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LOCAL EXISTENCE OF SOLUTIONS FOR A WAVE EQUATION WITH A VISCOELASTIC BOUNDARY CONDITION

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Abstract. The purpose of this paper is to study a wave equation with a viscoelastic boundary condition. Namely, we construct a iterative scheme in order to get a convergent sequence to a local unique weak solution of the problem.

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1 Introduction

In this paper, we are concerned with the following initial-boundary value problem

$$u_{tt} - u_{xx} + \mu(t, \|u(t)\|^2, \|u_t(t)\|^2)u_t = f(x, t, u), \tag{1.1}$$
$$u(1, t) = 0, \tag{1.2}$$
$$u_x(0, t) = g(t, u(0, t)) + \int_0^t k(t, s, u(0, s))ds, \tag{1.3}$$
$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ (x, t) \in (0, 1) \times (0, T), \tag{1.4}$$

where f, g, k, μ, u_0, u_1 are given functions. To achieve this study, we associate the problem with a recurrent sequence $\{v_m\}$ defined by

$$\begin{aligned} &v''_m - v_{mxx} + \mu(t, \|v_m(t)\|^2, \|v'_m(t)\|^2)v'_m \\ &= \sum\nolimits_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x, t, v_{m-1})(v_m - v_{m-1})^i, \end{aligned} \tag{1.5}$$
$$v_m(1, t) = 0, \tag{1.6}$$
$$v_{mx}(0, t) = g(t, v_m(0, t)) + \int_0^t k(t, s, v_{m-1}(0, s))ds, \tag{1.7}$$
$$v_m(x, 0) = u_0(x),\ v'_m(x, 0) = u_1(x), \tag{1.8}$$

where $(m, x, t) \in \mathbb{N}^* \times (0, 1) \times (0, T)$. The main result of this paper shows that the sequence $\{v_m\}$ converges to a local unique weak solution of the problem (1.1)-(1.4).

The wave equations with the different boundaries have been studied in many papers, for example [1]-[6], [8], [9], [11], [13]-[15], [17]-[26]. See below for some typical papers.

Q. Tiehu [21] proved the existence of a global smooth solution of the following problem

$$u_{tt} - (\rho(u_x))_x = 0, \tag{1.9}$$
$$u(0, t) = 0, \tag{1.10}$$
$$u(L, t) + \int_0^t k(t - s)\rho(u_x(L, s))ds = g(t), \tag{1.11}$$
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.12}$$

where $(x, t) \in (0, L) \times (0, \infty)$ and g, k, ρ, u_0, u_1 are given functions. In this case, the problem (1.9)-(1.12) is a mathematical model for a nonlinear one-dimensional motion of an elastic bar connected with a viscoelastic element at an end of the bar.

In [19], J. Rivera and D. Andrade gave the existence and exponential decay of the solutions of the wave equation with a viscoelastic boundary condition

$$u_{tt} - (\rho(u_x))_x + f(x, t) = 0, \tag{1.13}$$
$$u(0, t) = 0,\ u(1, t) = \int_0^t k(t - s)\rho(u_x(1, s))ds + g(t), \tag{1.14}$$
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.15}$$

where u_0, u_1, f, g, k, ρ are given functions.

M. Rammaha, T. Strei [18] considered weak solutions of an initial-boundary value problem for a nonlinear wave equation in one space dimension

$$u_{tt} - u_{xx} + |u|^{\delta} u_t = |u|^{\chi} u, \tag{1.16}$$

$$u(0,t) = 0, \quad u_x(1,t) = g(t), \tag{1.17}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.18}$$

where δ, $\chi > 1$ are constants and $g \in C^1(\mathbb{R}_+)$.

M. Li [13] mentioned the nonexistence of global solutions in time of the Emden-Fowler type semilinear wave equation

$$t^2 u_{tt} - u_{xx} = |u|^{\chi-2} u, \tag{1.19}$$

with boundary value null and initial values

$$u(x,1) = u_0(x), \; u_t(x,1) = u_1(x), \; (x,t) \in (a,b) \times (1,T), \tag{1.20}$$

where $\chi > 1$ is a constant and u_0, u_1 are given functions.

F. Ficken and B. Fleishman [8] established the global existence and stability of solutions for the following wave equation

$$u_{tt} - u_{xx} - \alpha u_t - \beta u = \varepsilon u^3 + \gamma, \tag{1.21}$$

where α, β, γ, ε are positive constants.

J. Pöschel [17] showed the existence of quasi-periodic solutions for the nonlinear wave equation

$$u_{tt} - u_{xx} + \sum_{j=1}^{n} K_j u^{2j-1} = 0, \tag{1.22}$$

on the finite x-interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0, \; t \in \mathbb{R}, \tag{1.23}$$

where $K_1 > 0$, $K_2 \neq 0$, $K_3, \ldots, K_n$ are constants.

Our paper can be regarded as the relative extension and improvement of the corresponding results of [11, 14, 20, 22, 24, 26].

2 Main result

Firstly, we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively the scalar product and the norm in $L^2(0,1)$.

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t)$ and $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$ and $\frac{\partial^2 u}{\partial x^2}(x,t)$ respectively.

Also, we define a closed subspace of the Sobolev space $H^1(0,1)$ below

$$W = \left\{ v \in H^1(0,1) : v(1) = 0 \right\}, \tag{2.1}$$

with the following scalar product and norm

$$\langle u, v \rangle_W = \langle u_x, v_x \rangle \quad \text{and} \quad \|u\|_W = \|u_x\|. \tag{2.2}$$

Then the following lemmas are well-known.

Lemma 2.1. *The embedding $W \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \leq \|v\|_W \leq \|v\|_{H^1(0,1)} \leq \sqrt{2}\|v\|_W, \ \forall v \in W. \tag{2.3}$$

Lemma 2.2. *The embedding $H^1(0,1) \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|^2_{C^0([0,1])} \leq \varepsilon\|v_x\|^2 + \left(1+\frac{1}{\varepsilon}\right)\|v\|^2, \ \forall v \in H^1(0,1), \ \varepsilon > 0. \tag{2.4}$$

Furthermore, we have the following results.

Lemma 2.3 (see [23])**.** *We always have the following inequality*

$$\left|\sum_{j=1}^{k} \frac{\sin(2j-1)x}{2j-1}\right| \leq \frac{\pi}{2} + 2, \ \forall (k,x) \in \mathbb{N}^* \times \mathbb{R}. \tag{2.5}$$

Lemma 2.4. *Let $(a,b,c_j,k) \in \mathbb{R}^3 \times \mathbb{N}^*$ and $\mu_j = (2j-1)\frac{\pi}{2}$, $j = \overline{1,k}$. Then*

$$\int_a^b \left(\sum_{j=1}^{k} c_j \sin(\mu_j s)\right)^2 ds \leq 2\left(|a|+|b|+2\right) \int_0^1 \left(\sum_{j=1}^{k} c_j \sin(\mu_j s)\right)^2 ds, \tag{2.6}$$

$$\int_a^b \left(\sum_{j=1}^{k} c_j \cos(\mu_j s)\right)^2 ds \leq 2\left(|a|+|b|+2\right) \int_0^1 \left(\sum_{j=1}^{k} c_j \cos(\mu_j s)\right)^2 ds. \tag{2.7}$$

The proof of Lemma 2.4 is straightforward, so we omit the details.
Now we make the following assumptions:

(A_1) $u_0 \in W \cap H^2(0,1)$, $u_1 \in W$;

(A_2) $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $N \in \mathbb{N}^*$;

(A_3) $g \in C^2(\mathbb{R}_+ \times \mathbb{R})$, there exist functions $g_1, g_2, g_3 \in C^0(\mathbb{R}_+)$, such that

$$\begin{cases} |g(t,u)| \leq g_1(t)(|u|+1), \ \forall (t,u) \in \mathbb{R}_+ \times \mathbb{R}, \\ |D_1 g(t,u)| + |D_2 g(t,u)| \leq g_2(t), \ \forall (t,u) \in \mathbb{R}_+ \times \mathbb{R}, \\ |D_1^2 g(t,u)| + |D_1 D_2 g(t,u)| + |D_2^2 g(t,u)| \leq g_3(t), \ \forall (t,u) \in \mathbb{R}_+ \times \mathbb{R}; \end{cases}$$

(A_4) $\mu \in C^1(\mathbb{R}^3_+)$, there exist constants $q > 1$, $\mu_* > 0$, such that

$$\left(|\mu| + \sum_{i=1}^{3} |D_i \mu|\right)(t,u,v) \leq \mu_*(u^p + v^p + 1), \ \forall t,u,v \in \mathbb{R}_+;$$

(A_5) $k \in C^2(\mathbb{R}^2_+ \times \mathbb{R})$;

(A_6) f, g, u_0 satisfy the following compatibility conditions

$$f(\cdot,0,u_0) = u_{0x}(0) = g(0,u_0(0)) = 0.$$

Fix $T_* > 0$, for each $\rho > 0$, $T \in (0, T_*]$, we put

$$\begin{cases} c(\rho, f) = \|f\|_{C^{N+1}([0,1]\times[0,T_*]\times[-\rho,\rho])}, \\ c(\rho, g) = \|g\|_{C^2([0,T_*]\times[-\rho,\rho])}, \\ c(\rho, k) = \|k\|_{C^2([0,T_*]^2\times[-\rho,\rho])}, \\ c(\rho, \mu) = \|\mu\|_{C^1([0,T_*]\times[0,\rho^2]^2)}, \end{cases}$$

and

$$\begin{cases} W(\rho, T) = \{v \in L^\infty(0, T; W) : v_t \in L^\infty(0, T; W), \\ v_{tt} \in L^\infty(0, T; L^2(0, 1)),\ v(0, \cdot) \in H^2(0, T), \text{ with } \|v\|_{L^\infty(0,T;W)}, \\ \|v_t\|_{L^\infty(0,T;W)},\ \|v_{tt}\|_{L^\infty(0,T;L^2(0,1))},\ \|v(0, \cdot)\|_{H^2(0,T)} \le \rho\}, \\ W_*(\rho, T) = \{v \in W(\rho, T) : v \in L^\infty(0, T; W \cap H^2(0, 1))\}. \end{cases}$$

Then we have the following theorem.

Theorem 2.5. *Let (A_1)-(A_6) hold. Then there exist positive constants ρ, T and a recurrent sequence $\{v_m\} \subset W_*(\rho, T)$ $(v_0 = 0)$ satisfying the following linear variational problem*

$$\begin{cases} \langle v_m''(t), \varphi\rangle + \langle v_{mx}(t), \varphi_x\rangle + w_m(t)\varphi(0) \\ \ + \mu_m(t) \langle v_m'(t), \varphi\rangle = \langle f_m(\cdot, t), \varphi\rangle,\ \forall\varphi \in W, \\ v_m(0) = u_0,\ \ v_m'(0) = u_1, \end{cases} \tag{2.8}$$

where

$$\begin{cases} f_m(x, t) = \sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x, t, v_{m-1})(v_m - v_{m-1})^i, \\ \mu_m(t) = \mu(t, \|v_m(t)\|^2, \|v_m'(t)\|^2), \\ w_m(t) = g(t, v_m(0, t)) + \int_0^t k(t, s, v_{m-1}(0, s))ds. \end{cases} \tag{2.9}$$

Proof of Theorem 2.5. The proof is based on the strong induction and Galerkin method. Indeed, it is not difficult to see that Theorem 2.5 holds for $m = 1$. Suppose that it holds for $2, \ldots, m-1$ $(m \ge 2)$. Now we show below that this theorem also holds for m.
Step 1. *Galerkin approximation.* We use an orthonormal basis of W as follows

$$\varphi_j(x) = \sqrt{2/(1 + \mu_j^2)}\cos(\mu_j x),\ \ \mu_j = (2j - 1)\frac{\pi}{2},\ \ j = 1, 2, \ldots. \tag{2.10}$$

Put $v_m^{(k)}(t) = \sum_{j=1}^k \omega_{mj}^{(k)}(t)\varphi_j$, where the functions $\omega_{mj}^{(k)}(t)$ satisfy the following system of differential equations

$$\begin{cases} \left\langle \ddot{v}_m^{(k)}(t), \varphi_j\right\rangle + \left\langle v_{mx}^{(k)}(t), \varphi_{jx}\right\rangle + w_m^{(k)}(t)\varphi_j(0) \\ \ + \mu_m^{(k)}(t) \left\langle \dot{v}_m^{(k)}(t), \varphi_j\right\rangle = \left\langle f_m^{(k)}(\cdot, t), \varphi_j\right\rangle,\ j = \overline{1, k}, \\ v_m^{(k)}(0) = \sum\limits_{j=1}^k a_{mj}^{(k)}\varphi_j = u_0,\ \ \dot{v}_m^{(k)}(0) = \sum\limits_{j=1}^k b_{mj}^{(k)}\varphi_j = u_1, \end{cases} \tag{2.11}$$

where

$$\begin{cases} f_m^{(k)}(x, t) = \sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x, t, v_{m-1})(v_m^{(k)} - v_{m-1})^i, \\ \mu_m^{(k)}(t) = \mu(t, \left\|v_m^{(k)}(t)\right\|^2, \left\|\dot{v}_m^{(k)}(t)\right\|^2), \\ w_m^{(k)}(t) = g(t, v_m^{(k)}(0, t)) + \int_0^t k(t, s, v_{m-1}(0, s))ds. \end{cases} \tag{2.12}$$

Accordingly

$$\begin{cases}\ddot{\omega}_{mj}^{(k)}(t)+\mu_j^2\omega_{mj}^{(k)}(t)=-\mu_m^{(k)}(t)\dot{\omega}_{mj}^{(k)}(t)+\frac{1}{\|\varphi_j\|^2}\left[\left\langle f_m^{(k)}(\cdot,t),\varphi_j\right\rangle-w_m^{(k)}(t)\varphi_j(0)\right],\\ \omega_{mj}^{(k)}(0)=a_{mj}^{(k)},\quad \dot{\omega}_{mj}^{(k)}(0)=b_{mj}^{(k)},\quad j=\overline{1,k}.\end{cases} \tag{2.13}$$

We deduce that

$$\begin{aligned}\omega_{mj}^{(k)}(t)&=a_{mj}^{(k)}\cos(\mu_j t)+b_{mj}^{(k)}\frac{\sin(\mu_j t)}{\mu_j}\\ &-\int_0^t\frac{\sin(\mu_j(t-s))}{\mu_j}\mu_m^{(k)}(s)\dot{\omega}_{mj}^{(k)}(s)ds\\ &-\frac{2}{\varphi_j(0)}\int_0^t ds\int_0^s\frac{\sin(\mu_j(t-s))}{\mu_j}k(s,\tau,v_{m-1}(0,\tau))d\tau\\ &+\frac{2}{\varphi_j^2(0)}\int_0^t\frac{\sin(\mu_j(t-s))}{\mu_j}\left\langle f_m^{(k)}(\cdot,s),\varphi_j\right\rangle ds\\ &-\frac{2}{\varphi_j(0)}\int_0^t\frac{\sin(\mu_j(t-s))}{\mu_j}h(s,v_m^{(k)}(0,s))ds,\ j=\overline{1,k}.\end{aligned} \tag{2.14}$$

By the Banach fixed-point theorem, it is not difficult to show that the system (2.14) has a unique solution $(\omega_{m1}^{(k)}(t),\ldots,\omega_{mk}^{(k)}(t))$ on the interval $[0,T]$ (see [7]).

Step 2. *A priori estimates.* This step includes two stages below.

Stage 1. *A priori estimates 1.* With the help of (2.14), thus $v_m^{(k)}(0,t)$ can be rewritten as follows

$$v_m^{(k)}(0,t)=g_m^{(k)}(t)-2\int_0^t\sum_{j=1}^k\frac{\sin(\mu_j s)}{\mu_j}w_m^{(k)}(t-s)ds, \tag{2.15}$$

where

$$\begin{aligned}g_m^{(k)}(t)&=\sum_{j=1}^k\varphi_j(0)\left(a_{mj}^{(k)}\cos(\mu_j t)+b_{mj}^{(k)}\frac{\sin(\mu_j t)}{\mu_j}\right)\\ &-\sum_{j=1}^k\varphi_j(0)\int_0^t\frac{\sin(\mu_j(t-s))}{\mu_j}\mu_m^{(k)}(s)\dot{\omega}_{mj}^{(k)}(s)ds\\ &+2\sum_{j=1}^k\int_0^t\frac{\sin(\mu_j(t-s))}{\mu_j}\left\langle f_m^{(k)}(\cdot,s),\varphi_j\right\rangle ds.\end{aligned} \tag{2.16}$$

Put $S_{1m}^{(k)}(t)=\left\|\dot{v}_m^{(k)}(t)\right\|^2+\left\|v_{mx}^{(k)}(t)\right\|^2$, we get the following lemma.

Lemma 2.6. *There exist positive constants $c_*(u)$ depending on u_0, u_1 and $c_*(\rho,f)$ depending on ρ, f such that*

$$\int_0^t\left|\dot{g}_m^{(k)}(s)\right|^2ds\le c_*(u)+c_*(\rho,f)\left[\int_0^t\left(S_{1m}^{(k)}\right)^M(s)ds+T\right],\ \forall t\in[0,T], \tag{2.17}$$

where $M=N+8p+1$.

Proof of Lemma 2.6. We define

$$\dot{g}_m^{(k)}(t)=a_m^{(k)}(t)+b_m^{(k)}(t)+c_m^{(k)}(t), \tag{2.18}$$

where

$$\begin{cases} a_m^{(k)}(t)=\sum\limits_{j=1}^{k}\varphi_j(0)[b_{mj}^{(k)}\cos(\mu_j t)-\mu_j a_{mj}^{(k)}\sin(\mu_j t)], \\ b_m^{(k)}(t)=-\sum\limits_{j=1}^{k}\varphi_j(0)\int_0^t\cos(\mu_j(t-s))\mu_m^{(k)}(s)\dot{\omega}_{mj}^{(k)}(s)ds, \\ c_m^{(k)}(t)=2\sum\limits_{j=1}^{k}\frac{1}{\varphi_j(0)}\int_0^t\frac{\sin(\mu_j(t-s))}{\mu_j}\left\langle\dot{f}_m^{(k)}(\cdot,s),\varphi_j\right\rangle ds. \end{cases} \tag{2.19}$$

Therefore

$$\int_0^t\left|\dot{g}_m^{(k)}(s)\right|^2ds\le 3\sum_{I\in\{a,b,c\}}\int_0^t\left|I_m^{(k)}(s)\right|^2ds=\sum_{i=1}^{3}I_i(t). \tag{2.20}$$

We will estimate each term on the right-hand side of this inequality.
First integral. Thanks to Lemma 2.4, we get

$$\begin{aligned} I_1(t)&\le 6\int_0^t\left(\sum_{j=1}^{k}\varphi_j(0)b_{mj}^{(k)}\cos(\mu_j s)\right)^2ds+6\int_0^t\left(\sum_{j=1}^{k}\varphi_j(0)\mu_j a_{mj}^{(k)}\sin(\mu_j s)\right)^2ds \\ &\le 12(T_*+2)\left(\|u_1\|^2+\|u_{0x}\|^2\right). \end{aligned} \tag{2.21}$$

Second integral. Then by the Cauchy-Schwarz inequality, we have

$$\left|b_m^{(k)}(t)\right|^2\le\int_0^t\left|\mu_m^{(k)}(s)\right|^2ds\int_0^t\left|\sum_{j=1}^{k}\varphi_j(0)\cos(\mu_j(t-s))\dot{\omega}_{mj}^{(k)}(s)\right|^2ds. \tag{2.22}$$

Using the Fubini theorem and Lemma 2.4, it follows from (2.22) that

$$\begin{aligned} I_2(t)&\le 3\int_0^t\left|\mu_m^{(k)}(\tau)\right|^2d\tau\int_0^t\int_0^s\left|\sum_{j=1}^{k}\varphi_j(0)\cos(\mu_j(s-\tau))\dot{\omega}_{mj}^{(k)}(\tau)\right|^2d\tau ds \\ &=3\int_0^t\left|\mu_m^{(k)}(\tau)\right|^2d\tau\int_0^t\int_\tau^t\left|\sum_{j=1}^{k}\varphi_j(0)\cos(\mu_j(s-\tau))\dot{\omega}_{mj}^{(k)}(\tau)\right|^2dsd\tau \\ &=3\int_0^t\left|\mu_m^{(k)}(\tau)\right|^2d\tau\int_0^t\left(\int_0^{t-\tau}\left|\sum_{j=1}^{k}\varphi_j(0)\cos(\mu_j z)\dot{\omega}_{mj}^{(k)}(\tau)\right|^2dz\right)d\tau \\ &\le 6(T_*+2)\int_0^t\left|\mu_m^{(k)}(\tau)\right|^2d\tau\int_0^t\left\|\dot{v}_m^{(k)}(\tau)\right\|^2d\tau \\ &\le 3T_*(T_*+2)\left(\int_0^t\left|\mu_m^{(k)}(\tau)\right|^4d\tau+\int_0^t\left\|\dot{v}_m^{(k)}(\tau)\right\|^4d\tau\right). \end{aligned} \tag{2.23}$$

On the other hand, using the following inequalities

$$\begin{cases} a^r\le a^s+1,\ \forall a\ge 0,\ s\ge r>0, \\ (b+c+d)^4\le 27(b^4+c^4+d^4),\ \forall b,c,d\in\mathbb{R}, \end{cases} \tag{2.24}$$

we obtain

$$\int_0^t \left\|\dot v_m^{(k)}(\tau)\right\|^4 d\tau \le \int_0^t (S_{1m}^{(k)})^M(s)ds + T, \tag{2.25}$$

$$\begin{aligned}\int_0^t \left|\mu_m^{(k)}(\tau)\right|^4 d\tau &\le \mu_*^4 \int_0^t \left(\left\|v_m^{(k)}(\tau)\right\|^{2p} + \left\|\dot v_m^{(k)}(\tau)\right\|^{2p} + 1\right)^4 ds \\ &\le 27\mu_*^4 \int_0^t \left(\left\|v_m^{(k)}(\tau)\right\|^{8p} + \left\|\dot v_m^{(k)}(\tau)\right\|^{8p} + 1\right) ds \\ &\le 54\mu_*^4 \int_0^t (S_{1m}^{(k)})^M(s)ds + 81\mu_*^4 T.\end{aligned} \tag{2.26}$$

Hence it follows from (2.23), (2.25) and (2.26) that

$$I_2(t) \le 3T_*(T_*+2)\left[(54\mu_*^4+1)\int_0^t (S_{1m}^{(k)})^M(s)ds + (81\mu_*^4+1)T\right]. \tag{2.27}$$

Third integral. Thanks to $(2.19)_3$ and Lemma 2.3, we deduce that

$$\begin{aligned}\left|c_m^{(k)}(t)\right| &\le \int_0^t \int_0^1 \left|\sum_{j=1}^k \frac{\sin(\mu_j(t-s+x))}{\mu_j}\right| \left|\dot f_m^{(k)}(x,s)\right| dxds \\ &+ \int_0^t \int_0^1 \left|\sum_{j=1}^k \frac{\sin(\mu_j(t-s-x))}{\mu_j}\right| \left|\dot f_m^{(k)}(x,s)\right| dxds \\ &\le 2\left(1+\frac{4}{\pi}\right)\int_0^t \left\|\dot f_m^{(k)}(\cdot,s)\right\|_{L^1(0,1)} ds.\end{aligned} \tag{2.28}$$

Moreover

$$\begin{aligned}&\left|\dot f_m^{(k)}(x,t)\right| \\ &\le |D_2 f[v]| + \left|D_3 f[v] v'_{m-1}\right| \\ &+ \sum\nolimits_{i=1}^{N-1} \frac{1}{i!}\left(\left|D_2 D_3^i f[v]\right| + \left|D_3^{i+1} f[v] v'_{m-1}\right|\right)\left(\left|v_m^{(k)}\right| + |v_{m-1}|\right)^i \\ &+ \sum\nolimits_{i=1}^{N-1} \frac{1}{(i-1)!}\left|D_3^i f[v]\right|\left(\left|v_m^{(k)}\right| + |v_{m-1}|\right)^{i-1}\left(\left|\dot v_m^{(k)}\right| + \left|v'_{m-1}\right|\right) \\ &\le c(\rho,f)(\rho+1)\left[1 + \sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\left(\sqrt{S_{1m}^{(k)}(t)} + \rho\right)^i\right] \\ &+ c(\rho,f)\sum\nolimits_{i=1}^{N-1}\frac{1}{(i-1)!}\left(\sqrt{S_{1m}^{(k)}(t)} + \rho\right)^{i-1}\left(\left|\dot v_m^{(k)}\right| + \rho\right),\end{aligned} \tag{2.29}$$

where $f[v] = f(x,t,v_{m-1})$. Next, using the inequalities $(a+b)^r \le 2^{r-1}(a^r+b^r)$, $\forall a,b \ge 0, r \ge 1$, and $(2.24)_1$, we arrive at

$$\left\|\dot f_m^{(k)}(\cdot,t)\right\|_{L^1(0,1)} \le c(\rho,f)(\rho+1)\left[1 + \sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\left(\sqrt{S_m^{(k)}(t)} + \rho\right)^i\right]$$

$$
\begin{aligned}
&+ c(\rho, f) \sum\nolimits_{i=1}^{N-1} \frac{1}{(i-1)!} \left(\sqrt{S_m^{(k)}(t)} + \rho \right)^i \\
&\leq c(\rho, f)(\rho+1) \left\{ 1 + \sum\nolimits_{i=1}^{N-1} \frac{2^{i-1}}{i!} \left[\left(\sqrt{S_m^{(k)}(t)} \right)^i + \rho^i \right] \right\} \\
&+ c(\rho, f) \sum\nolimits_{i=1}^{N-1} \frac{2^{i-1}}{(i-1)!} \left[\left(\sqrt{S_m^{(k)}(t)} \right)^i + \rho^i \right] \\
&\leq \widehat{c}(\rho, f) \left[\left(S_{1m}^{(k)}\right)^{M/2}(t) + 1 \right],
\end{aligned}
\tag{2.30}
$$

where

$$
\widehat{c}(\rho, f) = c(\rho, f)(\rho+1) \left[\sum\nolimits_{i=1}^{N-1} \frac{2^{i-1}}{(i-1)!}(\rho^i + 1) + 1 \right]. \tag{2.31}
$$

Therefore, we get from (2.28) and (2.30) that

$$
I_3(t) = 3 \int_0^t \left| c_m^{(k)}(s) \right|^2 ds \leq 4 \left(1 + \frac{4}{\pi} \right)^2 T_*^2 \widehat{c}^2(\rho, f) \left(3 \int_0^t \left(S_{1m}^{(k)}\right)^M (s) ds + 2T \right). \tag{2.32}
$$

From (2.20), (2.21), (2.27) and (2.32), we get

$$
\int_0^t \left| \dot{g}_m^{(k)}(s) \right|^2 ds \leq c_*(u) + c_*(\rho, f) \left[\int_0^t \left(S_{1m}^{(k)}\right)^M (s) ds + T \right], \ \forall t \in [0, T], \tag{2.33}
$$

where

$$
\begin{cases}
c_*(u) = 12(T_* + 2) \left(\|u_1\|^2 + \|u_{0x}\|^2 \right), \\
c_*(\rho, f) = 3T_*(T_*+2)(81\mu_*^4 + 1) + 12\left(1 + \frac{4}{\pi}\right)^2 T_*^2 \widehat{c}^2(\rho, f).
\end{cases}
\tag{2.34}
$$

The proof of Lemma 2.6 is completed. □

Remark 2.7. In the case of $\mu = 0$, from the proof of Lemma 2.6, we obtain the same result as Lemma 8 in [23].

With the help of Lemma 2.6, we establish the following lemma.

Lemma 2.8. *There exist positive constants $c_{**}(u)$ depending on u_0, u_1 and $c_{**}(\rho, f, k)$ depending on ρ, f, k such that*

$$
\int_0^t \left| \dot{v}_m^{(k)}(0, s) \right|^2 ds \leq c_{**}(u) + c_{**}(\rho, f, k) \left[\int_0^t \left(S_{1m}^{(k)}\right)^M (s) ds + T \right], \ \forall t \in [0, T]. \tag{2.35}
$$

Proof of Lemma 2.8. From $(2.12)_3$ and the assumptions (A_3), (A_5), we have

$$
\begin{aligned}
\left| \dot{w}_m^{(k)}(t) \right| &\leq \left| D_1 g(t, v_m^{(k)}(0,t)) \right| + \left| D_2 g(t, v_m^{(k)}(0,t)) \right| \left| \dot{v}_m^{(k)}(0,t) \right| \\
&+ \left| k(t, t, v_{m-1}(0,t)) \right| + \int_0^t \left| D_1 k(t, s, v_{m-1}(0,s)) \right| ds \\
&\leq h_* \left(\left| \dot{v}_m^{(k)}(0,t) \right| + 1 \right) + (T_* + 1) c(\rho, k),
\end{aligned}
\tag{2.36}
$$

with $h_* = \|g_1\|_{C^0([0,T_*])} + \|g_2\|_{C^0([0,T_*])} + \|g_3\|_{C^0([0,T_*])}$.
Using the assumption (A_6) and Lemma 2.3, we get from (2.15) and (2.36) that

$$\begin{aligned}
\left|\dot{v}_m^{(k)}(0,t)\right| &= \left|\dot{g}_m^{(k)}(t) - 2\int_0^t \sum_{j=1}^{k} \frac{\sin(\mu_j(t-s))}{\mu_j} \dot{w}_m^{(k)}(s)ds\right| \\
&\le \left|\dot{g}_m^{(k)}(t)\right| + 2\left(1+\frac{4}{\pi}\right)\int_0^t \left|\dot{w}_m^{(k)}(s)\right| ds \\
&\le 2\sqrt{2}\left(1+\frac{4}{\pi}\right) h_*\sqrt{T_*}\left[\int_0^t \left(\left|\dot{v}_m^{(k)}(0,s)\right|^2 + 1\right) ds\right]^{1/2} \\
&\quad + \left|\dot{g}_m^{(k)}(t)\right| + 2\left(1+\frac{4}{\pi}\right) T_*(T_*+1)c(\rho,k).
\end{aligned} \tag{2.37}$$

Combining (2.37) and Lemma 2.6 leads to

$$\begin{aligned}
&\int_0^t \left|\dot{v}_m^{(k)}(0,s)\right|^2 ds \\
&\le 3c_*(\rho,f)\int_0^t (S_{1m}^{(k)})^M(s)ds + 3c_*(u) \\
&+ 12T\left(1+\frac{4}{\pi}\right)^2 [c_*(\rho,f) + T_*(T_*+1)c(\rho,k)]^2 \\
&+ 24h_*^2 T_*\left(1+\frac{4}{\pi}\right)^2 \int_0^t\int_0^s \left(\left|\dot{v}_m^{(k)}(0,\tau)\right|^2 + 1\right) d\tau ds.
\end{aligned} \tag{2.38}$$

We now need the following well-known lemma.

Lemma 2.9 (Integral inequality of Gronwall type [16])**.** *Suppose that $\alpha \in C^1([0,T])$ and β, $v \in C^0([0,T])$, β is a nonnegative function. Moreover, if*

$$v(t) \le \alpha(t) + \int_0^t \beta(s)v(s)ds, \ \forall t \in [0,T], \tag{2.39}$$

then

$$v(t) \le \alpha(0)\exp\left(\int_0^t \beta(s)ds\right) + \int_0^t \exp\left(\int_s^t \beta(\tau)d\tau\right)\alpha'(s)ds, \ \forall t \in [0,T]. \tag{2.40}$$

Thanks to this lemma and (2.38), we obtain

$$\int_0^t \left|\dot{v}_m^{(k)}(0,s)\right|^2 ds \le c_{**}(u) + c_{**}(\rho,f,k)\left[\int_0^t (S_{1m}^{(k)})^M(s)ds + T\right], \forall t \in [0,T], \tag{2.41}$$

where

$$\begin{cases}
c_{**}(u) = 3c_*(u)\exp\left[12h_*^2T_*^2\left(1+\frac{4}{\pi}\right)^2\right], \\
c_{**}(\rho,f,k) = 12\left(1+\dfrac{4}{\pi}\right)^2 \exp\left[48h_*^2T_*^2\left(1+\dfrac{4}{\pi}\right)^2\right] \\
\qquad\qquad \times [c_*(\rho,f) + T_*(T_*+1)c(\rho,k) + T_*]^2.
\end{cases} \tag{2.42}$$

Which completes the proof of Lemma 2.8. □

Next, we replace φ_j in $(2.11)_1$ by $\dot{v}_m^{(k)}(t)$. Then integrating from 0 to t, we get after some calculations

$$\begin{aligned} S_{1m}^{(k)}(t) &\le S_{1m}^{(k)}(0) + 2\int_0^t \left|\mu_m^{(k)}(s)\right| S_{1m}^{(k)}(s)ds + 2\int_0^t \left\langle f_m^{(k)}(\cdot,s), \dot{v}_m^{(k)}(s)\right\rangle ds \\ &- 2\int_0^t h(s, v_m^{(k)}(0,s))\dot{v}_m^{(k)}(0,s)ds \\ &- 2\int_0^t \dot{v}_m^{(k)}(0,s)ds\int_0^s k(s,\tau,v_{m-1}(0,\tau))d\tau \\ &= \|u_1\|^2 + \|u_{0x}\|^2 + \sum_{i=1}^4 J_i. \end{aligned} \tag{2.43}$$

We now estimate the following terms in the right-hand side of (2.43) as follows. Estimating $J_1(t) = 2\int_0^t \left|\mu_m^{(k)}(s)\right| S_{1m}^{(k)}(s)ds$. Due to Using the assumption (A_4) and the inequality $(2.24)_1$, then

$$\begin{aligned} J_1(t) &\le 2\mu_*\int_0^t \left(\left\|v_m^{(k)}(s)\right\|^{2p} + \left\|\dot{v}_m^{(k)}(s)\right\|^{2p} + 1\right) S_{1m}^{(k)}(s)ds \\ &\le 2\mu_*\int_0^t \left[2(S_{1m}^{(k)})^p(s) + 1\right] S_{1m}^{(k)}(s)ds \\ &\le 6\mu_*\int_0^t (S_{1m}^{(k)})^M(s)ds + 6\mu_* T. \end{aligned} \tag{2.44}$$

Estimating $J_2(t) = 2\int_0^t \left\langle f_m^{(k)}(\cdot,s), \dot{v}_m^{(k)}(s)\right\rangle ds$. Note that, it follows from $(2.12)_1$ and $(2.24)_1$ that

$$\begin{aligned} \left|f_m^{(k)}(x,t)\right| &\le \sum\nolimits_{i=0}^{N-1} \frac{1}{i!}\left|D_3^i f(x,t,v_{m-1})\right| \left(\left|v_m^{(k)}\right| + |v_{m-1}|\right)^i \\ &\le c(\rho,f)\left[1 + \sum\nolimits_{i=1}^{N-1}\frac{1}{i!}\left(\sqrt{S_{1m}^{(k)}(t)} + \rho\right)^i\right] \\ &\le \widetilde{c}(\rho,f)\left[(S_{1m}^{(k)})^{M/2}(t) + 1\right], \end{aligned} \tag{2.45}$$

where

$$\widetilde{c}(\rho,f) = c(\rho,f)\left[\sum\nolimits_{i=1}^{N-1}\frac{2^{i-1}}{i!}(\rho^i + 1) + 1\right]. \tag{2.46}$$

Accordingly

$$\begin{aligned} J_2(t) &\le \int_0^t \left\|f_m^{(k)}(s)\right\|^2 ds + \int_0^t \left\|\dot{v}_m^{(k)}(s)\right\|^2 ds \\ &\le 2\widetilde{c}^2(\rho,f)\left[\int_0^t (S_{1m}^{(k)})^M(s)ds + T\right] + \int_0^t S_{1m}^{(k)}(s)ds \\ &\le \left[2\widetilde{c}^2(\rho,f) + 1\right]\left[\int_0^t (S_{1m}^{(k)})^M(s)ds + T\right]. \end{aligned} \tag{2.47}$$

Estimating $J_3(t) = -2\int_0^t h(s, v_m^{(k)}(0,s))\dot{v}_m^{(k)}(0,s)ds$. Since the assumption (A_3) and Lemma 2.8, we deduce that

$$\begin{aligned} J_3(t) &\le \int_0^t g_1^2(s)\left(\left|(v_m^{(k)}(0,s)\right|^2 + 1\right)ds + 2\int_0^t \left|\dot{v}_m^{(k)}(0,s)\right|^2 ds \\ &\le h_*^2\left(\int_0^t \left|v_m^{(k)}(0,s)\right|^2 ds + T\right) + 2\int_0^t \left|\dot{v}_m^{(k)}(0,s)\right|^2 ds \\ &\le 2c_{**}(u) + 4\left[c_{**}(\rho,f,k) + h_*^2\right]\left[\int_0^t \left(S_{1m}^{(k)}(s)\right)^M ds + T\right]. \end{aligned} \tag{2.48}$$

Estimating $J_4(t) = -2\int_0^t \dot{v}_m^{(k)}(0,s)ds\int_0^s k(s,\tau,v_{m-1}(0,\tau))d\tau$. Similarly, we get

$$\begin{aligned} J_4(t) &\le \int_0^t \left|\dot{v}_m^{(k)}(0,s)\right|^2 ds + T^3c^2(\rho,k) \\ &\le c_{**}(u) + c_{**}(\rho,f,k)\left[\int_0^t \left(S_{1m}^{(k)}\right)^M(s)ds + 2T\right]. \end{aligned} \tag{2.49}$$

Combining (2.43), (2.44) and (2.47)-(2.49) shows that

$$S_{1m}^{(k)}(t) \le d_*(u) + d_*(\rho,f,k)\left[\int_0^t \left(S_{1m}^{(k)}\right)^M(s)ds + T\right], \tag{2.50}$$

where

$$\begin{cases} d_*(u) = 3c_{**}(u) + c^2(g) + \|u_1\|^2 + \|u_{0x}\|^2, \\ d_*(\rho,f,g,k) = 6c_{**}(\rho,f,k) + 2\widetilde{c}^2(\rho,f) + 4h_*^2 + 6\mu_* + 1. \end{cases} \tag{2.51}$$

Stage 2. *A priori estimates 2.* Put $S_{2m}^{(k)}(t) = \left\|\ddot{v}_m^{(k)}(t)\right\|^2 + \left\|\dot{v}_{mx}^{(k)}(t)\right\|^2$. We need the following lemma.

Lemma 2.10. *There exist positive constants $c^*(u)$ depending on u_0, u_1 and $c^*(\rho,f)$ depending on ρ, f such that*

$$\int_0^t \left|\ddot{g}_m^{(k)}(s)\right|^2 ds \le c^*(u) + c^*(\rho,f)\left[\int_0^t \left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(s)ds + T\right],\ \forall t\in[0,T]. \tag{2.52}$$

Proof of Lemma 2.10. From (2.18), we deduce that

$$\ddot{g}_m^{(k)}(t) = \alpha_m^{(k)}(t) + \beta_m^{(k)}(t) + \gamma_m^{(k)}(t) + \delta_m^{(k)}(t), \tag{2.53}$$

where

$$\begin{cases} \alpha_m^{(k)}(t) = -\sum\limits_{j=1}^{k}\varphi_j(0)[\mu_j b_{mj}^{(k)}\sin(\mu_j t) + (\mu_j^2 a_{mj}^{(k)} + \mu_m^{(k)}(0)b_{mj}^{(k)})\cos(\mu_j t)], \\ \beta_m^{(k)}(t) = -\sum\limits_{j=1}^{k}\varphi_j(0)\int_0^t\cos(\mu_j(t-s))(\dot{\mu}_m^{(k)}\dot{\omega}_{mj}^{(k)} + \mu_m^{(k)}\ddot{\omega}_{mj}^{(k)})(s)ds, \\ \gamma_m^{(k)}(t) = 2\sum\limits_{j=1}^{k}\frac{1}{\varphi_j(0)}\frac{\sin(\mu_j t)}{\mu_j}\left\langle \dot{f}_m^{(k)}(\cdot,0),\varphi_j\right\rangle, \\ \delta_m^{(k)}(t) = 2\sum\limits_{j=1}^{k}\frac{1}{\varphi_j(0)}\int_0^t\frac{\sin(\mu_j(t-s))}{\mu_j}\left\langle \ddot{f}_m^{(k)}(\cdot,s),\varphi_j\right\rangle ds. \end{cases} \tag{2.54}$$

By the inequality $(\alpha+\beta+\gamma+\delta)^2 \le 4(\alpha^2+\beta^2+\gamma^2+\delta^2)$, $\forall \alpha,\beta,\gamma,\delta \in \mathbb{R}$, we get

$$\int_0^t \left|\ddot{g}_m^{(k)}(s)\right|^2 ds \le 4 \sum_{K\in\{\alpha,\beta,\gamma,\delta\}} \int_0^t \left|K_m^{(k)}(s)\right|^2 ds = \sum_{i=1}^4 K_i(t). \tag{2.55}$$

Set $\mu^* = \mu_m^{(k)}(0) = \mu(0, \|u_0\|^2, \|u_1\|^2)$. Similarly as in Lemma 2.6, we obtain

$$\begin{aligned} K_1(t) &\le 4\int_0^t \left(\sum_{j=1}^k \varphi_j(0)(\mu_j b_{mj}^{(k)} \sin(\mu_j s) + (\mu_j^2 a_{mj}^{(k)} + \mu^* b_{mj}^{(k)})\cos(\mu_j s))\right)^2 ds \\ &\le 8(T_*+2)\|u_{0xx} + u_{1x} - \mu^* u_1\|^2, \end{aligned} \tag{2.56}$$

$$\begin{aligned} K_2(t) &\le 8T_*(T_*+2)\left(\int_0^t \left|\dot{\mu}_m^{(k)}(\tau)\right|^4 d\tau + \int_0^t \left\|\dot{v}_m^{(k)}(\tau)\right\|^4 d\tau\right) \\ &\quad + 8T_*(T_*+2)\left(\int_0^t \left|\mu_m^{(k)}(\tau)\right|^4 d\tau + \int_0^t \left\|\ddot{v}_m^{(k)}(s)\right\|^4 ds\right), \end{aligned} \tag{2.57}$$

$$K_3(t) \le 16T\left(1+\frac{4}{\pi}\right)^2 \left\|\dot{f}_m^{(k)}(\cdot,0)\right\|_{L^1(0,1)}^2 \le 16T\left(1+\frac{4}{\pi}\right)^2 (\rho+1)^2 c^2(\rho,f), \tag{2.58}$$

$$K_4(t) \le 16\left(1+\frac{4}{\pi}\right)^2 \int_0^t \left(\int_0^s \left\|\ddot{f}_m^{(k)}(\cdot,\tau)\right\|_{L^1(0,1)} d\tau\right)^2 ds. \tag{2.59}$$

On the other hand, we get from the assumption (A_4) and Lemma 2.1 that

$$\begin{aligned} \left|\dot{\mu}_m^{(k)}(t)\right| &\le \left|D_1\mu(t, \left\|v_m^{(k)}(t)\right\|^2, \left\|\dot{v}_m^{(k)}(t)\right\|^2)\right| \\ &\quad + 2\left|D_2\mu(t, \left\|v_m^{(k)}(t)\right\|^2, \left\|\dot{v}_m^{(k)}(t)\right\|^2)\right| \left\|\dot{v}_m^{(k)}(t)\right\| \left\|v_m^{(k)}(t)\right\| \\ &\quad + 2\left|D_3\mu(t, \left\|v_m^{(k)}(t)\right\|^2, \left\|\dot{v}_m^{(k)}(t)\right\|^2)\right| \left\|\ddot{v}_m^{(k)}(t)\right\| \left\|\dot{v}_m^{(k)}(t)\right\| \\ &\le \mu_*\Big[4(S_{1m}^{(k)})^{p+1}(t) + 2(S_{1m}^{(k)})^p(t) + 2S_{1m}^{(k)}(t) \\ &\quad + 4(S_{1m}^{(k)})^p(t) S_{2m}^{(k)}(t) + 2S_{2m}^{(k)}(t) + 1\Big] \\ &\le 4\mu_*\Big[(S_{1m}^{(k)})^{2p}(t) + (S_{1m}^{(k)})^{p+1}(t) + (S_{1m}^{(k)})^p(t) \\ &\quad + S_{1m}^{(k)}(t) + (S_{2m}^{(k)})^2(t) + S_{2m}^{(k)}(t) + 1\Big]. \end{aligned} \tag{2.60}$$

Applying the inequalities (2.24), then (2.60) leads to

$$\begin{aligned} \left|\dot{\mu}_m^{(k)}(t)\right|^4 &\le 4^4\mu_*^4\Big[4(S_{1m}^{(k)})^{M/4}(t) + 2(S_{2m}^{(k)})^{M/4}(t) + 7\Big]^4 \\ &\le 84^4\mu_*^4\left[(S_{1m}^{(k)})^M(t) + (S_{2m}^{(k)})^M(t) + 1\right]. \end{aligned} \tag{2.61}$$

Consequently, from (2.25), (2.26), (2.57) and (2.61), we arrive at

$$\begin{aligned} K_2(t) &\le \mu_{**}\left[\int_0^t (S_{1m}^{(k)})^M(s)ds + \int_0^t (S_{2m}^{(k)})^M(s)ds + T\right] \\ &\le \mu_{**}\left[\int_0^t (S_{1m}^{(k)} + S_{2m}^{(k)})^M(s)ds + T\right], \end{aligned} \tag{2.62}$$

where

$$\mu_{**} = 8(2.84^4 \mu_*^4 + 2) T_* (T_* + 1). \tag{2.63}$$

Also, by the inequality $(2.24)_1$ again, we can estimate the term $\ddot{f}_m^{(k)}(x,t)$ as follows

$$\begin{aligned}
&\left|\ddot{f}_m^{(k)}(x,t)\right| \\
&\le c(\rho,f)\left[\left|v''_{m-1}\right| + \left(\left|v'_{m-1}\right|+1\right)^2\right]\left(\left|v_m^{(k)}\right| + \left|v_{m-1}\right| + 1\right) \\
&+ c(\rho,f)\left[\left(\left|v'_{m-1}\right|+1\right)\left(\left|\dot{v}_m^{(k)}\right| + \left|v'_{m-1}\right|\right) + \left(\left|\ddot{v}_m^{(k)}\right| + \left|v''_{m-1}\right|\right)\right] \\
&+ c(\rho,f)\left(\left|v'_{m-1}\right|+1\right)^2 \sum\nolimits_{i=2}^{N-1} \frac{1}{i!}\left(\left|v_m^{(k)}\right| + \left|v_{m-1}\right|\right)^i \\
&+ 2c(\rho,f)\left(\left|v'_{m-1}\right|+1\right) \sum\nolimits_{i=2}^{N-1} \frac{1}{(i-1)!}\left(\left|v_m^{(k)}\right| + \left|v_{m-1}\right|\right)^{i-1}\left(\left|\dot{v}_m^{(k)}\right| + \left|v'_{m-1}\right|\right) \\
&+ c(\rho,f) \sum\nolimits_{i=2}^{N-1} \frac{1}{(i-2)!}\left(\left|v_m^{(k)}\right| + \left|v_{m-1}\right|\right)^{i-2}\left(\left|\dot{v}_m^{(k)}\right| + \left|v'_{m-1}\right|\right)^2 \\
&+ c(\rho,f) \sum\nolimits_{i=2}^{N-1} \frac{1}{(i-1)!}\left(\left|v_m^{(k)}\right| + \left|v_{m-1}\right|\right)^{i-1}\left(\left|\ddot{v}_m^{(k)}\right| + \left|v''_{m-1}\right|\right) \\
&\le c(\rho,f)\left[\left|v''_{m-1}\right| + (\rho+1)^2\right]\left(\sqrt{S_{1m}^{(k)}(t)} + \rho + 1\right) \\
&+ c(\rho,f)\left[(\rho+1)\left(\sqrt{S_{2m}^{(k)}(t)} + \rho\right) + \left(\left|\ddot{v}_m^{(k)}\right| + \left|v''_{m-1}\right|\right)\right] \\
&+ c(\rho,f)(\rho+2)^2 \sum\nolimits_{i=2}^{N-1} \frac{1}{(i-2)!}\left(\sqrt{(S_{1m}^{(k)} + S_{2m}^{(k)})(t)} + \rho\right)^i \\
&+ c(\rho,f) \sum\nolimits_{i=2}^{N-1} \frac{1}{(i-1)!}\left(\sqrt{S_{1m}^{(k)}(t)} + \rho\right)^{i-1}\left(\left|\ddot{v}_m^{(k)}\right| + \left|v''_{m-1}\right|\right) \\
&\le c(\rho,f)\left[(\rho+2)\left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)(t) + \left|v''_{m-1}\right|^2 + \left|\ddot{v}_m^{(k)}\right| + (\rho+2)^4\right] \\
&+ c(\rho,f)(\rho+2)^2 \sum\nolimits_{i=2}^{N-1} \frac{2^{i-1}}{(i-2)!}\left[\left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^{i/2}(t) + \rho^i\right] \\
&+ c(\rho,f) \sum\nolimits_{i=2}^{N-1} \frac{2^{i-2}}{(i-1)!}\left[\left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^{(i-1)/2}(t) + \rho^{i-1}\right]\left(\left|\ddot{v}_m^{(k)}\right| + \left|v''_{m-1}\right|\right) \\
&\le \bar{c}(\rho,f)\left[\left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^{M/2}(t) + \left|\ddot{v}_m^{(k)}\right|^{M/2} + \left|v''_{m-1}\right|^2 + 1\right],
\end{aligned} \tag{2.64}$$

where

$$\bar{c}(\rho,f) = c(\rho,f)\left\{\sum\nolimits_{i=2}^{N-1} \frac{2^{i-1}}{(i-1)!}\left[\left(\rho^{i-1}+1\right)^2 + 4i(\rho+2)^2(\rho^i+1)\right] + (\rho+3)^4\right\}. \tag{2.65}$$

By means of the Cauchy-Schawrz inequality, it follows from (2.64) that

$$\begin{aligned}
&\left\|\ddot{f}_m^{(k)}(\cdot,t)\right\|_{L^1(0,1)}^2 \\
&\le \bar{c}^2(\rho,f)\left[\left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^{M/2}(t) + \left\|\left|\ddot{v}_m^{(k)}(t)\right|^{M/2}\right\|_{L^1(0,1)} + \rho^2 + 1\right]^2 \\
&\le 4\bar{c}^2(\rho,f)\left[2\left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(t) + \rho^2 + 1\right].
\end{aligned} \tag{2.66}$$

Thus

$$K_4(t) \leq 16\left(1+\frac{4}{\pi}\right)^2 T_* \int_0^t \int_0^s \left\| \ddot{f}_m^{(k)}(\cdot,\tau)\right\|_{L^1(0,1)}^2 d\tau ds$$
$$\leq 64\left(1+\frac{4}{\pi}\right)^2 T_* \bar{c}^2(\rho,f)\left[2\int_0^t \left(S_{1m}^{(k)}+S_{2m}^{(k)}\right)^M(s)ds + (\rho^2+1)T\right]. \tag{2.67}$$

In conclusion, from (2.55), (2.56), (2.58), (2.62) and (2.67), we obtain

$$\int_0^t \left|\ddot{g}_m^{(k)}(s)\right|^2 ds \leq c^*(u) + c^*(\rho,f)\left[\int_0^t \left(S_{1m}^{(k)}+S_{2m}^{(k)}\right)^M(s)ds + T\right],\ \forall t\in[0,T], \tag{2.68}$$

where

$$\begin{cases} c^*(u) = 8(T_*+2)\|u_{0xx}+u_{1x}-\mu^* u_1\|^2, \\ c^*(\rho,f) = 128\left(1+\frac{4}{\pi}\right)^2(\rho^2+1)\left[c^2(\rho,f)+T_*\bar{c}^2(\rho,f)\right]+\mu_{**}. \end{cases} \tag{2.69}$$

Lemma 2.10 is conpletely proved. □

Next, we construct the following lemma.

Lemma 2.11. *There exist positive constants $c^{**}(u)$ depending on u_0, u_1 and $c^{**}(\rho,f,k)$ depending on ρ, f, k such that*

$$\exp\left[-Tc^{**}(\rho,f,k)\right]\int_0^t \left|\ddot{v}_m^{(k)}(0,s)\right|^2 ds$$
$$\leq c^{**}(u) + Tc^{**}(\rho,f,k) + \int_0^t \left(S_{1m}^{(k)}+S_{2m}^{(k)}\right)^M(s)ds,\ \forall t\in[0,T]. \tag{2.70}$$

Proof of Lemma 2.11. Applying the assumptions (A_3), (A_5) and Lemma 2.1, we deduce from $(2.12)_3$ that

$$\begin{aligned} \left|\ddot{w}_m^{(k)}(t)\right| &= \Big|D_1^2 g(t,v_m^{(k)}(0,t)) + 2D_1D_2 g(t,v_m^{(k)}(0,t))\dot{v}_m^{(k)}(0,t) \\ &\quad + D_2 g(t,v_m^{(k)}(0,t))\ddot{v}_m^{(k)}(0,t) + D_2^2 g(t,v_m^{(k)}(0,t))\left|\dot{v}_m^{(k)}(0,t)\right|^2 \\ &\quad + D_1 k(t,t,v_{m-1}(0,t)) + D_2 k(t,t,v_{m-1}(0,t)) \\ &\quad + D_3 k(t,t,v_{m-1}(0,t))v'_{m-1}(0,t) + \int_0^t D_1^2 k(t,s,v_{m-1}(0,s))ds\Big| \\ &\leq 2h_*\left[S_{2m}^{(k)}(t) + \left|\ddot{v}_m^{(k)}(0,t)\right| + 1\right] + (\rho+T_*+2)c(\rho,k). \end{aligned} \tag{2.71}$$

Thus

$$\left(\int_0^t \left|\ddot{w}_m^{(k)}(s)\right| ds\right)^2 \leq 2[T_*(\rho+T_*+2)c(\rho,k)+2h_*]^2$$
$$+ 8h_*^2 T_*\left[\int_0^t \left(S_{2m}^{(k)}\right)^2(s)ds + \int_0^t \left|\ddot{v}_m^{(k)}(0,s)\right|^2 ds\right]. \tag{2.72}$$

By Lemma 2.3 and the assumption (A_6), we get from (2.15) that

$$\begin{aligned}
\left|\ddot{v}_m^{(k)}(0,t)\right| &= \left|\ddot{g}_m^{(k)}(t)\right| + 2\left|\sum_{j=1}^{k}\frac{\sin(\mu_j(t-s))}{\mu_j}\right|\left|\dot{w}_m^{(k)}(0)\right| \\
&\quad + 2\int_0^t\left|\sum_{j=1}^{k}\frac{\sin(\mu_j(t-s))}{\mu_j}\right|\left|\ddot{w}_m^{(k)}(s)\right|ds \\
&\le \left|\ddot{g}_m^{(k)}(t)\right| + 2\left(1+\frac{4}{\pi}\right)\left(\left|\dot{w}_m^{(k)}(0)\right| + \int_0^t\left|\ddot{w}_m^{(k)}(s)\right|\right) \\
&\le C_0\left(\left|\ddot{g}_m^{(k)}(t)\right| + \int_0^t\left|\ddot{w}_m^{(k)}(s)\right|ds + 1\right),
\end{aligned} \tag{2.73}$$

where

$$C_0 = 2\left(1+\frac{4}{\pi}\right)\left(|D_1 g(0,u_0(0))| + |D_2 g(0,u_0(0))|\,|u_1(0)| + |k(0,0,u_0(0))| + 1\right). \tag{2.74}$$

Owing to Lemma 2.10, it follows from (2.72) and (2.73) that

$$\begin{aligned}
&\int_0^t\left|\ddot{v}_m^{(k)}(0,s)\right|^2 ds \\
&\le 3C_0^2\left[\int_0^t\left|\ddot{g}_m^{(k)}(s)\right|^2 ds + \int_0^t\left(\int_0^s\left|\ddot{w}_m^{(k)}(\tau)\right|d\tau\right)^2 ds + T\right] \\
&\le 3C_0^2 c^*(u) + c^{**}(\rho,f,k)\left(\int_0^t\left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(s)ds\right. \\
&\quad \left. + \int_0^t ds\int_0^s\left|\ddot{v}_m^{(k)}(0,\tau)\right|^2 d\tau + T\right),
\end{aligned} \tag{2.75}$$

where

$$\begin{aligned}
c^{**}(\rho,f,k) &= 6C_0^2[T_*(\rho+T_*+2)c(\rho,k)+2h_*]^2 \\
&\quad + 6C_0^2\left[c^*(\rho,f) + 8h_*^2(T_*+1)^2 + 1\right].
\end{aligned} \tag{2.76}$$

By Lemma 2.9, from (2.75), we obtain

$$\begin{aligned}
&\exp\left[-Tc^{**}(\rho,f,k)\right]\int_0^t\left|\ddot{v}_m^{(k)}(0,s)\right|^2 ds \\
&\le 3C_0^2 c^*(u) + Tc^{**}(\rho,f,k) + \int_0^t\left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(s)ds,\ \forall t\in[0,T].
\end{aligned} \tag{2.77}$$

The proof of Lemma 2.11 is completed. □

Remark 2.12. Similarly to the proof of Lemma 2.11, we obtain the estimates for the sequences $\{u_{tt}^m(\xi_i,\cdot)\}$, $i=\overline{0,N}$ in the proof of Theorem 5 of [23] as follows:

$$\sum\nolimits_{i=0}^{N}\int_0^t |u_{tt}^m(\xi_i,s)|^2 ds \le C_T,\ \forall t\in[0,T], \tag{2.78}$$

where C_T is a positive constant independent of m.

By Lemma 2.8 and Lemma 2.11, we can estimate the terms $(S_{1m}^{(k)} + S_{2m}^{(k)})(t)$ and $\int_0^t \left(\left| \dot{v}_m^{(k)}(0,s) \right|^2 + \left| \ddot{v}_m^{(k)}(0,s) \right|^2 \right) ds$. Indeed, we differentiate the equations $(2.11)_1$ with respect to t, then

$$\begin{aligned} &\left\langle \ddot{v}_{mt}^{(k)}(t), \varphi_j \right\rangle + \left\langle \dot{v}_{mx}^{(k)}(t), \varphi_{jx} \right\rangle + \dot{w}_m^{(k)}(t)\varphi_j(0) \\ &+ \dot{\mu}_m^{(k)}(t) \left\langle \dot{v}_m^{(k)}(t), \varphi_j \right\rangle + \mu_m^{(k)}(t) \left\langle \ddot{v}_m^{(k)}(t), \varphi_j \right\rangle = \left\langle \dot{f}_m^{(k)}(\cdot, t), \varphi_j \right\rangle,\ j = \overline{1,k}. \end{aligned} \tag{2.79}$$

We multiply the j^{th} equation of (2.79) by $\ddot{\omega}_{mj}^{(k)}(t)$, then summing up with respect to $j = \overline{1,k}$ and integrating with respect to the time variable from 0 to t, we get

$$\begin{aligned} S_{2m}^{(k)}(t) &\leq S_{2m}^{(k)}(0) + 6\mu_* T + 6\mu_* \int_0^t (S_{2m}^{(k)})^M(s) ds \\ &+ 2\int_0^t \left|\dot{\mu}_m^{(k)}(s)\right| \left|\left\langle \ddot{v}_m^{(k)}(s), \dot{v}_m^{(k)}(s)\right\rangle\right| ds \\ &+ 2\int_0^t \left\langle \dot{f}_m^{(k)}(\cdot, s), \ddot{v}_m^{(k)}(s) \right\rangle ds - 2\int_0^t \dot{w}_m^{(k)}(s)\ddot{v}_m^{(k)}(0,s) ds \\ &= \|u_{1x}\|^2 + 6\mu_* T + 6\mu_* \int_0^t (S_{2m}^{(k)})^M(s) ds + \sum_{i=1}^4 L_i(t). \end{aligned} \tag{2.80}$$

Now we estimate the following terms in the right-hand side of (2.80) as follows. Estimating $L_1(t) = \left\| \ddot{v}_m^{(k)}(0) \right\|^2$. Using $(2.11)_1$ and the compatibility condition (A_6), then

$$\left\| \ddot{v}_m^{(k)}(0) \right\|^2 = \left\langle u_{0xx} - \mu_m^{(k)}(0) u_1, \ddot{v}_m^{(k)}(0) \right\rangle \leq \left\| u_{0xx} - \mu_m^{(k)}(0) u_1 \right\| \left\| \ddot{v}_m^{(k)}(0) \right\|. \tag{2.81}$$

Accordingly

$$L_1(t) \leq 2\|u_{0xx}\|^2 + 2\mu^2(0, \|u_0\|^2, \|u_1\|^2)\|u_1\|^2. \tag{2.82}$$

Estimating $L_2(t) = 2\int_0^t \left|\dot{\mu}_m^{(k)}(s)\right| \left|\left\langle \ddot{v}_m^{(k)}(s), \dot{v}_m^{(k)}(s)\right\rangle\right| ds$. We deduce from $(2.24)_1$ and (2.61) that

$$\begin{aligned} L_2(t) &\leq 2\int_0^t \left|\dot{\mu}_m^{(k)}(s)\right| \left\|\ddot{v}_m^{(k)}(s)\right\| \left\|\dot{v}_{mx}^{(k)}(s)\right\| ds \\ &\leq 168\mu_* \int_0^t \left[(S_{1m}^{(k)})^M(s) + (S_{2m}^{(k)})^M(s) + 1 \right]^{1/4} S_{2m}^{(k)}(s) ds \\ &\leq 168\mu_* \int_0^t \left[(S_{1m}^{(k)})^{M/4}(s) + (S_{2m}^{(k)})^{M/4}(s) + 1 \right] S_{2m}^{(k)}(s) ds \\ &\leq 504\mu_* \left[\int_0^t (S_{1m}^{(k)} + S_{2m}^{(k)})^M(s) ds + T \right]. \end{aligned} \tag{2.83}$$

Estimating $L_3(t) = 2\int_0^t \left\langle \dot{f}_m^{(k)}(\cdot, s), \ddot{v}_m^{(k)}(s) \right\rangle ds$. Similarly to (2.30), it is not difficult to show that

$$\left\| \dot{f}_m^{(k)}(\cdot, t) \right\| \leq 2\widehat{c}^2(\rho, f) \left[(S_{1m}^{(k)} + S_{2m}^{(k)})^M(t) + 1 \right]. \tag{2.84}$$

Therefore

$$
\begin{aligned}
L_3(t) &\le \int_0^t \left\| \dot{f}_m^{(k)}(\cdot,s)\right\|^2 ds + \int_0^t \left\|\ddot{v}_m^{(k)}(s)\right\|^2 ds \\
&\le \widehat{c}^2(\rho,f)\int_0^t \left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(s)ds + \int_0^t (S_{2m}^{(k)})(s)ds + \widehat{c}^2(\rho,f)T \\
&\le \left[\widehat{c}^2(\rho,f)+1\right]\left[\int_0^t \left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(s)ds + T\right].
\end{aligned}
\tag{2.85}
$$

Estimating $L_4(t) = -2\int_0^t \dot{w}_m^{(k)}(s)\ddot{v}_m^{(k)}(0,s)ds$. By Lemma 2.8 and Lemma 2.11, hence it follows from (2.36) that

$$
\begin{aligned}
L_4(t) &\le \int_0^t \left|\dot{w}_m^{(k)}(s)\right|^2 ds + \int_0^t \left|\ddot{v}_m^{(k)}(0,s)\right|^2 ds \\
&\le \int_0^t \left|\ddot{v}_m^{(k)}(0,s)\right|^2 ds + 3h_*^2\int_0^t \left|\dot{v}_m^{(k)}(0,s)\right|^2 ds \\
&+ 3h_*^2 + 3T(T_*+1)^2c^2(\rho,k) \\
&\le c^{***}(u)\left\{\exp\left[Tc^{**}(\rho,f,k)\right]+1\right\} \\
&+ c^{***}(\rho,f,k)\left[\int_0^t \left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(s)ds + T\right],
\end{aligned}
\tag{2.86}
$$

where

$$
\begin{cases}
c^{***}(u) = c^{**}(u) + 3h_*^2[c_{**}(u)+1], \\
c^{***}(\rho,f,k) = \exp\left[T_*c^{**}(\rho,f,k)\right]\left[c^{**}(\rho,f,k)+1\right] \\
\qquad\qquad + 3h_*^2c_{**}(\rho,f,k) + 3(T_*+1)^2c^2(\rho,k).
\end{cases}
\tag{2.87}
$$

From (2.80), (2.82), (2.83), (2.85) and (2.86), we arrive at

$$
\begin{aligned}
S_{2m}^{(k)}(t) &\le d^*(u)\left\{\exp\left[Tc^{**}(\rho,f,k)\right]+2\right\} \\
&+ d^*(\rho,f,k)\left[\int_0^t \left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(s)ds + T\right],
\end{aligned}
\tag{2.88}
$$

where

$$
\begin{cases}
d^*(u) = c^{***}(u) + 2\|u_{0xx}\|^2 + \|u_{1x}\|^2 + 2\mu^2(0,\|u_0\|^2,\|u_1\|^2)\|u_1\|^2, \\
d^*(\rho,f,k) = c^{***}(\rho,f,k) + \widehat{c}^2(\rho,f) + 516\mu_* + 1.
\end{cases}
\tag{2.89}
$$

Set

$$
S_m^{(k)}(t) = (S_{1m}^{(k)} + S_{2m}^{(k)})(t) + \int_0^t \left(\left|\dot{v}_m^{(k)}(0,s)\right|^2 + \left|\ddot{v}_m^{(k)}(0,s)\right|^2\right) ds. \tag{2.90}
$$

Combining (2.35), (2.50), (2.70) and (2.88) implies that

$$
\begin{aligned}
&\exp\left[-Tc^{**}(\rho,f,k)\right]S_m^{(k)}(t) \\
&\le d^{**}(u) + d^{**}(\rho,f,k)\left[\int_0^t \left(S_{1m}^{(k)} + S_{2m}^{(k)}\right)^M(s)ds + T\right],
\end{aligned}
\tag{2.91}
$$

where

$$\begin{cases} d^{**}(u) = 6d^{*}(u) + 6d_{*}(u), \\ d^{**}(\rho, f, k) = 3(c^{**} + c_{**} + d^{*} + d_{*})(\rho, f, k) + 3. \end{cases} \tag{2.92}$$

Next, we get the following result.

Lemma 2.13. *There exist positive constants ρ, T independent of k, m such that*

$$S_m^{(k)}(t) \leq \rho^2, \ \forall (t, k, m) \in [0, T] \times \mathbb{N}^2. \tag{2.93}$$

Proof of Lemma 2.13. We choose $\rho^2 = 2d^{**}(u) + 1$ and set

$$\begin{aligned} &\exp\left[-Tc^{**}(\rho, f, k)\right] S(t) \\ &= \frac{\rho^2}{2} + d^{**}(\rho, f, k) \left[\int_0^t (S_m^{(k)})^M (s) ds + T\right]. \end{aligned} \tag{2.94}$$

Using (2.91) and (2.94) gives that

$$S^{-M}(t) S'(t) \leq d^{**}(\rho, f, k) \exp\left[Tc^{**}(\rho, f, k)\right]. \tag{2.95}$$

Integrating both sides of (2.95) on $[0, T]$ leads to

$$\begin{aligned} S^{1-M}(t) &\geq \left[\rho^2/2 + Td^{**}(\rho, f, k)\right]^{1-M} \\ &\quad \times \exp\left[(1-M)Tc^{**}(\rho, f, k)\right] \\ &\quad + (1-M)Td^{**}(\rho, f, k) \exp\left[Tc^{**}(\rho, f, k)\right] \\ &\quad = d(T, \rho, f, k). \end{aligned} \tag{2.96}$$

On the other hand, we remark that

$$\lim_{T \to 0^+} d(T, \rho, f, k) = \left(\rho^2/2\right)^{1-M} > \rho^{2(1-M)}. \tag{2.97}$$

Thanks to (2.96) and (2.97), there exists a constant $T \in (0, T_*]$, such that

$$S^{1-M}(t) \geq \rho^{2(1-M)}, \ \forall t \in [0, T]. \tag{2.98}$$

Hence

$$S_m^{(k)}(t) \leq S(t) \leq \rho^2, \ \forall t \in [0, T]. \tag{2.99}$$

This completes the proof of Lemma 2.13. □

Applying Lemma 2.13, we get

$$v_m^{(k)} \in W_*(\rho, T), \ \forall (k, m) \in \mathbb{N}^2. \tag{2.100}$$

Remark 2.14. It is not difficutlt to show that the function

$$\begin{aligned} X(t) = \Big\{ &\exp\left[(1-M)Tc^{**}(\rho, f, k)\right] \left[\rho^2/2 + Td^{**}(\rho, f, k)\right]^{1-M} \\ &+ (1-M) \exp\left[Tc^{**}(\rho, f, k)\right] d^{**}(\rho, f, k) t \Big\}^{1/(1-M)} \end{aligned} \tag{2.101}$$

is the maximal solution of the following nonlinear Volterra integral equation [10]

$$\begin{aligned} &\exp\left[-Tc^{**}(\rho, f, k)\right] X(t) \\ &= \frac{\rho^2}{2} + d^{**}(\rho, f, k) \left[\int_0^t X^M(s) ds + T\right], \ t \in [0, T]. \end{aligned} \tag{2.102}$$

Step 3. *Limiting process.* Thanks to Lemma 2.13, we can extract a subsequence of sequence $\left\{v_m^{(k)}\right\}$, still labeled by the same notation, such that

$$\begin{cases} v_m^{(k)} \to v_m & \text{weakly* in } L^\infty(0,T;H^1(0,1)), \\ \dot{v}_m^{(k)} \to v_m' & \text{weakly* in } L^\infty(0,T;H^1(0,1)), \\ \ddot{v}_m^{(k)} \to v_m'' & \text{weakly* in } L^\infty(0,T;L^2(0,1)), \\ v_m^{(k)}(0,\cdot) \to v_m(0,\cdot) & \text{weakly in } H^2(0,T), \end{cases} \tag{2.103}$$

as $k \to \infty$, and $v_m \in W(\rho, T)$.
Using the compactness of the imbedding $H^2(0,T) \hookrightarrow C^1([0,T])$ and the lemma of J.L. Lions [12], p. 57, then (2.103) leads to the existence of a subsequence still denoted by $\left\{v_m^{(k)}\right\}$, such that

$$\begin{cases} v_m^{(k)} \to v_m & \text{strongly in } L^2((0,1)\times(0,T)) \text{ and a.e. in } (0,1)\times(0,T), \\ \dot{v}_m^{(k)} \to v_m' & \text{strongly in } L^2((0,1)\times(0,T)) \text{ and a.e. in } (0,1)\times(0,T), \\ v_m^{(k)}(0,\cdot) \to v_m(0,\cdot) & \text{strongly in } C^1([0,T]). \end{cases} \tag{2.104}$$

Using the following inequality

$$\left|a^i - b^i\right| \le i c^{i-1} \left|a-b\right|, \; \forall a,b \in [-c,c], \; c>0, \; i \in \mathbb{N}^*, \tag{2.105}$$

then it follows from $(2.9)_1$ and $(2.12)_1$ that

$$\left|(f_m^{(k)} - f_m)(x,t)\right| \le c(\rho, f) \sum\nolimits_{i=0}^{N} \frac{(2\rho)^i}{i!} \left|(v_m^{(k)} - v_m)(x,t)\right|. \tag{2.106}$$

On the other hand, we also have

$$\left|(\mu_m^{(k)} - \mu_m)(t)\right| \le 2\rho c(\rho,\mu) \left(\left\|(v_m^{(k)} - v_m)(t)\right\| + \left\|(\dot{v}_m^{(k)} - \dot{v}_m)(t)\right\|\right). \tag{2.107}$$

So, from (2.104), (2.106) and (2.107), we get

$$\begin{cases} f_m^{(k)}(x,t) \to f_m(x,t) & \text{strongly in } L^2((0,1)\times(0,T)), \\ \mu_m^{(k)}(t) \to \mu_m(t) & \text{strongly in } L^2(0,T), \\ w_m^{(k)}(t) \to w_m(t) & \text{strongly in } C^0([0,T]). \end{cases} \tag{2.108}$$

Passing to the limit in (2.11) by $(2.103)_{1,2,3}$ and (2.108), we obtain that v_m satisfies the problem (2.8)-(2.9).
Furthermore, it is not difficult to see from $(2.8)_1$ that

$$v_{mxx} = v_m'' + \mu_m(t) v_m' - f_m(x,t) \in L^\infty(0,T;L^2(0,1)). \tag{2.109}$$

Thus $v_m \in W_*(\rho,T)$. Theorem 2.5 is proved completely. □

Remark 2.15. Theorem 2.5 gives no conclusion of the existence of a recurrent sequence $\{v_m\} \subset W_*(\rho,T)$ when the term u_t in the equation (1.1) is replaced by the nonlinear term $\varphi(u_t)$. This is an open problem.

Next, we introduce the Banach space ([12], p. 26)

$$W(0,T)=\left\{v\in L^\infty(0,T;W): v'\in L^\infty(0,T;L^2(0,1))\right\}, \tag{2.110}$$

with the following norm

$$\|v\|_{W(0,T)}=\|v\|_{L^\infty(0,T;W)}+\|v'\|_{L^\infty(0,T;L^2(0,1))}. \tag{2.111}$$

Then the main theorem of this paper is shown by

Theorem 2.16. *Let* (A_1)-(A_6) *hold. Then there exist positive constants* ρ *and* T *such that the problem* (1.1)-(1.4) *has a unique weak solution* $v\in W_*(\rho,T)$.
Also, we have the following estimate

$$\|v_m-v\|_{W(0,T)}\le C_T\varepsilon^m,\ \forall m\in\mathbb{N}, \tag{2.112}$$

where the sequence $\{v_m\}$ *is given by Theorem 2.5,* $\varepsilon\in(0,1)$ *and* C_T *are constants depending only on* T, f, g, h, k, u_0, u_1.

Remark 2.17. By Theorem 2.16, the problem (1.1)-(1.4) has a unique weak solution v satisfying

$$\begin{cases} v\in L^\infty(0,T;W\cap H^2(0,1))\cap C^0(0,T;W)\cap C^1(0,T;L^2(0,1)),\\ v'\in L^\infty(0,T;W),\ v''\in L^\infty(0,T;L^2(0,1)),\ v(0,\cdot)\in H^2(0,T). \end{cases} \tag{2.113}$$

In addition, we can see that the solution $v\in H^2((0,1)\times(0,T))\cap L^\infty(0,T;H^2(0,1))$. Thus the solution v is almost classical which is rather natural since the initial data $(u_0,u_1)\notin C^2([0,1])\times C^1([0,1])$.

Proof of Theorem 2.16. First, we shall prove that $\{\omega_m=v_{m+1}-v_m\}$ is a Cauchy sequence in $W(0,T)$. Indeed, we can easily see that

$$\begin{cases} \langle\ddot\omega_m(t),\varphi\rangle+\langle\omega_{mx}(t),\varphi_x\rangle+(w_{m+1}-w_m)(t)\varphi(0)+\mu_{m+1}(t)\left\langle\dot\omega_m(t),\varphi\right\rangle\\ +(\mu_{m+1}-\mu_m)(t)\left\langle\dot v_m(t),\varphi\right\rangle=\left\langle(f_{m+1}-f_m)(\cdot,t),\varphi\right\rangle,\ \forall\varphi\in W,\\ \omega_m(x,0)=\dot\omega_m(x,0)=0. \end{cases} \tag{2.114}$$

Put

$$s_m(t)=\|\dot\omega_m(t)\|^2+\|\omega_{mx}(t)\|^2. \tag{2.115}$$

Substituting φ in (2.114) by $\dot\omega_m$, then integrating both sides of it on $[0,T]$, we obtain

$$\begin{aligned} s_m(t)=&-2\int_0^t\left\langle\mu_{m+1}(s)\dot\omega_m(s)+(\mu_{m+1}-\mu_m)(s)\dot v_m(s),\dot\omega_m(s)\right\rangle ds\\ &+2\int_0^t\left\langle(f_{m+1}-f_m)(\cdot,s),\dot\omega_m(s)\right\rangle ds\\ &-2\int_0^t\dot\omega_m(0,s)\left[h(s,v_{m+1}(0,s))-h(s,v_m(0,s))\right]ds\\ &-2\int_0^t\dot\omega_m(0,s)ds\int_0^s\left[k(s,\tau,v_m(0,\tau))-k(s,\tau,v_{m-1}(0,\tau))\right]d\tau\\ =&\,I_1(t)+I_2(t)+I_3(t)+I_4(t). \end{aligned} \tag{2.116}$$

Now we estimate the integrals on the right-hand side of (2.116) as follows.
First integral. Using the assumption (A_4) and Lemma 2.1 implies that

$$\begin{aligned} &I_1(t) \\ &\leq 2\int_0^t |\mu_{m+1}(s)| \, \|\dot{\omega}_m(s)\|^2 ds + 2\rho \int_0^t |(\mu_{m+1}-\mu_m)(s)| \, \|\dot{\omega}_m(s)\| \, ds \\ &\leq 2c(\rho,\mu)\int_0^t \|\dot{\omega}_m(s)\|^2 ds + 2\rho^2 c(\rho,\mu)\int_0^t \left(\|\omega_m(s)\|^2 + 3\|\dot{\omega}_m(s)\|^2\right) ds \\ &\leq 2(3\rho^2+1)c(\rho,\mu)\int_0^t s_m(s)ds. \end{aligned} \tag{2.117}$$

Second integral. Applying Taylor's formula for the function $f(x,t,v_m) = f(x,t,v_{m-1}+\omega_{m-1})$ about the point v_{m-1} up to the order N, we arrive at

$$\begin{aligned} f(x,t,v_m) &= \sum\nolimits_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,v_{m-1})\omega_{m-1}^i \\ &\quad + \frac{\omega_{m-1}^N}{(N-1)!}\int_0^1 (1-\theta)^{N-1} D_3^N f(x,t,v_{m-1}+\theta\omega_{m-1})d\theta \\ &= f_m(x,t) + \frac{\omega_{m-1}^N}{(N-1)!}\int_0^1 (1-\theta)^{N-1} D_3^N f(x,t,v_{m-1}+\theta\omega_{m-1})d\theta. \end{aligned} \tag{2.118}$$

Due to $(2.9)_1$ and (2.118), we deduce that

$$\begin{aligned} f_{m+1}(x,t) = f_m(x,t) &+ \sum\nolimits_{i=1}^{N-1} \frac{1}{i!} D_3^i f(x,t,v_m)\omega_m^i \\ &+ \frac{\omega_{m-1}^N}{(N-1)!}\int_0^1 (1-\theta)^{N-1} D_3^N f(x,t,v_{m-1}+\theta\omega_{m-1})d\theta. \end{aligned} \tag{2.119}$$

Consequently

$$\begin{aligned} &\|(f_{m+1}-f_m)(t)\| \\ &\leq c(\rho,f)\left[\sum\nolimits_{i=1}^{N-1} \frac{(2\rho)^{i-1}}{i!}\sqrt{s_m(t)} + \frac{(2\rho)^{N-1}}{N!}\|\omega_{m-1}\|_{W(0,T)}\right] \\ &\leq \sqrt{2}c(\rho,f)\sum\nolimits_{i=1}^{N} \frac{(2\rho)^{i-1}}{i!}\left[s_m(t) + \|\omega_{m-1}\|_{W(0,T)}^2\right]^{1/2}. \end{aligned} \tag{2.120}$$

This yields

$$\begin{aligned} I_2(t) &\leq \int_0^t \|(f_{m+1}-f_m)(\cdot,s)\|^2 ds + \int_0^t \|\dot{\omega}_m(s)\|^2 ds \\ &\leq \underline{c}(\rho,f)\left[\int_0^t s_m(s)ds + T\|\omega_{m-1}\|_{W(0,T)}^2\right], \end{aligned} \tag{2.121}$$

where

$$\underline{c}(\rho,f) = 2c^2(\rho,f)\left[\sum\nolimits_{i=1}^{N} \frac{(2\rho)^{i-1}}{i!}\right]^2 + 1. \tag{2.122}$$

Third integral. By the formula for integration by parts, we have

$$\begin{aligned} &I_3(t) \\ &= -2\int_0^t \dot{\omega}_m(0,s)\left[g(s,v_{m+1}(0,s)) - g(s,v_m(0,s))\right] ds \\ &= -2\int_0^t \dot{\omega}_m(0,s)ds \int_0^1 \frac{d}{d\theta} g(s,(v_m+\theta\omega_m)(0,s))d\theta \\ &= -\omega_m^2(0,t)\int_0^1 D_2 g(t,(v_m+\theta\omega_m)(0,t))d\theta \\ &+ \int_0^t \omega_m^2(0,s)ds \int_0^1 D_1D_2 g(s,(v_m+\theta\omega_m)(0,s))d\theta \\ &+ \int_0^t \omega_m^2(0,s)ds \int_0^1 D_2^2 g(s,(v_m+\theta\omega_m)(0,s))(\theta v'_{m+1} + (1-\theta)v'_m)(0,s)d\theta \\ &\le c(\rho,h)\left[\omega_m^2(0,t) + (\rho+1)\int_0^t s_m(s)ds\right]. \end{aligned} \tag{2.123}$$

On the other hand, from Lemma 2.2, we get

$$\omega_m^2(0,t) \le \varepsilon s_m(t) + (1+\varepsilon^{-1})T\int_0^t s_m(s)ds, \ \forall\varepsilon > 0. \tag{2.124}$$

Hence it follows from (2.123) and (2.124) that

$$I_3(t) \le \varepsilon c(\rho,g)s_m(t) + \left[(1+\varepsilon^{-1})T + \rho + 1\right] c(\rho,g)\int_0^t s_m(s)ds. \tag{2.125}$$

Fourth integral. Using integration by parts again, it follows that

$$\begin{aligned} &I_4(t) \\ &= -2\omega_m(0,t)\int_0^t \left[k(t,s,v_m(0,s)) - k(t,s,v_{m-1}(0,s))\right] ds \\ &+ 2\int_0^t \omega_m(0,s)\left[k(s,s,v_m(0,s)) - k(s,s,v_{m-1}(0,s))\right] ds \\ &+ 2\int_0^t \omega_m(0,s)ds \int_0^s \left[k(s,\tau,v_m(0,\tau)) - k(s,\tau,v_{m-1}(0,\tau))\right] d\tau \\ &= J_1(t) + J_2(t) + J_3(t). \end{aligned} \tag{2.126}$$

Moreover, we also have

$$\begin{aligned} J_1(t) &\le 2\sqrt{s_m(t)}\int_0^t c(\rho,k)\|\omega_{m-1}(s)\|_W ds \\ &\le \frac{1}{4}s_m(t) + 4T^2c^2(\rho,k)\left\|\omega_{m-1}\right\|_{W(0,T)}^2, \end{aligned} \tag{2.127}$$

$$\begin{aligned} J_2(t) &\le 2\int_0^t \sqrt{s_m(s)}c(\rho,k)\|\omega_{m-1}(s)\|_W ds \\ &\le \int_0^t s_m(s)ds + Tc^2(\rho,k)\left\|\omega_{m-1}\right\|_{W(0,T)}^2, \end{aligned} \tag{2.128}$$

$$J_3(t) \leq 2\int_0^t \sqrt{s_m(s)}ds \int_0^s c(\rho,k)\|\omega_{m-1}(\tau)\|_W d\tau$$
$$\leq \int_0^t s_m(s)ds + \frac{1}{3}T^3c^2(\rho,k)\,\|\omega_{m-1}\|^2_{W(0,T)}\,. \tag{2.129}$$

Combining (2.126)-(2.129) leads to

$$I_4(t) \leq \frac{1}{4}s_m(t) + 2\int_0^t s_m(s)ds + T(T+2)^2c^2(\rho,k)\,\|\omega_{m-1}\|^2_{W(0,T)}\,. \tag{2.130}$$

Choosing $4\varepsilon[c(\rho,g)+1]=1$, we get from (2.116), (2.117), (2.121), (2.125) and (2.130) that

$$s_m(t) \leq C(T,\rho,f,k)\,\|\omega_{m-1}\|^2_{W(0,T)} + C(\rho,f,h,\mu)\int_0^t s_m(s)ds, \tag{2.131}$$

where

$$\begin{cases} C(\rho,f,g,\mu) = 2\left[(1+\varepsilon^{-1})T_* + \rho + 1\right]c(\rho,g) + 4(3\rho^2+1)c(\rho,\mu) + \underline{c}(\rho,f) + 4, \\ C(T,\rho,f,k) = 2T\underline{c}(\rho,f) + 2T(T+2)^2c^2(\rho,k). \end{cases} \tag{2.132}$$

Applying the Gronwall inequality, we get

$$\|\omega_m\|_{W(0,T)} \leq \varepsilon\|\omega_{m-1}\|_{W(0,T)},\ \forall m \in \mathbb{N}^*, \tag{2.133}$$

where T is chosen small enough such that

$$\varepsilon^2 = 4C(T,\rho,f,k)\exp[TC(\rho,f,h,\mu)] < 1. \tag{2.134}$$

Therefore

$$\|v_{m+n} - v_m\|_{W(0,T)} \leq \frac{2\rho}{1-\varepsilon}\varepsilon^{m-1},\ \forall m,n \in \mathbb{N}^*. \tag{2.135}$$

From (2.135), $\{v_m\}$ is a Cauchy sequence in $W(0,T)$. Then there exists $v \in W(0,T)$ such that $v_m \to v$ strongly in $W(0,T)$. Let $n \to +\infty$, while fixing m, we get (2.112). Also, we remark that $\{v_m\} \subset W_*(\rho,T)$, by using a similar argument as Step 3 in the proof of Theorem 2.5, we obtain that v satisfies the following equation

$$\begin{cases} \langle v''(t),\varphi\rangle + \langle v_x(t),\varphi_x\rangle + w(t)\varphi(0) + \mu(t,\|v(t)\|^2,\|v'(t)\|^2)\,\langle v',\varphi\rangle \\ = \langle f(\cdot,t,v),\varphi\rangle\,,\ \forall\varphi \in W, \\ v(x,0) = u_0(x),\ v'(x,0) = u_1(x), \\ w(t) = g(t,(v(0,t)) + \int_0^t k(t,s,v(0,s))ds. \end{cases} \tag{2.136}$$

Finally, we prove the uniqueness of solutions of the problem (2.136). Indeed, let $v_1, v_2 \in W_*(\rho,T)$ be two solutions of the problem (1.1)-(1.4). Then $v = v_1 - v_2$ satisfies the following problem

$$\begin{cases} \langle v''(t),\varphi\rangle + \langle v_x(t),\varphi_x\rangle + (w_1-w_2)(t)\varphi(0) + \mu_1(t)\,\langle v'(s),\varphi\rangle \\ +\,(\mu_1-\mu_2)(t)\,\langle v_2'(s),\varphi\rangle = \langle (f_1-f_2)(\cdot,t),\varphi\rangle\,,\ \forall\varphi \in W, \\ v(x,0) = v'(x,0) = 0, \end{cases} \tag{2.137}$$

where

$$\begin{cases} \mu_i(t) = \mu(t, \|v_i(t)\|^2, \|v_i'(t)\|^2), \\ f_i(x,t) = f(x,t,v_i(t)), \\ w_i(t) = g(t, v_i(0,t)) + \int_0^t k(t,s,v_i(0,s))ds, \ i = 1,2. \end{cases} \tag{2.138}$$

Taking $\varphi = v'$ in $(2.137)_1$ and then integrating both sides of it from 0 to t, we get

$$\begin{aligned} S(t) = &-2\int_0^t \langle \mu_1(s)v'(s) + (\mu_1 - \mu_2)(s)v_2'(s), v'(s)\rangle\, ds \\ &+ 2\int_0^t \langle (f_1 - f_2)(\cdot, s), v'(s)\rangle\, ds \\ &- 2\int_0^t v'(0,s)\sum\nolimits_{i=1}^2 (-1)^{i-1} g(s, v_i(0,s))ds \\ &- 2\int_0^t v'(0,s)ds \int_0^s \sum_{i=1}^2 (-1)^{i-1} k(s,\tau,v_i(0,\tau))d\tau \\ = &\ J_1(t) + J_2(t) + J_3(t) + J_4(t), \end{aligned} \tag{2.139}$$

in which $S(t) = \|v'(t)\|^2 + \|v_x(t)\|^2$. Then we easily estimate the integrals on the right-hand side of (2.139) as follows.

$$J_1(t) \le 2(3\rho^2 + 1)c(\rho,\mu)\int_0^t S(s)ds, \tag{2.140}$$

$$J_2(t) \le 2c(\rho, f)\int_0^t S(s)ds, \tag{2.141}$$

$$J_3(t) \le \varepsilon c(\rho, g)S(t) + [(1+\varepsilon^{-1})T + \rho + 1]c(\rho, g)\int_0^t S(s)ds, \ \forall \varepsilon > 0, \tag{2.142}$$

$$J_4(t) \le \frac{1}{4}S(t) + [(T^2 + 4T)c^2(\rho,k) + 2c(\rho,k) + 1]\int_0^t S(s)ds. \tag{2.143}$$

Choosing $4\varepsilon[c(\rho, h) + 1] \le 1$, we get from (2.139)-(2.143) that

$$S(t) \le C(T, \rho, g, k, \mu)\int_0^t S(s)ds, \tag{2.144}$$

where

$$\begin{aligned} C(T,\rho,g,k,\mu) = &\ 4Tc^2(\rho,g) + (5T + \rho + 1)c(\rho,g) \\ &+ (T^2 + 4T)c^2(\rho,k) + 2c(\rho,k) + 2(3\rho^2+1)c(\rho,\mu) + 1. \end{aligned} \tag{2.145}$$

Using the Gronwall inequality, we obtain $S(t) = 0$, i.e. $v_1 = v_2$, completing the proof of Theorem 2.16. □

Remark 2.18. If we replace the term u_t in the equation (1.1) by the nonlinear term $\varphi(u_t)$, then we have no conclusion about the existence of weak solutions of the problem (1.1)-(1.4). This is an open problem.

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