,\theta)|_{\ell+6}\}\biggr) \\
&= O^*\biggl(\varepsilon + J\varepsilon \sup_{(\zeta,\theta)\in\mathbb{R}^2\times[0,\tau]} \{|\zeta|^{\frac{1}{2}} |\partial u(\zeta,\theta)|_{\ell+6}\}\biggr) \quad \text{in} \quad \mathbb{R}^2 \times \bigl[1/\varepsilon,\ T_B\bigr),
\end{aligned}
$$

if we take μ_1, μ_2, μ_3 to be $\mu_1+\mu_2+\mu_3 < 5/16$. Combining (4.27), (4.28), (4.29) and (4.30) and taking $\mu_1 = \eta$, $\varepsilon_0 \le 1$, $J\varepsilon_0 \le 1$, we have (4.25) and (4.26).

Now we show (4.23). It follows from (2.2), (4.26) and $k \ge 21$ that

$$
\begin{aligned}
&\sum_{i=1}^m \sum_{|b|\le[\frac{k+10}{2}]} [[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0 \\
&= O^*\biggl(\varepsilon + J^2 B \sup_{(y,s)\in\mathbb{R}^2\times[0,t]} |y|^{\frac{1}{2}} |\partial u(y,s)|_{[\frac{k+10}{2}]+6}\biggr) \\
&= O^*(\varepsilon + J^2[\partial u]_{k,t}) \\
&= O^*((1+J^3)\varepsilon) \\
&= O^*(\varepsilon^{\frac{1}{2}}) \quad \text{in} \quad \mathbb{R}^2 \times \bigl[1/\varepsilon,\ T_B\bigr),
\end{aligned}
$$

if we take ε_0 to be $J^6\varepsilon_0 \le 1$. This implies (4.23) and therefore (4.21). Furthermore, (4.21) implies that there exists a positive constant K_1 such that

$$[[[\partial u(s)]]]_{[\frac{k+10}{2}]} + \langle u(s)\rangle_{[\frac{k+10}{2}]+1} \le K_1\varepsilon^{\frac{1}{2}} \tag{4.31}$$

holds for $0 < \varepsilon < \varepsilon_0$. Hence, by (4.19) and (4.31), we have

$$\begin{aligned} &||\partial u(t)||_{k+9} \\ = \ & O\left(C_5||\partial u(0)||_{k+9}\exp\left(C_5\int_0^t \frac{[[[\partial u(s)]]]_{[\frac{k+10}{2}]} + \langle u(s)\rangle_{[\frac{k+10}{2}]+1}}{1+s}\, ds\right)\right) \\ = \ & O^*\left(\varepsilon\exp\left(\varepsilon^{\frac{1}{2}}\int_0^t \frac{C_5K_1}{1+s}\, ds\right)\right) \\ = \ & O^*\left(\varepsilon\exp\left(C_5K_1\varepsilon^{\frac{1}{2}}\log(1+t)\right)\right) \\ = \ & O^*\left(\varepsilon(1+t)^{C_5K_1\varepsilon^{\frac{1}{2}}}\right) \qquad \text{in} \qquad \big[1/\varepsilon,\ T_B\big). \end{aligned} \tag{4.32}$$

Therefore , by (4.11), (4.25), (4.26) and (4.32), we obtain

$$\begin{aligned} &\sum_{i=1}^m \sum_{|c|\le k+2} \langle\langle L_{c_i}(\Gamma^c F^i)(x,t)\rangle\rangle_0 \\ = \ & O^*\left(\varepsilon + J^2\varepsilon^2(1+t)^{\eta}\sup_{0\le s\le t}||\partial u(s)||_{k+9}\right) \\ = \ & O^*\left(\varepsilon + J^2\varepsilon^3(1+t)^{\eta+C_5K_1\varepsilon^{\frac{1}{2}}}\right) \\ = \ & O^*\big(\varepsilon(1+t)^{\frac{1}{16}}\big) \qquad \text{in} \qquad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big) \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} &\sum_{i=1}^m \sum_{|b|\le k+1} [[\nabla L_{c_i}(\Gamma^b F^i)(x,t)]]_0 \\ = \ & O^*\left(\varepsilon + J^2\varepsilon^2(1+\log(1+t))\sup_{0\le s\le t}||\partial u(s)||_{k+9}\right) \\ = \ & O^*\left(\varepsilon + J^2B\varepsilon(1+t)^{C_5K_1\varepsilon^{\frac{1}{2}}}\right) \\ = \ & O^*\big(\varepsilon^{\frac{127}{128}}(1+t)^{\frac{1}{256}}\big) \qquad \text{in} \qquad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big), \end{aligned} \tag{4.34}$$

if we choose η and ε_0 to be $0 < \eta + C_5K_1\varepsilon_0^{\frac{1}{2}} < 1/16$, $0 < C_5K_1\varepsilon_0^{\frac{1}{2}} < 1/256$ and $J^{256}\varepsilon_0 \le 1$. Hence, by (4.3), (4.7) and (4.33), we have

$$\langle\langle u(x,t)\rangle\rangle_{k+2} = O^*(\varepsilon(1+t)^{\frac{1}{16}}) \qquad \text{in} \qquad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big). \tag{4.35}$$

Hence, by (4.35),we obtain

$$\begin{aligned}\langle u(x,t)\rangle_{k+2} &= O^*((1+t)^{-\frac{1}{16}}\langle\langle u(x,t)\rangle\rangle_{k+2}) \\ &= O^*(\varepsilon) \qquad \text{in} \qquad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big)\end{aligned} \tag{4.36}$$

and therefore by (4.3), (4.7), (4.8), (4.17), (4.34) and (4.36), we have

$$[\partial u(x,t)]_{k+1} = O^*\big(\varepsilon^{\frac{127}{128}}(1+t)^{\frac{1}{256}}\big) \qquad \text{in} \qquad \mathbb{R}^2\times\big[1/\varepsilon,\ T_B\big). \tag{4.37}$$

Note that (4.36) and (4.37) are stronger than we needed with respect to the order of derivatives. We will make use of the strength of the estimates below.
On the other hand, in order to estimate ∂u, we introduce a subset of $\mathbb{R}^2\times[0,T_B)$ by

$$\tilde{\Lambda}_i(T) = \big\{(x,t)\ :\ ||x|-c_it|\le t^{\frac14},\ 1/\varepsilon\le t<T\big\} \qquad (i=1,2,\cdots,m),$$

and discuss by dividing the area $\mathbb{R}^2\times[0,T_B)$ into out-side and in-side of $\tilde{\Lambda}_i(T_B)$. We also introduce notations;

$$\tilde{\Lambda}_i^c(T) = \big\{(x,t)\ :\ (x,t)\notin\tilde{\Lambda}_i(T),\ \ 1/\varepsilon\le t<T\big\}$$

and

$$\begin{aligned}\partial\tilde{\Lambda}_i(T) = \big\{(x,t)\ :\ &||x|-c_it| = t^{\frac14} \ \text{ when }\ 1/\varepsilon<t<T \\ &\text{or }\ ||x|-c_it|<t^{\frac14}\ \text{ when }\ t=1/\varepsilon\ \big\}.\end{aligned}$$

Then we find that

$$(1+t)^{\frac14}\le C_6(1+||x|-c_i|) \qquad \text{for} \qquad (x,t)\in\tilde{\Lambda}_i^c(T_B) \tag{4.38}$$

holds for some constant $C_6>0$ and that

$$\tilde{\Lambda}_i(T_B)\subset\Lambda_i(T_B) \tag{4.39}$$

holds for sufficiently small $\varepsilon>0$. Hence, it follows from (4.34) and (4.38) that

$$\begin{aligned}\sum_{|b|\le k+1}[\nabla L_{c_i}(\Gamma^bF^i)(x,t)]_0 &= O^*\big(\varepsilon^{\frac{127}{128}}(1+t)^{\frac{1}{256}}(1+||x|-c_it|)^{-\frac{1}{16}}\big) \\ &= O^*\big(\varepsilon^{\frac{127}{128}}(1+t)^{-\frac{3}{256}}\big) \\ &= O^*(\varepsilon^{\frac{257}{256}}) \qquad \text{in} \qquad \tilde{\Lambda}_i^c(T_B),\end{aligned} \tag{4.40}$$

for $i=1,2,\cdots,m$. Hence, by (4.3), (4.7), (4.8), (4.16), (4.36) and (4.40), we obtain

$$[\partial u^i(x,t)]_{k+1} = O^*(\varepsilon) \qquad \text{in} \qquad \tilde{\Lambda}_i^c(T_B), \tag{4.41}$$

for $i=1,2,\cdots,m$. Especially, by (4.2), (4.3), (4.4), (4.7), (4.16), (4.36) and (4.40), we obtain

$$[\partial_0u^i(x,t)-\varepsilon\partial_0u_0^i(x,t)]_0 = O^*(\varepsilon^{\frac{257}{256}}) \qquad \text{on} \qquad \partial\tilde{\Lambda}_i(T_B) \tag{4.42}$$

and

$$[\partial_0^2 u^i(x,t) - \varepsilon\partial_0^2 u_0^i(x,t)]_0 \quad = \quad O^*(\varepsilon^{\frac{257}{256}}) \qquad \text{on} \qquad \partial\tilde{\Lambda}_i(T_B). \tag{4.43}$$

Now, the task left for us is to show

$$[\partial u^i(x,t)]_k = O^*(\varepsilon) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B), \tag{4.44}$$

for $i = 1, 2, \cdots, m$. We use the method of ordinary differential equation along the pseudo characteristic curves. Let $u(x,t) = (u^1(x,t), u^2(x,t), \cdots, u^m(x,t))$ be the solution to (1.1) and (1.2) and denote $x = r\omega$, $(r = |x|,\ \omega \in S^1)$. Then, for fixed $\lambda \in \mathbb{R}$ and $\omega \in S^1$, we define the i-th pseudo characteristic curve in (r,t)-plane by the solution $r = r^i(t;\lambda)$ of the Cauchy problem;

$$\begin{aligned} \frac{dr}{dt} \quad &= \quad \kappa_i(r,t) \equiv c_i + \frac{1}{2c_i^3}\Theta_i(-c_i,\omega)(\partial_0 u^i(r\omega,t))^2 \qquad t_0 \le t < T_B, \\ r(t_0) \quad &= \quad c_i t_0 + \lambda, \end{aligned} \tag{4.45}$$

where $t_0 = 1/\varepsilon$ when $|\lambda| < \varepsilon^{-\frac{1}{4}}$, and $t_0 = \lambda^4$ when $|\lambda| \ge \varepsilon^{-\frac{1}{4}}$. Namely, the initial point $(r^i(t_0;\lambda)\omega, t_0)$ is on $\partial\tilde{\Lambda}_i(T_B)$ for each $\lambda \in \mathbb{R}$ and $\omega \in S^1$. Denote

$$\mathcal{J}^i(\lambda;\omega) = \{(x,t) \ : \ x = r^i(t;\lambda)\omega, \quad t_0 \le t < T_B \ \},$$

then we find that

$$\tilde{\Lambda}_i(T_B) = \bigcup_{\lambda\in\mathbb{R},\ \omega\in S^1} \mathcal{J}^i(\lambda;\omega)$$

holds for each $i = 1, 2, \cdots, m$. For the details, see [6]. Now, we can transform the equation (1.1) into an ordinary differential equation along the pseudo characteristic curve. For a vector valued function $v = (v^1, v^2, \cdots, v^m)$, set

$$\mathcal{E}_i v = \Box_i v^i - \sum_{\ell=1}^{m}\sum_{\alpha,\beta=0}^{2} A_\ell^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta v^\ell,$$

then we obtain an identity

$$\begin{aligned} &(\partial_0 + \kappa_i\partial_r)\big(r^{\frac{1}{2}}\partial_0 v^i\big) \\ = \quad &\frac{r^{\frac{1}{2}}}{2}\mathcal{E}_i v + \frac{r^{\frac{1}{2}}}{2}(\partial_0 + c_i\partial_r)^2 v^i + \frac{r^{\frac{1}{2}}(\kappa_i - c_i)}{c_i}(\partial_0 + c_i\partial_r)\partial_0 v^i + \\ &+\frac{c_i^2}{2r^{\frac{3}{2}}}\Omega^2 v^i + \frac{1}{2r^{\frac{1}{2}}}(\kappa_i - c_i)\partial_0 v^i + \frac{c_i}{2r^{\frac{1}{2}}}(\partial_0 + c_i\partial_r)v^i - \\ &-\frac{r^{\frac{1}{2}}(\kappa_i - c_i)}{c_i}\partial_0^2 v^i + \frac{r^{\frac{1}{2}}}{2}\sum_{\ell=1}^{m}\sum_{\alpha,\beta=0}^{2} A_\ell^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta v^\ell. \end{aligned} \tag{4.46}$$

Note that the differential operator $\partial_0+\kappa_i\partial_r$ in the left hand side of (4.46) means the derivative along $r=r^i(t;\lambda)$ in (r,t)-plane. Furthermore, by (2.10), (2.11) and the definition of $\tilde{\Lambda}_i(T)$, we have for any $\alpha,\beta=0,1,2$ and $i=1,2,\cdots,m$

$$\begin{aligned}
\frac{1}{r^{\frac{1}{2}}}-\frac{1}{(c_it)^{\frac{1}{2}}} &= O\Big(\frac{|r-c_it|}{(1+t)^{\frac{3}{2}}}\Big)=O\Big(\frac{1}{(1+t)^{\frac{5}{4}}}\Big)\\
(\partial_0+c_i\partial_r)v &= O\Big(\frac{1+|r-c_it|}{1+t}|\partial v|_0+\frac{1}{1+t}|v|_1\Big)\\
\partial_\alpha v+\frac{\omega_\alpha}{c_i}\partial_0 v &= O\Big(\frac{1+|r-c_it|}{1+t}|\partial v|_0+\frac{1}{1+t}|v|_1\Big)\\
(\partial_0+c_i\partial_r)^2 v &= O\Big(\frac{(1+|r-c_it|)^2}{(1+t)^2}|\partial v|_1+\frac{1}{(1+t)^2}|v|_2\Big)\\
\partial_\alpha\partial_\beta v-\frac{\omega_\alpha\omega_\beta}{c_i^2}\partial_0^2 v &= O\Big(\frac{1+|r-c_it|}{1+t}|\partial v|_1+\frac{1}{(1+t)^2}|v|_2\Big)
\end{aligned}\tag{4.47}$$

in $\tilde{\Lambda}_i(T_B)$. By (2.2), (2.18) and (4.47), we have

$$\begin{aligned}
&-\frac{\kappa_i-c_i}{c_i}\partial_0^2 v^i+\frac{1}{2}\sum_{\ell=1}^{m}\sum_{\alpha,\beta=0}^{2}A_\ell^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta v^\ell\\
= &-\frac{1}{2c_i^4}\Theta_i(-c_i,\omega)(\partial_0 u^i)^2\partial_0^2 v^i+\frac{1}{2}\sum_{\alpha,\beta,\gamma,\delta=0}^{2}c_{iii}^{i,\alpha\beta\gamma\delta}\partial_\gamma u^i\partial_\delta u^i\partial_\alpha\partial_\beta v^i+\\
&+O^*\Big(\frac{|r-c_it|}{1+t}|\partial u^i|_0|\partial v^i|_1+\frac{1}{1+t}|u^i|_1|\partial v^i|_1+\frac{1}{(1+t)^2}|u^i|_1|v^i|_2+\\
&+\sum_{j\neq i}(|\partial u^j|_0|\partial u|_0|\partial v|_1+|\partial u|_0^2|\partial v^j|_1)+|\partial u|_0^3|\partial v|_1\Big)
\end{aligned}\tag{4.48}$$

$$\begin{aligned}
= \; & O^*\Big(\frac{|r-c_it|}{1+t}|\partial u^i|_0|\partial v^i|_1+\frac{1}{1+t}|u^i|_1|\partial v^i|_1+\frac{1}{(1+t)^2}|u^i|_1|v^i|_2+\\
&+\sum_{j\neq i}(|\partial u^j|_0|\partial u|_0|\partial v|_1+|\partial u|_0^2|\partial v^j|_1)+|\partial u|_0^3|\partial v|_1\Big) \quad \text{in} \quad \tilde{\Lambda}_i(T_B),
\end{aligned}$$

if we take ε_0 to be $J\varepsilon_0\leq 1$. Therefore, it follows from (4.46), (4.47) and (4.49) that

$$\begin{aligned}
&(\partial_0+\kappa_i\partial_r)\big(r^{\frac{1}{2}}\partial_0 v^i\big)-\frac{r^{\frac{1}{2}}}{2}\mathcal{E}_i v\\
= \; & O^*\Big(\frac{1}{1+t}|\partial v^i|_1+\frac{1}{(1+t)^{\frac{3}{2}}}|v^i|_2+\frac{1}{(1+t)^{\frac{1}{4}}}|\partial u^i|_0|\partial v^i|_1+\\
&+\frac{1}{(1+t)^{\frac{1}{2}}}|u^i|_1|\partial v^i|_1+r^{\frac{1}{2}}\sum_{j\neq i}(|\partial u^j|_0|\partial u|_0|\partial v|_1+|\partial u|_0^2|\partial v^j|_1)+\\
&+r^{\frac{1}{2}}|\partial u|_0^3|\partial v|_1\Big) \quad \text{in} \quad \tilde{\Lambda}_i(T_B).
\end{aligned}\tag{4.49}$$

Now we show (4.44) by induction with respect to k. Choose a point $(x,t)\in\tilde{\Lambda}_i(T_B)$, then there exist $\lambda\in\mathbb{R}$ and $\omega\in S^1$ such that $x=r\omega$ and $(r\omega,t)\in\mathcal{J}^i(\lambda;\omega)$. At first, we show

$$[\partial u^i(x,t)]_0=O^*(\varepsilon)\qquad\text{in}\qquad\tilde{\Lambda}_i(T_B)\tag{4.50}$$

for $i=1,2,\cdots,m$. Setting $v=u$ in (4.49), we have by (1.1), (2.2), (2.17) and (2.19),

$$\begin{aligned}
&\frac{d}{ds}\big\{\big(r^i(s;\lambda)\big)^{\frac12}\partial_0 u^i(r^i(s;\lambda)\omega,s)\big\}\\
=\;&O^*\bigg((r^i)^{\frac12}B^i(\partial u)+\frac{1}{1+s}|\partial u^i|_1+\frac{1}{(1+s)^{\frac32}}|u^i|_2+\frac{1}{(1+s)^{\frac14}}|\partial u^i|_1^2+\\
&+\frac{1}{(1+s)^{\frac12}}|u^i|_1|\partial u^i|_1+(r^i)^{\frac12}\sum_{j\neq i}|\partial u^j|_1|\partial u|_1^2+(r^i)^{\frac12}|\partial u|_1^4\bigg)\\
=\;&O^*\bigg(\frac{J\varepsilon}{(1+s)^{\frac54}}\bigg)\qquad\text{in}\qquad[t_0,t],
\end{aligned}\tag{4.51}$$

if we take ε_0 to be $J\varepsilon_0\le 1$. Integrating (4.50) from t_0 to t, we have

$$\begin{aligned}
&r^{\frac12}\partial_0 u^i(r\omega,t)\\
&=\big(r^i(t_0;\lambda)\big)^{\frac12}\partial_0 u^i(r^i(t_0;\lambda)\omega,t_0)+O^*\bigg(\frac{J\varepsilon}{(1+t_0)^{\frac14}}\bigg)\qquad\text{in}\qquad\tilde{\Lambda}_i(T_B),
\end{aligned}\tag{4.52}$$

which implies

$$r^{\frac12}\partial_0 u^i(r\omega,t)=O^*\big(\varepsilon\big)\qquad\text{in}\qquad\tilde{\Lambda}_i(T_B),\tag{4.53}$$

if we take ε_0 to be $J^4\varepsilon_0\le 1$. Moreover, integrating (4.45) and using (4.53), we have

$$\begin{aligned}
r-c_it\;&=\;r^i(t_0;\lambda)-c_it_0+O^*\big(\varepsilon^2\log(1+t)\big)\\
&=\;r^i(t_0;\lambda)-c_it_0+O^*(B)\\
&=\;O^*\big(1+|r^i(\lambda;t_0)-c_it_0|\big)\qquad\text{in}\qquad\tilde{\Lambda}_i(T_B).
\end{aligned}\tag{4.54}$$

Hence, by (4.40), (4.52) and (4.54), we obtain

$$\begin{aligned}
&[\partial_0 u^i(x,t)]_0\\
=\;&O\big((1+|r-c_it|)^{\frac{15}{16}}r^{\frac12}|\partial_0 u^i(r\omega,t)|\big)\\
=\;&O^*\bigg((1+|r^i(\lambda;t_0)-c_it_0|)^{\frac{15}{16}}\times\\
&\qquad\times\bigg\{\big(r^i(t_0;\lambda)\big)^{\frac12}|\partial_0 u^i(r^i(\lambda;t_0)\omega,t_0)|+\frac{J\varepsilon}{(1+t_0)^{\frac14}}\bigg\}\bigg)\\
=\;&O^*\bigg(\varepsilon+\frac{J\varepsilon}{(1+t_0)^{\frac{1}{64}}}\bigg)\\
=\;&O^*(\varepsilon)\qquad\text{in}\qquad\tilde{\Lambda}_i(T_B),
\end{aligned}\tag{4.55}$$

if we take ε_0 to be $J^{64}\varepsilon_0 \le 1$. It follows from (2.2), (4.47) and (4.55) that (4.50) holds. Note that (4.54) implies that there exists a positive constant C_7 independent of J such that

$$\frac{1}{C_7}(1+|r_0^i(t_0;\lambda)-c_it_0|) \le 1+|r-c_it| \le C_7(1+|r_0^i(t_0;\lambda)-c_it_0|) \tag{4.56}$$

for $(r\omega,t)\in\mathcal{J}^i(\lambda;\omega)$. Therefore, by (1.21), (4.42), (4.52) and (4.56), we have

$$\begin{aligned}&(r^i(s;\lambda))^{\frac12}\partial_0u^i(r^i(s;\lambda)\omega,s)\\ =\;& -c_i\varepsilon\partial_\rho\mathcal{F}(\lambda,\omega)+O^*\left(\frac{\varepsilon^{\frac{257}{256}}}{(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\right) \qquad \text{in} \qquad [t_0,t].\end{aligned} \tag{4.57}$$

Secondly, we will show

$$[\partial u^i(x,t)]_1 = O^*(\varepsilon) \qquad \text{in} \qquad \tilde\Lambda_i(T_B) \tag{4.58}$$

for $i=1,2,\cdots,m$. Set $v=\partial_0 u$ in (4.49), then we have

$$(\partial_0+\kappa_i\partial_r)((r^i)^{\frac12}\partial_0^2u^i) = \frac{(r^i)^{\frac12}}{2}\mathcal{E}_i\partial_0u+O^*\left(\frac{J\varepsilon}{(1+s)^{\frac54}}\right) \qquad \text{in} \qquad [t_0,t]. \tag{4.59}$$

By (1.1), (2.2), (2.17), (2.18), (2.19), (4.47) and (4.57), we have

$$\begin{aligned}&\mathcal{E}_i\partial_0u\\ =\;& \sum_{\ell=1}^{m}\sum_{\alpha,\beta=0}^{b}\partial_0\big(A_\ell^{i,\alpha\beta}(\partial u)\big)\partial_\alpha\partial_\beta u^\ell+\partial_0\big(B^i(\partial u)\big)\\ =\;& \sum_{\alpha,\beta,\gamma,\delta=0}^{2}c_{iii}^{i,\alpha\beta\gamma\delta}(\partial_0\partial_\gamma u^i\partial_\delta u^i+\partial_\gamma u^i\partial_0\partial_\delta u^i)\partial_\alpha\partial_\beta u^i+\\ &+O^*\Bigg(\frac{1+|r^i-c_is|}{1+s}|\partial u^i|_1^2+\frac{1}{(1+s)^2}|\partial_0u^i|_1|u^i|_2+\\ &\qquad+\sum_{j\ne i}(|\partial u^j|_0|\partial u|_1^2+|\partial u|_0|\partial u^j|_1|\partial u|_1)+|\partial u|_0^2|\partial u|_1^2\Bigg)\end{aligned} \tag{4.60}$$

$$\begin{aligned}=\;& \frac{2\Theta_i(-c_i,\omega)}{c_i^4(r^i)^{\frac32}}((r^i)^{\frac12}\partial_0u^i)((r^i)^{\frac12}\partial_0^2u^i)^2+O^*\left(\frac{J^2\varepsilon^2}{(1+s)^{\frac{39}{16}}}\right)\\ =\;& -\frac{2\varepsilon\Theta_i(-c_i,\omega)\partial_\rho\mathcal{F}^i(\lambda,\omega)}{c_i^4(r^i)^{\frac12}(1+s)}((r^i)^{\frac12}\partial_0^2u^i)^2+\\ &+O^*\left(\frac{J^2\varepsilon^2}{(1+s)^{\frac{39}{16}}}+\frac{J^2\varepsilon^{3+\frac{1}{256}}}{(1+s)^{\frac32}(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\right) \qquad \text{in} \qquad [t_0,t].\end{aligned}$$

Hence, we have

$$\begin{aligned} & (\partial_0+\kappa_i\partial_r)((r^i)^{\frac{1}{2}}\partial_0^2u^i) \\ = & -\frac{\varepsilon\Theta_i(-c_i,\omega)\partial_\rho\mathcal{F}^i(\lambda,\omega)}{c_i^4(1+s)}((r^i)^{\frac{1}{2}}\partial_0^2u^i)^2+ \\ & +O^*\left(\frac{J\varepsilon}{(1+s)^{\frac{5}{4}}}+\frac{J^2\varepsilon^2}{(1+s)^{\frac{31}{16}}}+\frac{J^2\varepsilon^{3+\frac{1}{256}}}{(1+s)(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\right) \qquad \text{in} \qquad [t_0,t]. \end{aligned} \tag{4.61}$$

Set

$$W(s)=(r^i(s;\lambda))^{\frac{1}{2}}\partial_0^2u^i(r^i(s;\lambda)\omega,s), \tag{4.62}$$

then (1.21), (4.43) and (4.62) imply the Cauchy problem of the ordinary differential equation;

$$W'(s)=-\frac{\varepsilon\Theta_i(-c_i,\omega)\partial_\rho\mathcal{F}^i(\lambda,\omega)}{c_i^4(1+s)}W(s)^2+Q(s), \qquad t_0\le s\le t\ (<T_B), \tag{4.63}$$

$$W(t_0)=(r^i(t_0,\lambda))^{\frac{1}{2}}\partial_0^2u^i(r^i(t_0,\lambda)\omega,t_0)=\varepsilon c_i^2\partial_\rho^2\mathcal{F}^i(\lambda,\omega)+O^*\big(\varepsilon^{\frac{257}{256}}\big), \tag{4.64}$$

where

$$Q(s)=O^*\left(\frac{J\varepsilon}{(1+s)^{\frac{5}{4}}}+\frac{J^2\varepsilon^2}{(1+s)^{\frac{31}{16}}}+\frac{J^2\varepsilon^{3+\frac{1}{256}}}{(1+s)(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\right). \tag{4.65}$$

Note that

$$\begin{aligned} \int_{t_0}^t|Q(s)|\,ds &= O^*\left(\frac{J\varepsilon}{(1+t_0)^{\frac{1}{4}}}+\frac{J^2\varepsilon^2}{(1+t_0)^{\frac{15}{16}}}+\frac{J^3\varepsilon^{3+\frac{1}{256}}\log(1+t)}{(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\right) \\ &= O^*\left(\frac{J\varepsilon^{\frac{17}{16}}}{(1+t_0)^{\frac{15}{64}}}+\frac{J^2\varepsilon^{\frac{173}{64}}}{(1+t_0)^{\frac{15}{64}}}+\frac{J^3B\varepsilon^{1+\frac{1}{256}}}{(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\right) \\ &= O^*\left(\frac{\varepsilon^{\frac{33}{32}}}{(1+t_0)^{\frac{15}{64}}}+\frac{\varepsilon^{\frac{513}{512}}}{(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\right) \qquad \text{in} \qquad [t_0,\ T_B), \end{aligned} \tag{4.66}$$

if we choose ε_0 to be $J^3\varepsilon_0^{\frac{1}{512}}<1$. Now we can show

$$[\partial_0^2u^i(x,t)]_0=O^*(\varepsilon) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B), \tag{4.67}$$

by using the following proposition.

Proposition 4.4 *Let $w(t)$ be the solution of the ordinary differential equation;*

$$w'(t)=\frac{\alpha}{1+t}w(t)^2+q(t) \qquad \textit{for} \qquad T_0\le t<T_1,$$

where α is a constant, T_0 and T_1 are positive constants and $q(t)$ is a continuous function in $[T_0,T_1)$. Assume

$$q_*=\int_{T_0}^{T_1}|q(t)|\,dt<\infty \qquad \textit{and} \qquad 2\alpha q_*\{\log(1+T_1)-\log(1+T_0)\}<1.$$

Then,

$$|w(t)| \le \left(1 + \frac{1}{1 - \alpha(w(T_0) + q_*)\{\log(1+t) - \log(1+T_0)\}}\right)(|w(T_0)| + q_*) \qquad (4.68)$$

holds, as long as the right hand side of (4.68) is well-defined.

For the proof of Proposition 4.4, see the proof of Proposition 3.4 in [5].

By (4.66), we have $q_* = \displaystyle\int_{t_0}^{T_B} |Q(s)|\, ds = O^*(\varepsilon^{\frac{513}{512}}) < \infty$ and

$$\begin{aligned}
& 2\alpha q_*(\log(1+T_B) - \log(1+t_0)) \\
= \;& -2\frac{\varepsilon\Theta_i(-c_i,\omega)\partial_\rho\mathcal{F}^i(\lambda,\omega)}{c_i^4} q_*(\log(1+T_B) - \log(1+t_0)) \\
\le \;& K_2 B \varepsilon^{\frac{1}{512}} < 1
\end{aligned}$$

for $0 < \varepsilon < \varepsilon_0$, if we take ε_0 to be $(K_2B)^{512}\varepsilon_0 < 1$. Hence, it follows from (4.63), (4.64), (4.66), (4.68) and $HB < 1$ that

$$\begin{aligned}
& |W(t)| \\
\le \;& \left(1 + \frac{1}{1 - \alpha(W(t_0) + q_*)\{\log(1+t) - \log(1+t_0)\}}\right)(|W(t_0)| + q_*) \\
\le \;& \left(1 + \frac{1}{1 - \left(\varepsilon^2 H + \alpha q_*\right)\{\log(1+t) - \log(1+t_0)\}}\right)(|W(t_0)| + q_*) \qquad (4.69) \\
\le \;& \left(1 + \frac{1}{1 - HB - K_2 B \varepsilon^{\frac{1}{512}}/2}\right)(|W(t_0)| + q_*) \\
\le \;& \left(1 + \frac{2}{1 - HB}\right)(|W(t_0)| + q_*) \qquad t_0 \le t < T_B
\end{aligned}$$

holds for $0 < \varepsilon < \varepsilon_0$, if we tale ε_0 to be $\varepsilon_0 \le \{(1 - HB)/(K_2B)\}^{512}$. Therefore, by (1.20), (4.56), (4.64) and (4.69), we obtain

$$\begin{aligned}
& (1 + |r - c_i t|)^{\frac{15}{16}} |W(t)| \\
= \;& O^*\left(\left(1 + \frac{2}{1 - HB}\right) C_7 (1 + |r^i(t_0;\lambda) - c_i t_0|)^{\frac{15}{16}} (|W(t_0)| + q_*)\right) \\
= \;& O^*(\varepsilon) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B).
\end{aligned}$$

This implies (4.67). Moreover, by (2.2), (4.47) and (4.67), we have

$$[\partial^2 u^i(x,t)]_0 = O^*(\varepsilon) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B). \qquad (4.70)$$

Now we show (4.58). Set $v = \Gamma_p u \;\; (p = 3, 4)$ in (4.49), then we have

$$(\partial_0 + \kappa_i \partial_r)((r^i)^{\frac{1}{2}} \partial_0 \Gamma_p u^i) = \frac{(r^i)^{\frac{1}{2}}}{2} \mathcal{E}_i \Gamma_p u + O^*\left(\frac{J\varepsilon}{(1+s)^{\frac{5}{4}}}\right) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B). \qquad (4.71)$$

By (1.1), (2.2), (2.17), (2.18), (2.19), (4.47), (4.50) and (4.70), we have

$$\begin{aligned}
&\mathcal{E}_i\Gamma_p u \\
=\ & \sum_{\ell=1}^{m}\sum_{\alpha,\beta=0}^{2}\left\{\Gamma_p\big(A_\ell^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u^\ell\big) - A_\ell^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma_p u^\ell\right\} - \\
&\quad -[\Gamma_p,\Box_i]u^i + \Gamma_p\big(B^i(\partial u)\big) \\
=\ & O^*\bigg(|\partial u|_0|\partial^2 u|_0|\partial_0\Gamma_p u^i|_0 + \frac{1+|r^i-c_is|}{1+s}|\partial u^i|_1^2 + \frac{1}{(1+s)^2}|\partial_0 u^i|_1|u^i|_2 + \qquad (4.72) \\
&\quad + \sum_{j\neq i}(|\partial u^j|_0|\partial u|_1^2 + |\partial u|_0|\partial u^j|_1|\partial u|_1) + |\partial u|_0^2|\partial u|_1^2\bigg) \\
=\ & O^*\left(\frac{\varepsilon^2}{(r^i)^{\frac{1}{2}}(1+s)}|r^{\frac{1}{2}}\partial_0\Gamma_p u^i|_0 + \frac{J^2\varepsilon^2}{(1+s)^{\frac{39}{16}}}\right) \qquad \text{in} \qquad [t_0,t].
\end{aligned}$$

Hence, by setting

$$V_1(s) = (r^i(s;\lambda))^{\frac{1}{2}}\partial_0\Gamma_p u^i(r^i(s;\lambda)\omega,s), \tag{4.73}$$

we have

$$V_1'(s) = O^*\left(\frac{\varepsilon^2}{1+s}|V_1(s)| + \frac{J\varepsilon}{(1+s)^{\frac{5}{4}}}\right) \qquad \text{in} \qquad [t_0,t], \tag{4.74}$$

if we take ε_0 to be $J\varepsilon_0 \leq 1$. Thus, the Gronwall inequality implies

$$\begin{aligned}
&|V_1(t)| \\
=\ & O^*\left(\left\{|V_1(t_0)| + \int_{t_0}^{t}\left(\frac{J\varepsilon}{(1+s)^{\frac{5}{4}}}\right)ds\right\}\exp\left(\int_{t_0}^{t}\frac{K_3\varepsilon^2}{1+s}ds\right)\right) \qquad (4.75) \\
=\ & O^*\left(\left(|V_1(t_0)| + \frac{J\varepsilon}{(1+t_0)^{\frac{1}{4}}}\right)e^{K_3B}\right) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B).
\end{aligned}$$

Hence, by (4.41), (4.54) and (4.75), we have

$$\begin{aligned}
&(1+|r-c_it|)^{\frac{15}{16}}|V_1(t)| \\
=\ & O^*\left(C_7(1+|r^i(t_0,\lambda)-c_it_0|)^{\frac{15}{16}}\left(|V_1(t_0)| + \frac{J\varepsilon}{(1+t_0)^{\frac{1}{4}}}\right)e^{K_3B}\right) \\
=\ & O^*\left(\varepsilon + \frac{J\varepsilon}{(1+t_0)^{\frac{1}{64}}}\right) \qquad (4.76) \\
=\ & O^*(\varepsilon) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B),
\end{aligned}$$

if we take ε_0 to be $J^{64}\varepsilon_0 \leq 1$. Therefore, (4.36), (4.47) and (4.76) imply (4.58).

Finally, for any integer h so that $2 \leq h \leq k$, we show

$$[\partial u^i(x,t)]_h = O^*(\varepsilon) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B), \tag{4.77}$$

for $i=1,\cdots,m$, under the assumption

$$[\partial u(x,t)]_{h-1}=O^*(\varepsilon) \qquad \text{in} \qquad \mathbb{R}^2\times[1/\varepsilon,T_B). \tag{4.78}$$

Set $v=\Gamma^a u$ with $|a|\le h$ in (4.49), then (1.1), (2.17), (2.19), (4.36), (4.37), (4.50), (4.70) and (4.78) imply

$$\begin{aligned}
&(\partial_0+\kappa_i\partial_r)((r^i)^{\frac12}\partial_0\Gamma^a u^i)\\
=\ &\frac{(r^i)^{\frac12}}{2}\mathcal{E}_i\Gamma^a u+O^*\Bigg(\frac{1}{1+s}|\partial u^i|_{h+1}+\frac{1}{(1+s)^{\frac32}}|u^i|_{h+2}+\\
&+\frac{1}{(1+s)^{\frac14}}|\partial u^i|_0|\partial u^i|_{h+1}+\frac{1}{(1+s)^{\frac12}}|u^i|_1|\partial u^i|_{h+1}+\\
&+(r^i)^{\frac12}\sum_{j\ne i}(|\partial u^j|_0|\partial u|_0|\partial u|_{h+1}+|\partial u|_0^2|\partial u^j|_{h+1})+(r^i)^{\frac12}|\partial u|_0^3|\partial u|_{h+1}\Bigg)\\
=\ &\frac{(r^i)^{\frac12}}{2}\mathcal{E}_i\Gamma^a u+O^*\Bigg(\frac{\varepsilon}{(1+s)^{1+\frac{63}{256}}}\Bigg) \qquad \text{in} \qquad [t_0,t]
\end{aligned} \tag{4.79}$$

and

$$\begin{aligned}
&\mathcal{E}_i\Gamma^a u\\
=\ &\sum_{\ell=1}^{m}\sum_{\alpha,\beta=0}^{2}\{\Gamma^a(A_\ell^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u^\ell)-A_\ell^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^\ell\}+\Gamma^a B^i(\partial u)+[\Box_i,\Gamma^a]u^i\\
=\ &O^*\Bigg(|\partial u^i|_0|\partial^2 u^i|_0\sum_{|a|\le h}|\partial_0\Gamma^a u^i|+\frac{|r^i-c_i s|}{1+s}(|\partial u^i|_h+|\partial u^i|_h^2)|\partial u^i|_{h+1}+\\
&+\frac{1}{1+s}(|\partial u^i|_{h+1}+|\partial u^i|_{h+1}^2)|u^i|_{h+2}+|\partial u^i|_{h-1}^3+\\
&+\sum_{j\ne i}(|\partial u^j|_h|\partial u|_h|\partial u|_{h+1}+|\partial u|_h^2|\partial u^j|_{h+1})+|\partial u|_h^3|\partial u|_{h+1}\Bigg)\\
=\ &O^*\Bigg(\frac{\varepsilon^2}{(r^i)^{\frac12}(1+s)}|(r^i)^{\frac12}\partial_0\Gamma^a u^i|+\frac{\varepsilon^{1+\frac{127}{128}}}{(r^i)^{\frac12}(1+s)^{\frac{127}{256}}}+\\
&+\frac{\varepsilon^3}{(r^i)^{\frac12}(1+s)(1+|r^i-c_i s|)^{\frac{15}{16}}}\Bigg) \qquad \text{in} \qquad [t_0,t].
\end{aligned} \tag{4.80}$$

Thus, by setting

$$V_h(s)=\sum_{|a|\le h}(r^i(s;\lambda))^{\frac12}\partial_0\Gamma^a u^i(r^i(s;\lambda)\omega,s) \tag{4.81}$$

and by (4.56), (4.79) and (4.80), we have

$$\begin{aligned}
&V_h'(s)\\
=\ &O^*\Bigg(\frac{\varepsilon^2}{1+s}|V_h(s)|+\frac{\varepsilon}{(1+s)^{1+\frac{63}{256}}}+\frac{\varepsilon^3}{(1+s)(1+|r^i(t_0;\lambda)-c_i t_0|)^{\frac{15}{16}}}\Bigg) \qquad \text{in} \quad [t_0,t].
\end{aligned}$$

The gronwall inequality implies

$$\begin{aligned}
&|V_h(t)| \\
= \;& O^*\Bigg(\Bigg\{|V_h(t_0)| + \int_{t_0}^{t}\Bigg(\frac{\varepsilon}{(1+s)^{1+\frac{63}{256}}} + \frac{\varepsilon^3}{(1+s)(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\Bigg)ds\Bigg\} \times \\
&\qquad \times \exp\Bigg(\int_{t_0}^{t}\frac{K_4\varepsilon^2}{1+s}ds\Bigg)\Bigg) \\
= \;& O^*\Bigg(\Bigg\{|V_h(t_0)| + \frac{\varepsilon}{(1+t_0)^{\frac{63}{256}}} + \frac{B\varepsilon}{(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\Bigg\}e^{K_4B}\Bigg)
\end{aligned}$$

Hence, by (4.56), we have

$$\begin{aligned}
&(1+|r-c_it|)^{\frac{15}{16}}|V_h(t)| \\
= \;& O^*\Bigg(C_7(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}} \times \\
&\qquad \times\Bigg\{|V_h(t_0)| + \frac{\varepsilon}{(1+t_0)^{\frac{63}{256}}} + \frac{B\varepsilon}{(1+|r^i(t_0;\lambda)-c_it_0|)^{\frac{15}{16}}}\Bigg\}e^{K_4B}\Bigg) \qquad (4.82) \\
= \;& O^*\Bigg(\varepsilon + \frac{\varepsilon}{(1+t_0)^{\frac{3}{256}}}\Bigg) \\
= \;& O^*(\varepsilon) \qquad \text{in} \qquad \tilde{\Lambda}_i(T_B).
\end{aligned}$$

It follows from (4.41) and (4.83) that (4.77) holds.

References

[1] R. Agemi and K. Yokoyama, The null condition and global existence of solutions to systems of wave equations with different speeds, Advances in Nonlinear Partial Differential Equations and Stcastics, Ser. Adv. Math. Appl. Sci., **48**, World Scientific, River Edge, NJ, (1998), pp. 43-86.

[2] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions I, Invent. Math., **145** (2001), pp. 597-618.

[3] L. Hoı̈rmander, The lifespan of classical solutions of nonlinear hyperbolic equations, Lecture Note in Math., **1256** (1987), pp. 214-280.

[4] A. Hoshiga, The asymptotic behaviour of the radially symmetric solutions to quasilinear wave equations in two space dimensions, Hokkaido Math. Journal, **24(3)** (1995), pp. 575-615.

[5] A. Hoshiga, The lifespan of solutions to quasilinear hyperbolic systems in two-space dimensions, Nonlinear Analysis., **42** (2000), pp. 543-560.

[6] A. Hoshiga, Existence and blowing up of solutions to systems of quasilinear wave equations in two space dimensions, Advances in Math. Sci. and Appli., **15** (2005), pp. 69-110.

[7] A. Hoshiga, The existence of global solutions to systems of quasilinear wave equations with quadratic nonlinearities in 2-dimensional space, Funkcialaj Ekvaciaj, **49** (2006), pp. 357-384.

[8] A. Hoshiga and H. Kubo, Global small amplitude solutions of nonlinear hyperbolic systems with a critical exponent under the null condition, SIAM J. Math., **31(3)** (2000), pp. 486-513.

[9] A. Hoshiga and H. Kubo, Global solvability for systems of nonlinear wave equations with multiple speeds in two space dimensions, Diff. and Int. Eqs., **17** (2004), pp. 593-622.

[10] F. John, Blow-up for quasi-linear wave equations in three space dimensions, Comm. Pure Appli. Math., **34** (1981), pp. 29-51.

[11] S. Klainerman, Remarks on the global Sobolev inequalities in the Minkowski space $\mathbb{R}^{n+1}$, Comm. Pure Appli. Math., **40** (1987), pp. 111-117.

[12] K. Kubota and K. Yokoyama, Global existence of classical solutions to systems of nonlinear wave equations with different speeds of propagation, Japanese J. Math., **27** (2001), pp. 113-202.

[13] A. Majda, *Compressible fluid flow and systems of conservation laws*, Springer Appli. Math. Sci., **53** (1984).