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A DIRECT APPROACH TO QUASILINEAR PARABOLIC EQUATIONS ON UNBOUNDED DOMAINS BY BRÉZIS'S THEORY FOR SUBDIFFERENTIAL OPERATORS

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Abstract. This paper deals with nonlinear diffusion equations and their approximate equations under homogeneous Neumann boundary conditions in unbounded domains with smooth bounded boundary. Colli and Fukao [8] studied similar equations in bounded domains by applying an abstract theory for doubly nonlinear evolution inclusions; however, the proof is based on compactness methods and hence the case of unbounded domains is excluded from the framework. The present paper asserts that one can solve the original problem and the approximate problem individually and directly in unbounded domains by applying Brézis theory.

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1 Introduction

1.1 Two problems

We consider applications of Brézis's theory for subdifferential operators proposed in [5] to quasilinear parabolic equations on *unbounded* domains. In [5, Theorem 3.6] it is explained that there exists a unique solution of the following Cauchy problem for abstract evolution equations:

$$\begin{cases} u'(t) + \partial\psi(u(t)) \ni \tilde{f}(t) \quad \text{in } X \quad \text{for a.a. } t \in (0,T), \\ u(0) = u_0 \quad \text{in } X, \end{cases}$$

where X is a Hilbert space, $\partial\psi$ is a subdifferential operator of a proper lower semicontinuous convex function ψ, $u : [0,T] \to X$ is an unknown function and $\tilde{f} \in L^2(0,T;X)$ is a given function. The theory is often applied to problems on *bounded* domains (see some examples given in [5]). The theme of this paper is to apply the theory in [5] *directly* to two quasilinear parabolic partial differential equations on *unbounded* domains.

The first purpose is that we apply the above Brézis's theory to show existence and uniqueness of solutions to the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta + 1)\beta(u) = g & \text{in } \Omega \times (0,T), \\ \partial_\nu \beta(u) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{P}$$

where Ω is an *unbounded* domain in $\mathbb{R}^N$ with smooth bounded boundary $\partial\Omega$, $N \in \mathbb{N}$, $T > 0$, g, u_0 is given functions, and ∂_ν denotes differentiation with respect to the outward normal of $\partial\Omega$. If $N = 2, 3$, Ω is bounded and $-\Delta + 1$ is replaced with $-\Delta$, then (P) represents the porous media equation (see, e.g., [1, 18, 22, 23]), the Stefan problem (see, e.g., [4, 10, 13, 14, 16]), the fast diffusion equation (see, e.g., [11, 20, 22]), etc. In this case, existence and uniqueness of solutions to these problems can be proved by a direct application of [5]. However, since the proof of the existence depends on boundedness of Ω, there seems to be no work on the problem on unbounded domains via [5]. In this paper we mainly study the case such as $\beta(u) = |u|^{q-1}u + u$ ($q > 1$).

The second purpose is to show that the theory in [5] is directly applicable to the following problem for the Cahn–Hilliard type system:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + (-\Delta + 1)\mu_\varepsilon = 0 & \text{in } \Omega \times (0,T), \\ \mu_\varepsilon = \varepsilon(-\Delta + 1)u_\varepsilon + \beta(u_\varepsilon) + \pi_\varepsilon(u_\varepsilon) - f & \text{in } \Omega \times (0,T), \\ \partial_\nu \mu_\varepsilon = \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon} & \text{in } \Omega, \end{cases} \tag{P$_\varepsilon$}$$

where π_ε is an anti-monotone function with $\varepsilon > 0$, f is a function determined by g and $u_{0\varepsilon}$ is a given function. If $N = 2, 3$, Ω is bounded and $-\Delta+1$ is reduced to $-\Delta$, then (P)$_\varepsilon$

represents the Cahn–Hilliard system (see e.g., [6, 7, 12]) and is regarded as an approximate problem to (P) (see [8, 14]). In particular, in the proof of existence of solutions to the problem in [8], *one more approximation* $(\mathrm{P})_{\varepsilon,\lambda}$ of $(\mathrm{P})_\varepsilon$ was essentially required, where existence of solutions to $(\mathrm{P})_{\varepsilon,\lambda}$ was proved by applying the abstract theory by Colli and Visintin [9] for doubly nonlinear evolution inclusions of the form

$$Au'(t) + \partial\psi(u(t)) \ni k(t)$$

with some bounded monotone operator A and some proper lower semicontinuous convex function ψ. Since the theory is based on compactness methods, boundedness of Ω is necessary and hence the case of unbounded domains is excluded from their frameworks.

The relation between (P) and $(\mathrm{P})_\varepsilon$ was recently studied by Colli and Fukao [8] in the case stated above. More precisely, in [8], existence of weak solutions to (P) and $(\mathrm{P})_\varepsilon$ with error estimates was established under the condition that $N = 2, 3$, Ω is a bounded domain with smooth boundary and $-\Delta + 1$ is replaced with $-\Delta$ in (P) and $(\mathrm{P})_\varepsilon$. In particular, they considered the case of degenerate diffusion and their approach to degenerate diffusion equations from the Cahn–Hilliard system made a new development. They established the error estimate that the solution of $(\mathrm{P})_\varepsilon$ converges to solution of (P) in the order $\varepsilon^{1/2}$ as $\varepsilon \searrow 0$. Their proof was also based on one more approximation $(\mathrm{P})_{\varepsilon,\lambda}$, while in this paper we will directly establish an error estimate without using $(\mathrm{P})_{\varepsilon,\lambda}$.

1.2 Main result for (P)

Before stating the main result for (P), we give some conditions, notations and definitions. We will assume that β, g, f, and u_0 satisfy the following conditions:

(C1) $\beta : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function and $\beta(r) = \hat{\beta}\,'(r) = \partial\hat{\beta}(r)$, where $\hat{\beta}\,'$ and $\partial\hat{\beta}$ are the differential and subdifferential of a proper differentiable (lower semicontinuous) convex function $\hat{\beta} : \mathbb{R} \to [0, +\infty]$ satisfying $\hat{\beta}(0) = 0$. This entails $\beta(0) = 0$. There exists a constant $c_1 > 0$ such that

$$\hat{\beta}(r) \geq c_1|r|^2 \quad \text{for all } r \in \mathbb{R}.$$

For all $z \in L^2(\Omega)$, if $\hat{\beta}(z) \in L^1(\Omega)$, then $\beta(z) \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, for all $z \in L^2(\Omega)$ and for all $\psi \in C_{\mathrm{c}}^\infty(\Omega)$, if $\hat{\beta}(z) \in L^1(\Omega)$, then $\hat{\beta}(z + \psi) \in L^1(\Omega)$.

(C2) $g \in L^2\big(0, T; L^2(\Omega)\big)$. Then we fix a solution $f \in L^2\big(0, T; H^2(\Omega)\big)$ of

$$\begin{cases} (-\Delta + 1)f(t) = g(t) & \text{a.e. in } \Omega, \\ \partial_\nu f(t) = 0 & \text{in the sense of traces on } \partial\Omega \end{cases} \tag{1.1}$$

for a.a. $t \in (0, T)$, that is,

$$\int_\Omega \nabla f(t) \cdot \nabla z + \int_\Omega f(t) z = \int_\Omega g(t) z \quad \text{for all } z \in H^1(\Omega). \tag{1.2}$$

(C3) $u_0 \in L^2(\Omega)$ and $\hat{\beta}(u_0) \in L^1(\Omega)$.

We put the Hilbert spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega) \tag{1.3}$$

with inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$, respectively. Moreover, we use

$$W := \{z \in H^2(\Omega) \mid \partial_\nu z = 0 \quad \text{a.e. on } \partial\Omega\}. \tag{1.4}$$

The notation V^* denotes the dual space of V with duality pairing $\langle \cdot,\cdot\rangle_{V^*,V}$. Moreover, we define a bijective mapping $F : V \to V^*$ and the inner product in V^* as

$$\langle Fv_1, v_2\rangle_{V^*,V} := (v_1, v_2)_V \quad \text{for all } v_1, v_2 \in V, \tag{1.5}$$

$$(v_1^*, v_2^*)_{V^*} := \langle v_1^*, F^{-1}v_2^*\rangle_{V^*,V} \quad \text{for all } v_1^*, v_2^* \in V^*; \tag{1.6}$$

note that $F : V \to V^*$ is well-defined by the Riesz representation theorem. We remark that (C2) implies

$$Ff(t) = g(t) \quad \text{for a.a. } t \in (0,T). \tag{1.7}$$

We define weak solutions of (P) as follows.

Definition 1.1. A pair (u, μ) with

$$u \in H^1(0,T;V^*) \cap L^\infty(0,T;H),$$
$$\mu \in L^2(0,T;V)$$

is called a *weak solution* of (P) if (u, μ) satisfies

$$\langle u'(t), z\rangle_{V^*,V} + (\mu(t), z)_V = 0 \quad \text{for all } z \in V \text{ and a.a. } t \in (0,T), \tag{1.8}$$

$$\mu(t) = \beta(u(t)) - f(t) \quad \text{in } V \quad \text{for a.a. } t \in (0,T), \tag{1.9}$$

$$u(0) = u_0 \quad \text{a.e. on } \Omega. \tag{1.10}$$

Now the main result for (P) reads as follows.

Theorem 1.1. *Assume* (C1)-(C3). *Then there exists a unique weak solution* (u, μ) *of* (P), *satisfying*

$$u \in H^1(0,T;V^*) \cap L^\infty(0,T;H), \quad \mu \in L^2(0,T;V).$$

Moreover, for all $t \in [0,T]$,

$$\int_0^t |u'(s)|_{V^*}^2 \, ds + 2c_1|u(t)|_H^2 \le M_1, \tag{1.11}$$

$$\int_0^t |\mu(s)|_V^2 \, ds \le M_1, \tag{1.12}$$

$$\int_0^t |\beta(u(s))|_V^2 \, ds \le 2\big(M_1 + |f|_{L^2(0,T;V)}^2\big), \tag{1.13}$$

where $M_1 := 2\int_\Omega \hat\beta(u_0) + |f|_{L^2(0,T;V)}^2$.

1.3 Main result for $(\mathrm{P})_\varepsilon$

We will assume that π_ε and $u_{0\varepsilon}$ satisfy the following conditions:

(C4) $\pi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $\pi_\varepsilon(0) = 0$ for all $\varepsilon \in (0,1]$. Moreover, there exists a constant $c_2(\varepsilon) > 0$ depending on ε such that there exists $\overline{\varepsilon} \in (0,1]$ satisfying $c_2(\varepsilon) < 2c_1$ for all $\varepsilon \in (0, \overline{\varepsilon}]$ and

$$\left|\pi_\varepsilon'\right|_{L^\infty(\mathbb{R})} \le c_2(\varepsilon) \quad \text{for all } \varepsilon \in (0,1]. \tag{1.14}$$

Moreover, $r \mapsto \frac{\varepsilon}{2}r^2 + \hat{\pi}_\varepsilon(r)$ is convex, where $\hat{\pi}_\varepsilon(r) := \int_0^r \pi_\varepsilon(s)\,ds$.

(C5) Let $u_{0\varepsilon} \in H^1(\Omega)$ fulfill $\hat{\beta}(u_{0\varepsilon}) \in L^1(\Omega)$ and

$$|u_{0\varepsilon}|^2_{L^2(\Omega)} \le c_3(\varepsilon), \quad \int_\Omega \hat{\beta}(u_{0\varepsilon}) \le c_3(\varepsilon), \quad \varepsilon|u_{0\varepsilon}|^2_{H^1(\Omega)} \le c_3(\varepsilon), \tag{1.15}$$

where $c_3(\varepsilon) > 0$ is a constant depending on ε.

Let H, V and W be as in Section 1.2. Then we define weak solutions of $(\mathrm{P})_\varepsilon$ as follows.

Definition 1.2. A pair $(u_\varepsilon, \mu_\varepsilon)$ with

$$u_\varepsilon \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W),$$
$$\mu_\varepsilon \in L^2(0,T;V)$$

is called a *weak solution* of $(\mathrm{P})_\varepsilon$ if $(u_\varepsilon, \mu_\varepsilon)$ satisfies

$$\left\langle u_\varepsilon'(t), z\right\rangle_{V^*,V} + \left(\mu_\varepsilon(t), z\right)_V = 0 \quad \text{for all } z \in V \text{ and a.a. } t \in (0,T), \tag{1.16}$$

$$\mu_\varepsilon(t) = \varepsilon(-\Delta + I)u_\varepsilon(t) + \beta(u_\varepsilon(t)) + \pi_\varepsilon(u_\varepsilon(t)) - f(t) \quad \text{in } V \quad \text{for a.a. } t \in (0,T), \tag{1.17}$$

$$u_\varepsilon(0) = u_{0\varepsilon} \quad \text{a.e. on } \Omega. \tag{1.18}$$

Now the main result for $(\mathrm{P})_\varepsilon$ reads as follows.

Theorem 1.2. *Assume* (C1)-(C5)*. Then there exists $\overline{\varepsilon} \in (0,1]$ such that for every $\varepsilon \in (0,\overline{\varepsilon}]$ there exists a unique weak solution $(u_\varepsilon, \mu_\varepsilon)$ of $(\mathrm{P})_\varepsilon$, satisfying*

$$u_\varepsilon \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \quad \mu_\varepsilon \in L^2(0,T;V).$$

Moreover, for all $t \in [0,T]$ and $\varepsilon \in (0,\overline{\varepsilon}]$,

$$\int_0^t \left|u_\varepsilon'(s)\right|^2_{V^*} ds + \varepsilon|u_\varepsilon(t)|^2_V + \bigl(2c_1 - c_2(\varepsilon)\bigr)|u_\varepsilon(t)|^2_H \le M_2(\varepsilon), \tag{1.19}$$

$$\int_0^t |\mu_\varepsilon(s)|^2_V\, ds \le M_2(\varepsilon), \tag{1.20}$$

$$\int_0^t |\beta(u_\varepsilon(s))|^2_H\, ds \le 3\left(M_2(\varepsilon) + \frac{c_2(\varepsilon)^2 M_2(\varepsilon)T}{2c_1 - c_2(\varepsilon)} + |f|^2_{L^2(0,T;V)}\right), \tag{1.21}$$

$$\int_0^t |\varepsilon u_\varepsilon(s)|^2_W\, ds \le 16L^2\left(M_2(\varepsilon) + \frac{c_2(\varepsilon)^2 M_2(\varepsilon)T}{2c_1 - c_2(\varepsilon)} + |f|^2_{L^2(0,T;V)}\right), \tag{1.22}$$

where $M_2(\varepsilon) := 3c_3(\varepsilon) + c_2(\varepsilon)c_3(\varepsilon) + |f|^2_{L^2(0,T;V)}$ and L is a positive constant appearing in the elliptic regularity estimate $|w|_W \le L|(-\Delta + I)w|_H$ for all $w \in W$.

1.4 Outline of this paper

The strategy in the proofs of the main theorems is as follows. As to Theorem 1.1, by setting a proper lower semicontinuous convex function ϕ well, we can rewrite (P) as an abstract nonlinear evolution equation with simple form by the subdifferential of ϕ:

$$u'(t) + \partial\phi(u(t)) = g(t) \quad \text{in } V^*,$$

so that we can solve (P) even on unbounded domains directly with monotonicity methods (Lemma 2.3). Moreover, from this, Colli and Fukao [8] proved apriori estimates for solutions of (P) by the limit of apriori estimates for solutions of $(\mathrm{P})_\varepsilon$ as $\varepsilon \searrow 0$, while we can obtain apriori estimates for solutions of (P) directly. The proof of Theorem 1.2 is parallel to that of Theorem 1.1, and hence we need not consider one more approximation problem $(\mathrm{P})_{\varepsilon,\lambda}$ which cannot be used when Ω is unbounded. In Theorem 5.1 we can establish an error estimate between the solution of (P) and the solution of $(\mathrm{P})_\varepsilon$ without one more approximation of $(\mathrm{P})_\varepsilon$ even on unbounded domains.

This paper is organized as follows. In Section 2 we give the definition and basic results for subdifferentials of proper lower semicontinuous convex functions and useful results for proving the main theorems. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2. In Section 5 we prove an error estimate between the solution of (P) and the solution of $(\mathrm{P})_\varepsilon$. In Section 6 we give examples similar to the porous media and the fast diffusion equations.

2 Preliminaries

We first give the definition and basic results for subdifferentials of convex functions.

Definition 2.1. Let X be a Hilbert space. Given a proper lower semicontinuous (l.s.c. for short) convex function $\phi : X \to \overline{\mathbb{R}}$, the mapping $\partial\phi : X \to X$ defined by

$$\partial\phi(z) := \big\{\tilde{z} \in X \mid (\tilde{z}, w - z)_X \le \phi(w) - \phi(z) \quad \text{for all } w \in X\big\}$$

is called the *subdifferential* operator of ϕ, with domain $D(\partial\phi) := \{z \in X \mid \partial\phi(z) \neq \emptyset\}$.

The following lemma is well-known (see e.g., Barbu [3, Theorem 2.8]).

Lemma 2.1. *Let X be a Hilbert space and let $\phi : X \to \overline{\mathbb{R}}$ be a proper l.s.c. convex function. Then $\partial\phi$ is maximal monotone in X.*

The next asserts the chain rule. For the proof see e.g., Showalter [21, Lemma IV.4.3].

Lemma 2.2. *Let $\psi : X \to \overline{\mathbb{R}}$ be a proper, convex and l.s.c. function on a Hilbert space X. If $u \in H^1(0,T;X)$ and there exists $v \in L^2(0,T;X)$ such that $v \in \partial\psi(u)$ a.e. on $[0,T]$, then the function $\psi \circ u$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{dt}\psi(u(t)) = \big(w(t), u'(t)\big)_X \quad \textit{for a.a. } t \in [0,T]$$

for any function w satisfying $w(t) \in \partial\psi(u(t))$ for a.a. $t \in [0,T]$.

The following lemma plays a key role in the direct proof of existence of solutions to (P) and $(\mathrm{P})_\varepsilon$ individually.

Lemma 2.3 (Brézis [5, Theoreme 3.6])**.** *Let X be a Hilbert space and let $\psi : X \to \overline{\mathbb{R}}$ be a proper l.s.c. convex function. If $u_0 \in D(\psi)$ and $\tilde{f} \in L^2(0,T;X)$, then there exists a unique function u such that $u \in H^1(0,T;X)$, $u(t) \in D(\partial\psi)$ for a.a. $t \in (0,T)$ and u solves the following initial value problem:*

$$\begin{cases} u'(t) + \partial\psi(u(t)) \ni \tilde{f}(t) \quad \text{in } X \quad \text{for a.a. } t \in (0,T), \\ u(0) = u_0 \quad \text{in } X. \end{cases}$$

3 Existence of solutions to (P)

3.1 Convex function for Proof of Theorem 1.1

Let H, V and W be as in (1.3) and (1.4). We define a function $\phi : V^* \to \overline{\mathbb{R}}$ as

$$\phi(z) = \begin{cases} \displaystyle\int_\Omega \hat{\beta}(z(x))\,dx & \text{if } z \in D(\phi) := \{z \in H \mid \hat{\beta}(z) \in L^1(\Omega)\}, \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 3.1. *Let ϕ be as above. Then ϕ is a proper l.s.c. convex function on V^*.*

Proof. It follows that ϕ is proper and convex since $0 \in D(\phi)$ and $\hat{\beta}$ is convex. To prove the lower semicontinuity of ϕ on V^* let $\{z_n\}$ be a sequence in $D(\phi)$ such that $z_n \to z$ in V^* as $n \to +\infty$. We put $\alpha := \liminf_{n\to+\infty} \phi(z_n)$. If $\alpha = +\infty$, then $\phi(z) \le +\infty = \alpha = \liminf_{n\to+\infty} \phi(z_n)$. We assume that $\alpha < +\infty$. Then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\phi(z_{n_k}) \nearrow \alpha$ as $k \to +\infty$ and hence, $\alpha \ge \phi(z_{n_k}) = \int_\Omega \hat{\beta}(z_{n_k}) \ge c_1 |z_{n_k}|_H^2$ by (C1). Thus $z_{n_k} \rightharpoonup z$ weakly in H as $k \to +\infty$. Now let $\phi_H := \phi|_H$. Since $\hat{\beta}$ is proper l.s.c. convex, the function ϕ_H is also proper l.s.c. convex on H and hence ϕ_H is weakly l.s.c. on H. So it follows that

$$\phi_H(z) \le \liminf_{k\to+\infty} \phi_H(z_{n_k}) = \liminf_{k\to+\infty} \int_\Omega \hat{\beta}(z_{n_k}) \le \alpha < +\infty.$$

Thus we see that $z \in D(\phi_H) = D(\phi)$ and $\phi(z) = \phi_H(z) \le \alpha = \liminf_{n\to+\infty} \phi(z_n)$. □

The following lemma plays an important role in our proof (cf. [15, Lemma 4.1]).

Lemma 3.2. *Let $z \in D(\partial\phi) := \{z \in D(\phi) \mid \emptyset \ne \partial\phi(z) \subset V^*\} \subset D(\phi)$. Then $z^* \in \partial\phi(z)$ in V^* if and only if*

$$F^{-1}z^* = \beta(z) \tag{3.1}$$

Consequently, $\partial\phi$ is single-valued and for all $z \in D(\partial\phi)$ it holds that $\beta(z) \in V$ and

$$\partial\phi(z) = F\beta(z). \tag{3.2}$$

Proof. Let $z \in D(\partial\phi)$ and $z^* \in \partial\phi(z)$. Then it follows from the inclusion $D(\partial\phi) \subset D(\phi)$ that $z \in D(\phi)$. Hence we have by the definition of $\partial\phi$,

$$(z^*, w - z)_{V^*} \le \int_\Omega \bigl(\hat\beta(w) - \hat\beta(z)\bigr) \quad \text{for all } w \in D(\phi).$$

Here, choose $w = z \pm \lambda\psi$ $(\lambda > 0)$ in the above inequality for each $\psi \in C_c^\infty(\Omega)$. Noting by (C1) that $z \pm \lambda\psi \in D(\phi)$, we obtain

$$\int_\Omega \frac{\hat\beta(z) - \hat\beta(z - \lambda\psi)}{\lambda} \le (z^*, \psi)_{V^*} \le \int_\Omega \frac{\hat\beta(z + \lambda\psi) - \hat\beta(z)}{\lambda}. \tag{3.3}$$

Here, since $\beta = \partial\hat\beta$, it follows from the definition of subdifferentials and the convexity and nonnegativity of $\hat\beta$ that

$$\beta(z)\psi \le \frac{\hat\beta(z + \lambda\psi) - \hat\beta(z)}{\lambda} = \frac{\hat\beta\bigl(\lambda(z + \psi) + (1 - \lambda)z\bigr) - \hat\beta(z)}{\lambda} \le \hat\beta(z + \psi),$$

$$-\hat\beta(z - \psi) \le \frac{\hat\beta(z) - \hat\beta\bigl(\lambda(z - \psi) + (1 - \lambda)z\bigr)}{\lambda} = \frac{\hat\beta(z) - \hat\beta(z - \lambda\psi)}{\lambda} \le \beta(z)\psi,$$

and hence we observe

$$\left|\frac{\hat\beta(z + \lambda\psi) - \hat\beta(z)}{\lambda}\right| \le |\beta(z)\psi| + |\hat\beta(z + \psi)|,$$

$$\left|\frac{\hat\beta(z) - \hat\beta(z - \lambda\psi)}{\lambda}\right| \le |\beta(z)\psi| + |\hat\beta(z - \psi)|.$$

Noting that $|\beta(z)\psi| + |\hat\beta(z \pm \psi)| \in L^1(\Omega)$ and $\hat\beta$ is differentiable because of (C1) and passing to the limit $\lambda \searrow 0$ in (3.3), we infer from Lebesgue's convergence theorem that

$$(z^*, \psi)_{V^*} = \int_\Omega \hat\beta\,'(z)\psi = \int_\Omega \beta(z)\psi \quad \text{for all } \psi \in C_c^\infty(\Omega).$$

Writing as $(z^*, \psi)_{V^*} = \bigl(F^{-1}z^*, \psi\bigr)_H$ by (1.6), we see that

$$\int_\Omega \bigl(F^{-1}z^*\bigr)\psi = \int_\Omega \beta(z)\psi \quad \text{for all } \psi \in C_c^\infty(\Omega).$$

Thus, since $\beta(z) \in L^1_{\text{loc}}(\Omega)$ by (C1), it follows from du Bois Reymond's lemma that

$$F^{-1}z^* = \beta(z) \quad \text{a.e. on } \Omega.$$

That is, (3.1) holds. Conversely, if (3.1) holds, then for all $w \in D(\phi)$,

$$(z^*, w - z)_{V^*} = \bigl(F^{-1}z^*, w - z\bigr)_H = \int_\Omega \beta(z)(w - z) \le \int_\Omega \bigl(\hat\beta(w) - \hat\beta(z)\bigr),$$

where we have used $\beta = \partial\hat\beta$, and hence $z^* \in \partial\phi(z)$. Therefore we conclude that $\partial\phi$ is single-valued and for all $z \in D(\partial\phi)$, $\beta(z) \in V$ and (3.2) holds. □

Now we prove the first main theorem.

3.2 Proof of Theorem 1.1

Proof of Theorem 1.1. To prove existence of weak solutions to (P) we turn our eyes to the following initial value problem (3.4):

$$\begin{cases} u'(t) + \partial\phi(u(t)) = Ff(t) \quad \text{in } V^* \quad \text{for a.e. } t \in [0,T], \\ u(0) = u_0 \quad \text{in } V^*. \end{cases} \tag{3.4}$$

Thanks to Lemma 2.3, there exists a unique solution $u \in H^1(0,T;V^*)$ of (3.4) such that $u(t) \in D(\partial\phi)$ for a.a. $t \in (0,T)$. Putting $\mu(t) := -F^{-1}(u'(t))$, we deduce from (1.5), (1.6) and (3.2) that $\mu \in L^2(0,T;V)$ and (u,μ) satisfies (1.8)-(1.10).

Next we show (1.11). It follows from the equation in (3.4) that

$$\begin{aligned} |u'(s)|_{V^*}^2 &= \big(u'(s), u'(s)\big)_{V^*} \\ &= \big(u'(s), -\partial\phi(u(s)) + Ff(s)\big)_{V^*} \\ &= -\big(u'(s), \partial\phi(u(s))\big)_{V^*} + \big(u'(s), Ff(s)\big)_{V^*}. \end{aligned}$$

Here Lemma 2.2 gives

$$\big(u'(s), \partial\phi(u(s))\big)_{V^*} = \frac{d}{ds}\phi(u(s)),$$

and (1.6) and Young's inequality yield

$$(u'(s), Ff(s))_{V^*} = \langle u'(s), f(s)\rangle_{V^*,V} \le \frac{1}{2}|u'(s)|_{V^*}^2 + \frac{1}{2}|f(s)|_V^2.$$

Therefore we obtain

$$\frac{1}{2}|u'(s)|_{V^*}^2 \le -\frac{d}{ds}\phi(u(s)) + \frac{1}{2}|f(s)|_V^2.$$

Integrating this inequality yields

$$\frac{1}{2}\int_0^t |u'(s)|_{V^*}^2\, ds \le -\phi(u(t)) + \phi(u_0) + \frac{1}{2}|f|_{L^2(0,T;V)}^2,$$

i.e.,

$$\frac{1}{2}\int_0^t |u'(s)|_{V^*}^2\, ds + \int_\Omega \hat{\beta}(u(t)) \le \int_\Omega \hat{\beta}(u_0) + \frac{1}{2}|f|_{L^2(0,T;V)}^2.$$

Since (C1) implies

$$\int_\Omega \hat{\beta}(u(t)) \ge c_1|u(t)|_H^2,$$

we see that

$$\int_0^t |u'(s)|_{V^*}^2\, ds + 2c_1|u(t)|_H^2 \le 2\int_\Omega \hat{\beta}(u_0) + |f|_{L^2(0,T;V)}^2 =: M_1.$$

This implies (1.11). Moreover, (1.11) shows that $u \in L^\infty(0,T;H)$.

Next we show (1.12). Since $\mu(s) = -F^{-1}\big(u'(s)\big)$, we have from (1.5) and (1.6) that

$$\int_0^t |\mu(s)|_V^2 \, ds = \int_0^t \big|F^{-1}\big(u'(s)\big)\big|_V^2 \, ds = \int_0^t \big|u'(s)\big|_{V^*}^2 \, ds.$$

Thus we obtain (1.12) from (1.11).

Next we verify (1.13). From (1.9) and Young's inequality we infer

$$\begin{aligned} |\beta(u(s))|_V^2 &= \big(\beta(u(s)), \beta(u(s))\big)_V \\ &= \big(\mu(s) + f(s), \beta(u(s))\big)_V \\ &\leq |\mu(s)|_V^2 + |f(s)|_V^2 + \frac{1}{2}|\beta(u(s))|_V^2. \end{aligned}$$

Therefore,

$$\int_0^t |\beta(u(s))|_V^2 \, ds \leq 2 \int_0^t |\mu(s)|_V^2 \, ds + 2|f|_{L^2(0,T;V)}^2.$$

Consequently, (1.13) holds from (1.12). □

4 Existence of solutions to $(\mathrm{P})_\varepsilon$

4.1 Preliminaries for $(\mathrm{P})_\varepsilon$

We first give a useful inequality.

Lemma 4.1. *Let β be a single-valued maximal monotone function as in Section 1. Then*

$$\big(-\Delta u, \beta_\lambda(u)\big)_H \geq 0 \quad \textit{for all } u \in W,$$
$$\big(-\Delta u, \beta(u)\big)_H \geq 0 \quad \textit{for all } u \in W \textit{ with } \beta(u) \in H,$$

where $W = \big\{z \in H^2(\Omega) \mid \partial_\nu z = 0 \quad \text{a.e. on } \partial\Omega\big\}$ and $\{\beta_\lambda\}_{\lambda>0}$ is the Yosida approximation of β: $\beta_\lambda := \lambda^{-1}\big(I - (I + \lambda\beta)^{-1}\big)$.

Proof. It follows from Okazawa [19, Proof of Theorem 3 with $a = b = 0$] that

$$\big(-\Delta u, \beta_\lambda(u)\big)_H \geq 0 \quad \text{for all } u \in W \text{ and } \lambda > 0.$$

Noting that $\beta_\lambda(u) \to \beta(u)$ in H as $\lambda \searrow 0$ if $\beta(u) \in H$ (see e.g., [5, Proposition 2.6] or [21, Theorem IV.1.1]), we can obtain the second inequality. □

The above and the next lemmas will be used in order to regard $(\mathrm{P})_\varepsilon$ as a problem of the form stated in Lemma 2.3.

Lemma 4.2. *Let A and B be maximal monotone operators in H such that*

(i) *$D(A) \cap D(B) \neq \emptyset$,*

(ii) *$(Av,\ B_\lambda v)_H \geq 0 \quad$ for all $v \in D(A)$ and $\lambda > 0$,*

where $\{B_\lambda\}_{\lambda>0}$ is the Yosida approximation of B. Then $A + B$ is maximal monotone.

Proof. We can show this lemma by applying Barbu [2, Theoreme II.3.6]. □

4.2 Convex function for Proof of Theorem 1.2

Let $\varepsilon > 0$. Then we define a function $\phi_\varepsilon : V^* \to \overline{\mathbb{R}}$ as

$$\phi_\varepsilon(z) = \begin{cases} \dfrac{\varepsilon}{2}\displaystyle\int_\Omega \big(|z(x)|^2 + |\nabla z(x)|^2\big)\,dx + \int_\Omega \hat{\beta}(z(x))\,dx + \int_\Omega \hat{\pi}_\varepsilon(z(x))\,dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } z \in D(\phi_\varepsilon) := \{z \in V \mid \hat{\beta}(z) \in L^1(\Omega)\}, \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

Lemma 4.3. *Let ϕ_ε be as above. Then there exists $\overline{\varepsilon} \in (0,1]$ such that for all $\varepsilon \in (0,\overline{\varepsilon}]$, ϕ_ε is a proper l.s.c. convex function on V^*.*

Proof. Since 0 belongs to $D(\phi_\varepsilon)$ and $r \mapsto \hat{\beta}(r)$, $r \mapsto \frac{\varepsilon}{2}r^2 + \hat{\pi}_\varepsilon(r)$ are convex, it follows that ϕ_ε is proper and convex. To prove the lower semicontinuity of ϕ_ε in V^* let $\{z_n\}$ be a sequence in $D(\phi_\varepsilon)$ such that $z_n \to z$ in V^* as $n \to +\infty$. We put $\alpha := \liminf_{n\to+\infty} \phi_\varepsilon(z_n)$. If $\alpha = +\infty$, then $\phi_\varepsilon(z) \le +\infty = \alpha = \liminf_{n\to+\infty} \phi_\varepsilon(z_n)$. We assume that $\alpha < +\infty$. Then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\phi_\varepsilon(z_{n_k}) \nearrow \alpha$ as $k \to +\infty$ and hence,

$$\alpha \ge \phi_\varepsilon(z_{n_k}) = \int_\Omega \hat{\beta}(z_{n_k}) + \int_\Omega \hat{\pi}_\varepsilon(z_{n_k}) + \frac{\varepsilon}{2}|z_{n_k}|_V^2 \ge c_1 \int_\Omega |z_{n_k}|^2 + \int_\Omega \hat{\pi}_\varepsilon(z_{n_k}) + \frac{\varepsilon}{2}|z_{n_k}|_V^2.$$

Here, we deduce from (C4) that there exists $\overline{\varepsilon} \in (0,1]$ such that $c_2(\varepsilon) < 2c_1$ for all $\varepsilon \in (0,\overline{\varepsilon}]$. The definition of $\hat{\pi}_\varepsilon$ shows that for all $\varepsilon \in (0,\overline{\varepsilon}]$,

$$|\hat{\pi}_\varepsilon(r)| = \left|\int_0^r (\pi_\varepsilon(s) - \pi_\varepsilon(0))\,ds\right| \le |\pi'_\varepsilon|_{L^\infty(\mathbb{R})}\left|\int_0^r |s|\,ds\right| \le \frac{1}{2}c_2(\varepsilon)|r|^2 \le c_1|r|^2.$$

Hence $\alpha \ge \frac{\varepsilon}{2}|z_{n_k}|_V^2$. Thus $z_{n_k} \rightharpoonup z$ weakly in V as $k \to +\infty$. Now let $\phi_{\varepsilon,V} := \phi_\varepsilon|_V$. Since $\hat{\beta}$ is proper l.s.c. convex and $r \mapsto \frac{\varepsilon}{2}r^2 + \hat{\pi}_\varepsilon(r)$ is convex, the function $\phi_{\varepsilon,V}$ is also proper l.s.c. convex on V and hence $\phi_{\varepsilon,V}$ is weakly l.s.c. on V. So it follows that

$$\phi_{\varepsilon,V}(z) \le \liminf_{k\to+\infty} \phi_{\varepsilon,V}(z_{n_k}) \le \alpha < +\infty.$$

Consequently, $z \in D(\phi_{\varepsilon,V}) = D(\phi_\varepsilon)$ and $\phi_\varepsilon(z) = \phi_{\varepsilon,V}(z) \le \alpha = \liminf_{n\to+\infty} \phi_\varepsilon(z_n)$. □

Lemma 4.4. *Define a function $\phi_\varepsilon^H : H \to \overline{\mathbb{R}}$ as*

$$\phi_\varepsilon^H(w) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |w|^2 + \frac{\varepsilon}{2}\int_\Omega \big(|w|^2 + |\nabla w|^2\big) + \int_\Omega \hat{\beta}(w) + \int_\Omega \hat{\pi}_\varepsilon(w) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } w \in D(\phi_\varepsilon^H) := \{w \in V \mid \hat{\beta}(w) \in L^1(\Omega)\}, \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \textit{otherwise.} \end{cases}$$

Then ϕ_ε^H is a proper l.s.c. convex function on H and

$$D(\partial\phi_\varepsilon^H) \subset W. \tag{4.1}$$

Proof. As in the proof of Lemma 4.3, we first observe that ϕ_ε^H is a proper l.s.c. convex function on H. Next we set $\phi_H^{(1)} : H \to \overline{\mathbb{R}}$ as

$$\phi_H^{(1)}(w) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |w|^2 + \dfrac{\varepsilon}{2}\int_\Omega \big(|w|^2 + |\nabla w|^2\big) + \int_\Omega \hat{\pi}_\varepsilon(w) & \text{if } w \in D(\phi_H^{(1)}) := V, \\ +\infty & \text{otherwise} \end{cases}$$

and $\phi_H^{(2)} : H \to \overline{\mathbb{R}}$ as

$$\phi_H^{(2)}(w) = \begin{cases} \displaystyle\int_\Omega \hat{\beta}(w) & \text{if } w \in D(\phi_H^{(2)}) := \{w \in H \mid \hat{\beta}(w) \in L^1(\Omega)\}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\phi_H^{(1)}$ and $\phi_H^{(2)}$ are proper l.s.c. convex functions on H and

$$w \in D(\partial\phi_H^{(1)}) \implies w \in W \text{ and } \partial\phi_H^{(1)}(w) = w + \varepsilon(-\Delta + I)w + \pi_\varepsilon(w), \tag{4.2}$$

$$w \in D(\partial\phi_H^{(2)}) \implies \partial\phi_H^{(2)}(w) = \beta(w). \tag{4.3}$$

Since (4.3) is well-known (see e.g., [5, Example 2.8.3], [21, Example II.8.B]), we verify only (4.2). Let $w \in D(\partial\phi_H^{(1)})$ and $w^* \in \partial\phi_H^{(1)}(w)$. Then it follows from the inclusion $D(\partial\phi_H^{(1)}) \subset D(\phi_H^{(1)})$ that $w \in D(\phi_H^{(1)})$. Hence we have from the definition of $\partial\phi_H^{(1)}$ that

$$\begin{aligned}(w^*, \tilde{w} - w)_H \le \frac{1}{2}\int_\Omega \big(|\tilde{w}|^2 - |w|^2\big) + \frac{\varepsilon}{2}\int_\Omega \big(|\tilde{w}|^2 - |w|^2\big) + \frac{\varepsilon}{2}\int_\Omega \big(|\nabla\tilde{w}|^2 - |\nabla w|^2\big) \\ + \int_\Omega \big(\hat{\pi}_\varepsilon(\tilde{w}) - \hat{\pi}_\varepsilon(w)\big).\end{aligned}$$

Here, choose $\tilde{w} = w \pm \lambda v$ ($\lambda > 0$) in the above inequality for each $v \in V$ and divide the both sides by λ and finally pass to the limit $\lambda \searrow 0$. Then we obtain

$$(w^*, v)_H = \int_\Omega wv + \varepsilon\left(\int_\Omega wv + \int_\Omega \nabla w \cdot \nabla v\right) + \int_\Omega \pi_\varepsilon(w)v \quad \text{for all } v \in V.$$

Hence we see that

$$\int_\Omega wv + \int_\Omega \nabla w \cdot \nabla v = \int_\Omega \frac{w^* - w - \pi_\varepsilon(w)}{\varepsilon} v \quad \text{for all } v \in V.$$

Thus we derive that $w \in W$ and

$$w^* = w + \varepsilon(-\Delta + I)w + \pi_\varepsilon(w).$$

That is, (4.2) holds. Now we show that (4.1). If

$$\big(\partial\phi_H^{(1)}(w),\ \beta_\lambda(w)\big)_H \ge 0 \quad \text{for all } w \in D(\partial\phi_H^{(1)}), \tag{4.4}$$

then $\partial\phi_H^{(1)} + \partial\phi_H^{(2)}$ is maximal monotone by Lemma 4.2 and hence we have

$$\partial\phi_\varepsilon^H = \partial(\phi_H^{(1)} + \phi_H^{(2)}) = \partial\phi_H^{(1)} + \partial\phi_H^{(2)},$$

with

$$D(\partial\phi_\varepsilon) = D(\partial\phi_H^{(1)}) \cap D(\partial\phi_H^{(2)}); \tag{4.5}$$

note that $\phi_H = \phi_H^{(1)} + \phi_H^{(2)}$ is also proper l.s.c. convex and $\partial(\phi_H^{(1)} + \phi_H^{(2)}) \subset \partial\phi_H^{(1)} + \partial\phi_H^{(2)}$. We can show (4.4) by using Lemma 4.1. Indeed, since the function $r \mapsto \frac{\varepsilon}{2}r^2 + \hat{\pi}_\varepsilon(r)$ is convex, it follows that $r \mapsto \varepsilon r + \pi_\varepsilon(r)$ is monotone, so that the monotonicity of β_λ yields

$$(\varepsilon w + \pi_\varepsilon(w),\ \beta_\lambda(w))_H \geq 0,$$

and hence we see from Lemma 4.1 that

$$\begin{aligned}\big(\partial\phi_H^{(1)}(w),\ \beta_\lambda(w)\big)_H &= (w + \varepsilon(-\Delta + I)w + \pi_\varepsilon(w),\ \beta_\lambda(w))_H \\ &= (w,\ \beta_\lambda(w))_H + \varepsilon(-\Delta w,\ \beta_\lambda(w))_H + (\varepsilon w + \pi_\varepsilon(w),\ \beta_\lambda(w))_H \\ &\geq 0.\end{aligned}$$

Therefore we obtain (4.5). On the other hand, we infer from (4.2) that

$$\begin{aligned}D(\partial\phi_H^{(1)}) &= \big\{w \in D(\phi_H^{(1)}) \ \big|\ \partial\phi_H^{(1)}(w) \in H\big\} \\ &= \big\{w \in V \ \big|\ w \in W,\ w + \varepsilon(-\Delta + I)w + \pi_\varepsilon(w) \in H\big\} \\ &= W \end{aligned} \tag{4.6}$$

and from (4.3) that

$$\begin{aligned}D(\partial\phi_H^{(2)}) &= \big\{w \in D(\phi_H^{(2)}) \ \big|\ \partial\phi_H^{(2)}(w) \in H\big\} \\ &= \{w \in H \ \big|\ \beta(w) \in H\}.\end{aligned} \tag{4.7}$$

Thus, connecting (4.6) and (4.7) to (4.5) gives (4.1). □

Lemma 4.5. *Let $z \in D(\partial\phi_\varepsilon) := \{z \in D(\phi_\varepsilon) \mid \emptyset \neq \partial\phi_\varepsilon(z) \subset V^*\} \subset D(\phi_\varepsilon)$. Then $z^* \in \partial\phi_\varepsilon(z)$ in V^* if and only if $z \in W$ and*

$$F^{-1}z^* = \varepsilon(-\Delta + I)z + \beta(z) + \pi_\varepsilon(z) \tag{4.8}$$

Consequently, $\partial\phi_\varepsilon$ is single-valued and for all $z \in D(\partial\phi_\varepsilon)$ it holds that

$$\begin{aligned}&z \in W, \quad \varepsilon(-\Delta + I)z + \beta(z) + \pi_\varepsilon(z) \in V \quad \text{and} \\ &\partial\phi_\varepsilon(z) = F(\varepsilon(-\Delta + I)z + \beta(z) + \pi_\varepsilon(z)).\end{aligned} \tag{4.9}$$

Proof. Let $z \in D(\partial\phi_\varepsilon)$ and $z^* \in \partial\phi_\varepsilon(z)$. Noting that $D(\partial\phi_\varepsilon) \subset D(\phi_\varepsilon)$, we see from the definition of $\partial\phi_\varepsilon$ that for all $w \in D(\phi_\varepsilon)$,

$$\begin{aligned}(z^*, w - z)_{V^*} \leq{}& \frac{\varepsilon}{2}\int_\Omega \big(|w|^2 - |z|^2\big) + \frac{\varepsilon}{2}\int_\Omega \big(|\nabla w|^2 - |\nabla z|^2\big) \\ &+ \int_\Omega \big(\hat{\beta}(w) - \hat{\beta}(z)\big) + \int_\Omega \big(\hat{\pi}_\varepsilon(w) - \hat{\pi}_\varepsilon(z)\big).\end{aligned}$$

Here, choose $w = z \pm \lambda\psi$ ($\lambda > 0$) in the above inequality for each $\psi \in \mathcal{D}(\Omega) := C_{\mathrm{c}}^\infty(\Omega)$ and divide the both sides by λ and finally pass to the limit $\lambda \searrow 0$. Then for all $\psi \in \mathcal{D}(\Omega)$, we obtain

$$(z^*, \psi)_{V^*} = \varepsilon \left(\int_\Omega z\psi + \int_\Omega \nabla z \cdot \nabla \psi \right) + \int_\Omega \beta(z)\psi + \int_\Omega \pi_\varepsilon(z)\psi.$$

The relation $(z^*,\ \psi)_{V^*} = \big(F^{-1}z^*,\ \psi)_H$ and the arbitrariness of $\psi \in \mathcal{D}(\Omega)$ yield

$$\int_\Omega z(-\Delta + I)\psi = \int_\Omega \frac{F^{-1}z^* - \beta(z) - \pi_\varepsilon(z)}{\varepsilon}\psi \quad \text{for all } \psi \in \mathcal{D}(\Omega).$$

This implies that

$$(-\Delta + I)_{\mathcal{D}'(\Omega)} z = \frac{F^{-1}z^* - \beta(z) - \pi_\varepsilon(z)}{\varepsilon} \quad \text{in } \mathcal{D}'(\Omega),$$

where $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. Thus we see that

$$\partial\phi_\varepsilon(z) = F\big(\varepsilon(-\Delta + I)_{\mathcal{D}'(\Omega)} z + \beta(z) + \pi_\varepsilon(z)\big) \quad \text{for all } z \in D(\partial\phi_\varepsilon).$$

It suffices from Lemma 4.4 to prove the following inclusion relation:

$$D(\partial\phi_\varepsilon) \subset D(\partial\phi_\varepsilon^H). \tag{4.10}$$

It holds that

$$\begin{aligned} D(\partial\phi_\varepsilon) &= \{w \in D(\phi_\varepsilon) \mid \partial\phi_\varepsilon(w) \in V^*\} \\ &= \left\{ w \in V \;\middle|\; \hat{\beta}(w) \in L^1(\Omega),\ F\big(\varepsilon(-\Delta + I)_{\mathcal{D}'(\Omega)} w + \beta(w) + \pi_\varepsilon(w)\big) \in V^* \right\} \\ &= \left\{ w \in V \;\middle|\; \hat{\beta}(w) \in L^1(\Omega),\ \varepsilon(-\Delta + I)_{\mathcal{D}'(\Omega)} w + \beta(w) + \pi_\varepsilon(w) \in V \right\} \end{aligned}$$

and it follows that

$$\begin{aligned} D(\partial\phi_\varepsilon^H) &= \{w \in D(\phi_\varepsilon^H) \mid \partial\phi_\varepsilon^H(w) \in H\} \\ &= \left\{ w \in V \;\middle|\; \hat{\beta}(w) \in L^1(\Omega),\ w + \varepsilon(-\Delta + I)_{\mathcal{D}'(\Omega)} w + \beta(w) + \pi_\varepsilon(w) \in H \right\} \\ &= \left\{ w \in V \;\middle|\; \hat{\beta}(w) \in L^1(\Omega),\ \varepsilon(-\Delta + I)_{\mathcal{D}'(\Omega)} w + \beta(w) + \pi_\varepsilon(w) \in H \right\}. \end{aligned}$$

That is, (4.10) holds. □

4.3 Proof of Theorem 1.2

We are now in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. To show existence of weak solutions to $(\mathrm{P})_\varepsilon$ we consider

$$\begin{cases} u_\varepsilon'(t) + \partial\phi_\varepsilon(u_\varepsilon(t)) = Ff(t) \quad \text{in } V^* \quad \text{for a.a. } t \in [0,T], \\ u_\varepsilon(0) = u_{0\varepsilon} \quad \text{in } V^*. \end{cases} \tag{4.11}$$

In light of Lemma 2.3, there exists a unique solution $u_\varepsilon \in H^1(0,T;V^*)$ of (4.11) such that $u_\varepsilon(t) \in D(\partial\phi_\varepsilon)$ for a.a. $t \in (0,T)$. Putting $\mu_\varepsilon(t) := -F^{-1}\big(u_\varepsilon'(t)\big)$, we deduce from (1.5), (1.6) and (4.9) that $\mu_\varepsilon \in L^2(0,T;V)$ and $(u_\varepsilon, \mu_\varepsilon)$ satisfies (1.16)-(1.18).

Next we show (1.19). It follows from the equation in (4.11) that

$$\begin{aligned} |u_\varepsilon'(s)|_{V^*}^2 &= \big(u_\varepsilon'(s), u_\varepsilon'(s)\big)_{V^*} \\ &= \big(u_\varepsilon'(s), -\partial\phi_\varepsilon(u_\varepsilon(s)) + Ff(s)\big)_{V^*} \\ &= -\big(u_\varepsilon'(s), \partial\phi_\varepsilon(u_\varepsilon(s))\big)_{V^*} + (u_\varepsilon'(s), Ff(s))_{V^*}. \end{aligned}$$

Here, we have by Lemma 2.2,

$$\big(u_\varepsilon'(s), \partial\phi_\varepsilon(u_\varepsilon(s))\big)_{V^*} = \frac{d}{ds}\phi_\varepsilon(u_\varepsilon(s)),$$

and (1.6) and by Young's inequality yield

$$(u_\varepsilon'(s), Ff(s))_{V^*} = \langle u_\varepsilon'(s), f(s)\rangle_{V^*,V} \le \frac{1}{2}|u_\varepsilon'(s)|_{V^*}^2 + \frac{1}{2}|f(s)|_V^2.$$

Therefore we obtain

$$\frac{1}{2}|u_\varepsilon'(s)|_{V^*}^2 \le -\frac{d}{ds}\phi_\varepsilon(u_\varepsilon(s)) + \frac{1}{2}|f(s)|_V^2.$$

Integrating this inequality yields

$$\frac{1}{2}\int_0^t |u_\varepsilon'(s)|_{V^*}^2\, ds \le -\phi_\varepsilon(u_\varepsilon(t)) + \phi_\varepsilon(u_{0\varepsilon}) + \frac{1}{2}|f|_{L^2(0,T;V)}^2,$$

i.e.,

$$\begin{aligned} &\frac{1}{2}\int_0^t |u_\varepsilon'(s)|_{V^*}^2\, ds + \frac{\varepsilon}{2}|u_\varepsilon(t)|_V^2 + \int_\Omega \hat{\beta}(u_\varepsilon(t)) + \int_\Omega \hat{\pi}_\varepsilon(u_\varepsilon(t)) \\ &\le \frac{\varepsilon}{2}|u_{0\varepsilon}|_V^2 + \int_\Omega \hat{\beta}(u_{0\varepsilon}) + \int_\Omega \hat{\pi}_\varepsilon(u_{0\varepsilon}) + \frac{1}{2}|f|_{L^2(0,T;V)}^2. \end{aligned}$$

Here (C1) implies

$$\int_\Omega \hat{\beta}(u_\varepsilon(t)) \ge c_1|u_\varepsilon(t)|_H^2. \tag{4.12}$$

Recalling (1.14), we infer that

$$|\hat{\pi}_\varepsilon(r)| \le \frac{1}{2}|\pi_\varepsilon'|_{L^\infty(\mathbb{R})}|r|^2 \le \frac{1}{2}c_2(\varepsilon)|r|^2 \tag{4.13}$$

for all $r \in \mathbb{R}$. Now, from (C4) we deduce that there exists $\overline{\varepsilon} \in (0,1]$ such that $c_2(\varepsilon) < 2c_1$ for all $\varepsilon \in (0, \overline{\varepsilon}]$. Thus combining (4.12) and (4.13) gives

$$\int_\Omega \hat{\beta}(u_\varepsilon(t)) + \int_\Omega \hat{\pi}_\varepsilon(u_\varepsilon(t)) \geq \frac{1}{2}\bigl(2c_1 - c_2(\varepsilon)\bigr)|u_\varepsilon(t)|_H^2$$

for a.a. $t \in (0,T)$. Moreover, using (1.15) of (C5) leads to

$$\begin{aligned}\frac{\varepsilon}{2}|u_{0\varepsilon}|_V^2 + \int_\Omega \hat{\beta}(u_{0\varepsilon}) + \int_\Omega \hat{\pi}_\varepsilon(u_{0\varepsilon}) &\leq \frac{c_3(\varepsilon)}{2} + c_3(\varepsilon) + \frac{1}{2}c_2(\varepsilon)|u_{0\varepsilon}|_H^2 \\ &\leq \frac{3}{2}c_3(\varepsilon) + \frac{1}{2}c_2(\varepsilon)c_3(\varepsilon).\end{aligned}$$

Therefore we see that

$$\int_0^t |u_\varepsilon'(s)|_{V^*}^2\,ds + \varepsilon|u_\varepsilon(t)|_V^2 + \bigl(2c_1 - c_2(\varepsilon)\bigr)|u_\varepsilon(t)|_H^2 \leq 3c_3(\varepsilon) + c_2(\varepsilon)c_3(\varepsilon) + |f|_{L^2(0,T;V)}^2.$$

This implies (1.19) with $M_2(\varepsilon) := 3c_3(\varepsilon) + c_2(\varepsilon)c_3(\varepsilon) + |f|_{L^2(0,T;V)}^2$.

Next we prove (1.20). Since $\mu_\varepsilon(s) = -F^{-1}\bigl(u_\varepsilon'(s)\bigr)$, it follows that

$$\int_0^t |\mu_\varepsilon(s)|_V^2\,ds = \int_0^t \bigl|-F^{-1}\bigl(u_\varepsilon'(s)\bigr)\bigr|_V^2\,ds = \int_0^t \bigl|u_\varepsilon'(s)\bigr|_{V^*}^2\,ds.$$

Therefore we arrive at (1.20) via (1.19).

Next we show (1.21). Noting by Lemma 4.5 that $u_\varepsilon(s) \in W$ for a.a. $s \in (0,T)$ and recalling the definition of $\mu(\cdot)$, the monotonicity of β and Lemma 4.1, we have

$$\begin{aligned}|\beta(u_\varepsilon(s))|_H^2 &= \bigl(\beta(u_\varepsilon(s)), \beta(u_\varepsilon(s))\bigr)_H \\ &= \bigl(\mu_\varepsilon(s) - \varepsilon(-\Delta + I)u_\varepsilon(s) - \pi_\varepsilon(u_\varepsilon(s)) + f(s), \beta(u_\varepsilon(s))\bigr)_H \\ &= \bigl(\mu_\varepsilon(s) - \pi_\varepsilon(u_\varepsilon(s)) + f(s), \beta(u_\varepsilon(s))\bigr)_H \\ &\quad - \varepsilon\bigl(-\Delta u_\varepsilon(s), \beta(u_\varepsilon(s))\bigr)_H - \varepsilon\bigl(u_\varepsilon(s), \beta(u_\varepsilon(s))\bigr)_H \\ &\leq (|\mu_\varepsilon(s)|_V + |\pi_\varepsilon(u_\varepsilon(s))|_H + |f(s)|_V)|\beta(u_\varepsilon(s))|_H,\end{aligned}$$

where Young's inequality and (C4) yield

$$\begin{aligned}&\bigl(|\mu_\varepsilon(s)|_V + |\pi_\varepsilon(u_\varepsilon(s))|_H + |f(s)|_V\bigr)|\beta(u_\varepsilon(s))|_H \\ &\leq \frac{1}{2}(|\mu_\varepsilon(s)|_V + |\pi_\varepsilon'|_{L^\infty(\mathbb{R})}|u_\varepsilon(s)|_H + |f(s)|_V)^2 + \frac{1}{2}|\beta(u_\varepsilon(s))|_H^2 \\ &\leq \frac{3}{2}\bigl(|\mu_\varepsilon(s)|_V^2 + c_2(\varepsilon)^2|u_\varepsilon(s)|_H^2 + |f(s)|_V^2\bigr) + \frac{1}{2}|\beta(u_\varepsilon(s))|_H^2.\end{aligned}$$

Therefore,

$$\int_0^t |\beta(u_\varepsilon(s))|_H^2\,ds \leq 3\Bigl(\int_0^t |\mu_\varepsilon(s)|_V^2\,ds + c_2(\varepsilon)^2\int_0^t |u_\varepsilon(s)|_H^2\,ds + |f|_{L^2(0,T;V)}^2\Bigr).$$

Thus we obtain (1.21) by virtue of (1.19) and (1.20).

Next we show (1.22). It follows from (1.17) that

$$\begin{aligned}
&\int_0^t |\varepsilon(-\Delta + I)u_\varepsilon(s)|_H^2\, ds \\
&= \int_0^t |\mu_\varepsilon(s) - \beta(u_\varepsilon(s)) - \pi_\varepsilon(u_\varepsilon(s)) + f(s)|_H^2\, ds \\
&\le 4\Bigl(\int_0^t |\mu_\varepsilon(s)|_V^2\, ds + \int_0^t |\beta(u_\varepsilon(s))|_H^2\, ds + \int_0^t |\pi_\varepsilon(u_\varepsilon(s))|_H^2\, ds + \int_0^t |f(s)|_V^2\, ds\Bigr) \\
&\le 16\Bigl(\int_0^t |\mu_\varepsilon(s)|_V^2\, ds + c_2(\varepsilon)^2 \int_0^t |u_\varepsilon(s)|_H^2\, ds + |f|_{L^2(0,T;V)}^2\Bigr).
\end{aligned}$$

Hence, by the standard elliptic regularity estimate that there exists a constant $L > 0$ such that $|w|_W \le L|(-\Delta + I)w|_H$ for all $w \in W$, we infer

$$\int_0^t |\varepsilon u_\varepsilon(s)|_W^2\, ds \le 16L^2\Bigl(\int_0^t |\mu_\varepsilon(s)|_V^2\, ds + c_2(\varepsilon)^2 \int_0^t |u_\varepsilon(s)|_H^2\, ds + |f|_{L^2(0,T;V)}^2\Bigr)$$

for all $t \in [0, T]$. Therefore (1.22) follows from (1.19) and (1.20).

Moreover, we see from (1.19) and (1.22) that $u_\varepsilon \in L^\infty(0, T; V)$ and $u_\varepsilon \in L^2(0, T; W)$, respectively. □

5 Error estimates

Regarding $(\mathrm{P})_\varepsilon$ as approximate problems of (P) as $\varepsilon \searrow 0$, we can obtain the following theorem which gives an information about the error estimate between the solution of (P) and the solution of $(\mathrm{P})_\varepsilon$. Our proof is based on a direct estimate and hence it is simpler than that in [8].

Theorem 5.1. *In* (C4) *and* (C5) *assume further that*

$$c_2(\varepsilon) = \tilde{c}_2\varepsilon, \quad c_3(\varepsilon) \equiv \tilde{c}_3 \tag{5.1}$$

and

$$|u_{0\varepsilon} - u_0|_{V^*} \le c_4\varepsilon^{1/4} \tag{5.2}$$

for some constants $\tilde{c}_2$, $\tilde{c}_3$ and $c_4 > 0$ independent of ε. Let $(u_\varepsilon, \mu_\varepsilon)$ and (u, μ) be weak solutions of $(\mathrm{P})_\varepsilon$ *and* (P)*, respectively. Then there exist constants $C^* > 0$ and $\overline{\varepsilon} \in (0, 1]$, independent of ε, such that*

$$|u_\varepsilon - u|_{C([0,T];V^*)}^2 + \int_0^T \bigl(\beta(u_\varepsilon(s)) - \beta(u(s)), u_\varepsilon(s) - u(s)\bigr)_H\, ds \le C^*\varepsilon^{1/2} \tag{5.3}$$

for all $\varepsilon \in (0, \overline{\varepsilon}]$.

Proof of Theorem 5.1. Under the additional condition (5.1) we have from Theorems 1.1 and 1.2 that there exist constants $M_1 > 0$, $M_2 > 0$ and $\overline{\varepsilon} \in (0,1]$, independent of ε, such that $2c_1 - \tilde{c_2}\overline{\varepsilon} \geq c_1$ and

$$\int_0^T |u'(s)|_{V*}^2 \, ds + 2c_1|u(t)|_H^2 \leq M_1, \tag{5.4}$$

$$\int_0^T |u_\varepsilon'(s)|_{V*}^2 \, ds + \varepsilon|u_\varepsilon(t)|_V^2 + c_1|u_\varepsilon(t)|_H^2 \leq M_2 \tag{5.5}$$

for all $t \in [0,T]$ and $\varepsilon \in (0, \overline{\varepsilon}]$. Now we see from (1.8) and (1.16) that

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}|u_\varepsilon(s) - u(s)|_{V^*}^2 &= (u_\varepsilon(s) - u(s), u_\varepsilon'(s) - u'(s))_{V^*} \\ &= \big\langle u_\varepsilon(s) - u(s), F^{-1}u_\varepsilon'(s) - F^{-1}u'(s)\big\rangle_{V^*,V} \\ &= -\langle u_\varepsilon(s) - u(s), \mu_\varepsilon(s) - \mu(s)\rangle_{V^*,V}.\end{aligned}$$

Here, since $u_\varepsilon \in L^\infty(0,T;V)$, $u \in L^\infty(0,T;H)$ and $\mu_\varepsilon, \mu \in L^2(0,T;V)$, we derive

$$\langle u_\varepsilon(s) - u(s), \mu_\varepsilon(s) - \mu(s)\rangle_{V^*,V} = (u_\varepsilon(s) - u(s), \mu_\varepsilon(s) - \mu(s))_H.$$

Thus, by (1.17) and since the function $r \mapsto \varepsilon r + \pi_\varepsilon(r)$ is a monotone increasing function, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}|u_\varepsilon(s) - u(s)|_{V^*}^2 + \big(u_\varepsilon(s) - u(s), \beta(u_\varepsilon(s)) - \beta(u(s))\big)_H \\ &= -\big(u_\varepsilon(s) - u(s), \varepsilon(-\Delta + I)u_\varepsilon(s) + \pi_\varepsilon(u_\varepsilon(s))\big)_H \\ &\leq \varepsilon\big(u(s), (-\Delta + I)u_\varepsilon(s)\big)_H + \big(u(s), \pi_\varepsilon(u_\varepsilon(s))\big)_H.\end{aligned}$$

Integrating this inequality yields

$$\begin{aligned}&\frac{1}{2}|u_\varepsilon(t) - u(t)|_{V^*}^2 + \int_0^t \big(u_\varepsilon(s) - u(s), \beta(u_\varepsilon(s)) - \beta(u(s))\big)_H \, ds \\ &\leq \frac{1}{2}|u_{0\varepsilon} - u_0|_{V^*}^2 + \varepsilon\int_0^t \big(u(s), (-\Delta + I)u_\varepsilon(s)\big)_H \, ds + \int_0^t \big(u(s), \pi_\varepsilon(u_\varepsilon(s))\big)_H \, ds \\ &=: A(\varepsilon) + B_\varepsilon(t) + C_\varepsilon(t).\end{aligned} \tag{5.6}$$

From (5.2) we have

$$A(\varepsilon) \leq \frac{1}{2}c_4^2\varepsilon^{1/2}. \tag{5.7}$$

By (1.14), (5.1), (5.4) and (5.5) there exist a constant $C_1 > 0$ such that

$$C_\varepsilon(t) \leq \tilde{c_2}\varepsilon\int_0^t |u_\varepsilon(s)|_H|u(s)|_H \, ds \leq \tilde{c_2}\sqrt{\frac{M_2}{c_1}}\sqrt{\frac{M_1}{2c_1}}T\varepsilon \leq C_1\varepsilon. \tag{5.8}$$

Moreover, Schwarz's inequality and (5.4) give

$$B_\varepsilon(t) \le \varepsilon^{1/2}\Big(\varepsilon\int_0^t |(-\Delta+I)u_\varepsilon(s)|_H^2\,ds\Big)^{1/2}\Big(\int_0^t |u(s)|_H^2\,ds\Big)^{1/2}$$
$$\le \sqrt{\frac{M_1T}{2c_1}}\Big(\varepsilon\int_0^t |(-\Delta+I)u_\varepsilon(s)|_H^2\,ds\Big)^{1/2}\varepsilon^{1/2}$$

Here, by (1.16), (1.17), Young's inequality, (1.7), (5.4) and (5.5), it follows that

$$\begin{aligned}\varepsilon|(-\Delta+I)u_\varepsilon(s)|_H^2 &= \big(\mu_\varepsilon'(s),\ (-\Delta+I)u_\varepsilon(s)\big)_H - \big(\beta(u_\varepsilon(s)),\ (-\Delta+I)u_\varepsilon(s)\big)_H\\
&\quad - \big(\pi_\varepsilon(u_\varepsilon(s)),\ (-\Delta+I)u_\varepsilon(s)\big)_H + \big(f(s),\ (-\Delta+I)u_\varepsilon(s)\big)_H\\
&= -\langle u_\varepsilon'(s),\ u_\varepsilon(s)\rangle_{V^*,V} - \big(\beta(u_\varepsilon(s)),\ -\Delta u_\varepsilon(s)\big)_H\\
&\quad - \big(\beta(u_\varepsilon(s)),\ u_\varepsilon(s)\big)_H - \big(\pi_\varepsilon(u_\varepsilon(s)),\ u_\varepsilon(s)\big)_V + (g(s),\ u_\varepsilon(s))_H\\
&\le -\frac{1}{2}\frac{d}{ds}|u_\varepsilon(s)|_H^2 + \tilde{c_2}\varepsilon|u_\varepsilon(s)|_V^2 + \frac{1}{2}|g(s)|_H^2 + \frac{1}{2}|u_\varepsilon(s)|_H^2\\
&\le -\frac{1}{2}\frac{d}{ds}|u_\varepsilon(s)|_H^2 + \tilde{c_2}M_2 + \frac{1}{2}|g(s)|_H^2 + \frac{M_2}{2c_1},\end{aligned}$$

and hence there exists a constant $C_2 > 0$ such that

$$\varepsilon\int_0^t |(-\Delta+I)u_\varepsilon(s)|_H^2\,ds \le \frac{1}{2}|u_{0\varepsilon}|_H^2 + \tilde{c_2}M_2T + \frac{1}{2}|g|_{L^2(0,T;H)}^2 + \frac{M_2}{2c_1}T \le C_2.$$

Thus, there exists a constant $C_3 > 0$ such that

$$B_\varepsilon(t) \le C_3\varepsilon^{1/2}. \tag{5.9}$$

Plugging (5.7), (5.8) and (5.9) into (5.6), we have

$$|u_\varepsilon - u|_{C([0,T];V^*)}^2 \le c_4^2\varepsilon^{1/2} + 2C_3\varepsilon^{1/2} + 2C_1\varepsilon,$$

and

$$\int_0^T \big(u_\varepsilon(s)-u(s),\ \beta(u_\varepsilon(s))-\beta(u(s))\big)_H\,ds \le \frac{1}{2}c_4^2\varepsilon^{1/2} + C_3\varepsilon^{1/2} + C_1\varepsilon,$$

that is, there exists $C^* > 0$ such that the error estimate (5.3) holds. □

6 Examples

In this section we apply Theorems 1.1, 1.2 and 5.1 to the following two examples.

Example 6.1 (porous media and Cahn–Hilliard type equations). We consider

$$\beta(r) = |r|^{q-1}r + r \quad (q > 1), \qquad \pi_\varepsilon(r) = -\varepsilon r.$$

This β is the function obtained by adding the correction term r to $|r|^{q-1}r$ in the porous media equation (see, e.g., [1, 18, 22, 23]). On the other hand, π_ε is the function appearing in Cahn–Hilliard type equations.

Example 6.2 (fast diffusion and Cahn–Hilliard type equations). Consider

$$\beta(r) = |r|^{q-1}r + r \quad (0 < q < 1), \qquad \pi_\varepsilon(r) = -\varepsilon r$$

This β is the function obtained by adding the correction term r to $|r|^{q-1}r$ in the fast diffusion equation (see, e.g., [11, 20, 22]).

In both examples we can show that β and π_ε satisfy (C1), (C4) and (C5) as follows. Let $q > 0$. Since

$$\beta(r) = |r|^{q-1}r + r = \hat{\beta}\,'(r) = \partial\hat{\beta}(r),$$

where $\hat{\beta}(r) := \frac{1}{q+1}|r|^{q+1} + \frac{1}{2}|r|^2$, we see that (C1) is satisfied.

Next it follows that $\pi_\varepsilon(r) = -\varepsilon r$ is Lipschitz continuous and

$$\begin{aligned}
&\pi_\varepsilon'(r) = -\varepsilon,\\
&\frac{\varepsilon}{2}r^2 + \hat{\pi}_\varepsilon(r) = \frac{\varepsilon}{2}r^2 + \int_0^r \pi_\varepsilon(s)\,ds = \frac{\varepsilon}{2}r^2 - \frac{\varepsilon}{2}r^2 = 0.
\end{aligned}$$

Hence (C4) holds.

To verify (C5) we assume (C3), i.e., $u_0 \in L^2(\Omega) \cap L^{q+1}(\Omega)$. Then we put

$$\begin{aligned}
&A_{L^2} := -\Delta + I : D(A_{L^2}) := W \subset L^2(\Omega) \to L^2(\Omega),\\
&(J_{L^2})_\lambda := (I + \lambda A_{L^2})^{-1},\\
&A_{L^{q+1}} := -\Delta + I : D(A_{L^{q+1}}) := Y \subset L^{q+1}(\Omega) \to L^{q+1}(\Omega),\\
&(J_{L^{q+1}})_\lambda := (I + \lambda A_{L^{q+1}})^{-1},
\end{aligned}$$

where $Y := \{z \in W^{2,q+1}(\Omega) \mid \partial_\nu z = 0 \quad \text{a.e. on } \partial\Omega\}$. There exists $u_{0\varepsilon} \in W \cap Y$ such that

$$\begin{cases}
u_{0\varepsilon} + \varepsilon(-\Delta + 1)u_{0\varepsilon} = u_0 \quad \text{in } \Omega,\\
\partial_\nu u_{0\varepsilon} = 0 \quad \text{on } \partial\Omega,
\end{cases}$$

that is,

$$u_{0\varepsilon} = (J_{L^2})_\varepsilon u_0 = (J_{L^{q+1}})_\varepsilon u_0.$$

From the properties of $(J_{L^2})_\varepsilon$ and $(J_{L^{q+1}})_\varepsilon$ we have

$$\begin{aligned}
&u_{0\varepsilon} = (J_{L^2})_\varepsilon u_0 \to u_0 \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \searrow 0,\\
&|u_{0\varepsilon}|_{L^2(\Omega)} = |(J_{L^2})_\varepsilon u_0|_{L^2(\Omega)} \le |u_0|_{L^2(\Omega)},\\
&\|u_{0\varepsilon}\|_{L^{q+1}(\Omega)} = \|(J_{L^{q+1}})_\varepsilon u_0\|_{L^{q+1}(\Omega)} \le \|u_0\|_{L^{q+1}(\Omega)},
\end{aligned}$$

and hence

$$\begin{aligned}
\int_\Omega \hat{\beta}(u_{0\varepsilon}) &= \frac{1}{q+1}\|u_{0\varepsilon}\|^{q+1}_{L^{q+1}(\Omega)} + \frac{1}{2}|u_{0\varepsilon}|^2_{L^2(\Omega)} \le \frac{1}{q+1}\|u_0\|^{q+1}_{L^{q+1}(\Omega)} + \frac{1}{2}|u_0|^2_{L^2(\Omega)},\\
\varepsilon|u_{0\varepsilon}|^2_{H^1(\Omega)} &= \big(\varepsilon(-\Delta + I)u_{0\varepsilon}, u_{0\varepsilon}\big)_{L^2(\Omega)} = (u_0 - u_{0\varepsilon}, u_{0\varepsilon})_{L^2(\Omega)} \le |u_0|^2_{L^2(\Omega)}. \qquad (6.1)
\end{aligned}$$

Hence there exists $u_{0\varepsilon}$ satisfying (C5). Moreover, we observe that

$$|u_{0\varepsilon} - u_0|_{H^{-1}(\Omega)} \le \varepsilon^{1/2}|u_0|_{L^2(\Omega)}.$$

Indeed, it follows from (6.1) that

$$|u_{0\varepsilon} - u_0|^2_{H^{-1}(\Omega)} = |\varepsilon(-\Delta + I)u_{0\varepsilon}|^2_{H^{-1}(\Omega)} = \varepsilon^2|Fu_{0\varepsilon}|^2_{H^{-1}(\Omega)} = \varepsilon^2|u_{0\varepsilon}|^2_{H^1(\Omega)} \le \varepsilon|u_0|^2_{L^2(\Omega)}.$$

Finally, letting $g \in L^2\big(0,T;L^2(\Omega)\big)$, we find a function $f \in L^2(0,T;H^2(\Omega))$ satisfying (C2). From the above, (C1), (C2), (C4) and (C5) hold and we obtain Theorems 1.1, 1.2 and 5.1 for the functions β and π_ε in Examples 6.1 and 6.2.

Remark 6.1. In this paper, since the increasing condition of $\hat{\beta}$ is quadratic, we can only deal with the case of nondegenerate diffusion terms adding the correction term "$+u$" to $\beta(u)$. We can exclude such the correction term by translation with a constant when Ω is bounded; however, we cannot do it when Ω is unbounded. By revising the increasing condition of $\hat{\beta}$ with the m-th power ($m > 1$), we can deal with the porous media equation and the fast diffusion equation without the correction term "$+u$". We will discuss this in the continuation [17].

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