Advances in Mathematical
Sciences and Applications
Vol. 26, No. 1 (2017), pp. 143–173

GAKKOTOSHO
TOKYO JAPAN

STRONGLY DEGENERATE PARABOLIC EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. We consider the initial value problem for a degenerate parabolic equation of the form:

$$\partial_t u + \nabla \cdot A(x,t,u) + B(x,t,u) = \Delta\beta(x,t,u). \tag{DP}$$

This equation has both properties of hyperbolic equations and those of parabolic equations. Thus, this is also called hyperbolic-parabolic equation. Since (DP) has nonlinear convection and diffusion terms, this describes various nonlinear convective diffusion phenomena such as filtration problems, Stefan problems and so on.

One of the features of this article is that the equation (DP) has variable coefficients. In this case, it is difficult to prove a BV-estimate for approximate solutions to (DP). We overcome this difficulty to use the assumption which was derived by Wu-Zhao (1983). To prove the uniqueness of generalized solutions to (DP), we refer to the method of Chen-Karlsen (2005). Using the arguments, we prove the existence and uniqueness of entropy solutions in the space BV to the problem.

Communicated by Editors; Received July 31, 2016.
This author is supported by Grant-in-Aid No. 25800086 and No. 26400138, JSPS.
AMS Subject Classification: 35K65, 35K55, 35L65.
Keywords: strongly degenerate parabolic equation, variable coefficients, BV-entropy solution, conservation laws.

1 Introduction

We consider the initial value problem for a degenerate parabolic equation of the form

$$\begin{cases} \partial_t u + \nabla \cdot A(x,t,u) + B(x,t,u) = \Delta\beta(x,t,u), & (x,t) \in \mathbb{R}^N_T = \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N). \end{cases} \tag{P}$$

Here $\partial_t := \partial/\partial t$, $\nabla := (\partial/\partial x_1, \dots, \partial/\partial x_N)$ and $\Delta := \sum_{i=1}^N \partial^2/\partial x_i^2$ are the spatial nabla and the Laplacian in $\mathbb{R}^N$, respectively. $[0,T]$ is a fixed time interval. $A(x,t,\xi) = (A^1, ..., A^N)(x,t,\xi)$ is an $\mathbb{R}^N$-valued function on $\mathbb{R}^N \times [0,T] \times \mathbb{R}$ and $B(x,t,\xi)$ and $\beta(x,t,\xi)$ are $\mathbb{R}$-valued functions on $\mathbb{R}^N \times [0,T] \times \mathbb{R}$. The function $\beta(x,t,\xi)$ is supposed to be monotone nondecreasing and locally Lipschitz continuous with respect to ξ for any $(x,t) \in \mathbb{R}^N_T$. Since β is assumed to be monotone nondecreasing, the set of points ξ where $\partial_\xi \beta(x,t,\xi) = 0$ may have a positive measure for any $(x,t) \in \mathbb{R}^N_T$. In this sense, we say that the equation posed as (DP):

$$\partial_t u + \nabla \cdot A(x,t,u) + B(x,t,u) = \Delta\beta(x,t,u) \tag{1.1}$$

is a *strongly degenerate parabolic equation*. The equation (1.1) can be applied to several mathematical models; hyperbolic conservation laws [8], porous medium [15], Stefan problem [21], filtration problem [4], sedimentation process [3], traffic flow [20], blood flow [13], and so on. Moreover, (1.1) is regarded as a linear combination of the time dependent conservation laws (quasilinear hyperbolic equation) and the porous medium equation (nonlinear degenerate parabolic equation). Thus, (1.1) has both properties of hyperbolic equations and those of parabolic equations. In particular, up to the assumptions on β, (1.1) has the following properties:

. If β is strictly increasing, then "parabolicity" is majorant to "hyperbolicity".

. If β is monotone nondecreasing, then "parabolicity" and "hyperbolicity" are not necessarily comparable.

Our mathematical treatment of the equation (1.1) is L^1-framework. More specifically, we consider (1.1) in the space $L^1(\mathbb{R}^N)$ and construct solutions to (1.1) in the space $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover, solutions to (1.1) should be defined in generalized sense. To ensure the existence and uniqueness of it, it is necessitate to consider distributional solutions satisfying a special condition. This framework was first treated by Vol'pert-Hudjaev [16].

It is crucial problem that (1.1) have either regular coefficient $A(\cdot,t,\xi)$ or discontinuous coefficient $A(\cdot,t,\xi)$ with respect to x. In this paper, we only treat the regular case of the coefficient. In this case, it is well known that compactness in the space BV and the Kružkov's doubling variable method [11] are available. To see this, it is necessarily to get the following estimates:

$$\max\{||u_\varepsilon(\cdot,t)||_{L^\infty},\ TV(u_\varepsilon(\cdot,t)),\ ||\partial_t u_\varepsilon(\cdot,t)||_{L^1}\} \le C,$$

where u_ε is an approximate solution to (P) and $TV(u_\varepsilon)$ denotes the total variation of u_ε. In fact, Carrillo [5] proved the existence and uniqueness of entropy solutions to the

Dirichlet problem for strongly degenerate parabolic equations. Moreover, Karlsen-Risebro [10] generalized the uniqueness results of Carrillo by showing that it holds for the Cauchy problem with a non-smooth flux function. On the other hand, Watanabe-Oharu [18, 19] and Watanabe [17] formulated BV-entropy solutions and proved the existence and uniqueness of BV-entropy solutions to the Neumann type problem for the equation (1.1). Here, entropy solutions are weak solutions satisfying an entropy inequality which is derived by Kružkov [11]. When we consider the existence and uniqueness of generalized solutions to (1.1), we usually choose the concept of entropy solutions as generalized solutions to (1.1), because (1.1) has hyperbolicity on nondegenerate intervals. Also, BV-entropy solutions are entropy solutions in the space BV. In fact, the space $L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ can be regarded as an invariant space of (P). More specifically, if we take the initial function in $L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$, then we may construct the entropy solution in the space.

Features of the present paper are to consider the equation (1.1) with variable coefficients and BV-entropy solutions to (P). Previous works [6, 10] treated the type of (DP) and prove the uniqueness and continuous dependence of entropy solutions. In this article, we focus our attention not only on the uniqueness of entropy solutions but also on the existence of it in the space BV. If the diffusion term depends on the space and time variables, then it is difficult to prove a BV-estimate and a time regularity estimate. To overcome this difficulty, we use an additional assumption for the diffusion term which was derived by Wu-Zhao [22]. This is the first feature of this article. As is mentioned the above, our first objective is to specify assumptions to prove the existence of BV-entropy solutions. Moreover, we next use the method of Chen-Karlsen [6] to consider the uniqueness of BV-entropy solutions. They considered equations with quasi-linear type diffusion. In this article, we consider the nonlinear type diffusion $\Delta\beta(x,t,u)$. This is the second feature of this article.

Throughout this paper, we employ the following notations and terminologies. For $1 \le p \le \infty$, the Lebesgue space of real-valued Lebesgue-measurable functions on $\mathbb{R}^N$ equipped with the usual norm $||\cdot||_p$ is denoted by $L^p(\mathbb{R}^N)$. The space of functions of bounded variation in $\mathbb{R}^N$ is denoted by $BV(\mathbb{R}^N)$ (cf. [2, 9, 23]). Furthermore, the function $\mathrm{sgn}(\xi)$ means the usual *signum* function.

2 Assumptions and the main result

In this section, we present some assumptions and the main result. Before that, we write the nabla of the function $A(x,t,u)$ and the laplacian of the function $\beta(x,t,u)$ as follows:

$$\nabla \cdot A(x,t,u) = [\nabla \cdot A](x,t,u) + [\partial_\xi A](x,t,u) \cdot \nabla u,$$

and

$$\begin{aligned} \Delta\beta(x,t,u) &= \nabla \cdot ([\nabla\beta](x,t,u) + [\partial_\xi \beta](x,t,u) \cdot \nabla u) \\ &= [\Delta\beta](x,t,u) + 2[\partial_\xi \nabla\beta](x,t,u) \cdot \nabla u + [\partial_\xi^2 \beta](x,t,u)|\nabla u|^2 + [\partial_\xi \beta](x,t,u)\Delta u, \end{aligned} \tag{2.1}$$

for $(x,t) \in \mathbb{R}^N_T$ and some regular function u. These are based on the chain rule formulas in [1] (see also [2, Theorem 3.99], [23, Theorem 2.1.11]).

Throughout this paper, we impose the following assumptions on the functions A, B, β and u_0. Here, we write $\partial_{x_i} := \partial/\partial x_i$ for $i = 1, \ldots, N$, $\partial_{x_{N+1}} := \partial/\partial t$, $\widehat{\nabla} := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N}, \partial_{x_{N+1}})$ and $\mathcal{U} := [-U, U]$ for any $U > 0$. For any $U > 0$, the following conditions hold:

{A0} $u_0(x) \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$.

{A1} $\begin{cases} A \in L^1(\mathbb{R}^N_T \times \mathcal{U})^N \cap L^\infty(\mathbb{R}^N_T \times \mathcal{U})^N \cap L^\infty(\mathcal{U}; L^2(\mathbb{R}^N_T))^N, \\ \partial_\xi A \in L^1(\mathbb{R}^N_T \times \mathcal{U})^N \cap L^\infty(\mathbb{R}^N_T \times \mathcal{U})^N, \quad \nabla \cdot A,\ \partial_\xi \nabla \cdot A \in L^1(\mathbb{R}^N_T \times \mathcal{U}). \end{cases}$

{A2} $\begin{cases} B \in L^1(\mathbb{R}^N_T \times \mathcal{U}) \cap L^\infty(\mathbb{R}^N_T \times \mathcal{U}) \cap L^\infty(\mathcal{U}; L^1(\mathbb{R}^N_T)), \\ |\widehat{\nabla} B| \in L^\infty(\mathcal{U}; L^1(\mathbb{R}^N_T)), \quad \partial_\xi B \in L^\infty(\mathbb{R}^N_T \times \mathcal{U}). \end{cases}$

{A3} $\begin{cases} \beta \in L^1(\mathbb{R}^N_T \times \mathcal{U}) \cap L^\infty(\mathbb{R}^N_T \times \mathcal{U}), \quad \nabla\beta \in L^1(\mathbb{R}^N_T \times \mathcal{U})^N \cap L^\infty(\mathcal{U}; L^2(\mathbb{R}^N_T))^N, \\ \partial_t \beta \in L^\infty(\mathcal{U}; L^1(\mathbb{R}^N_T)), \quad \partial_\xi \beta \in L^\infty(\mathbb{R}^N_T \times \mathcal{U}), \\ \partial_\xi \nabla\beta \in L^1(\mathbb{R}^N_T \times \mathcal{U})^N, \quad \varDelta\beta,\ \partial_\xi \varDelta\beta \in L^1(\mathbb{R}^N_T \times \mathcal{U}). \end{cases}$

{A4} $B(x,t,0) = 0$ and $\nabla\beta(x,t,0) - A(x,t,0) = \vec{0}$ for $(x,t) \in \mathbb{R}^N \times [0,T]$.

{A5} Let $\Psi(x,t,\xi) := \nabla \cdot A(x,t,\xi) - \varDelta\beta(x,t,\xi) + B(x,t,\xi)$. Then, there exist positive constants c_0, c_1 such that

$$\sup_{(x,t) \in \mathbb{R}^N \times [0,T]} |\Psi(x,t,0)| \le c_0, \qquad \sup_{(x,t,\xi) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}} (-\partial_\xi \Psi(x,t,\xi)) \le c_1.$$

{A6} For $i = 1, 2, \ldots, N+1$,

$$\begin{cases} \partial_\xi \partial_{x_i} (\nabla\beta - A) \in L^\infty(\mathbb{R}^N_T \times \mathcal{U})^N, \\ \nabla \cdot (\nabla\beta - A),\ |\widehat{\nabla}(\varDelta\beta - \nabla \cdot A)| \in L^\infty(\mathcal{U}; L^1(\mathbb{R}^N_T)). \end{cases}$$

{A7} For $(x,t,\xi) \in \mathbb{R}^N \times [0,T] \times \mathcal{U}$ and $\lambda = (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$, there exists a constant $\kappa > 0$ such that

$$\sum_{i,k=1}^{N+1} (\partial_\xi \beta(x,t,\xi)\lambda_i\lambda_k - \kappa(\partial_{x_i}\partial_\xi\beta(x,t,\xi)\lambda_k)^2) \ge 0.$$

The conditions $\{A1\}$-$\{A3\}$ are regularity assumptions for the functions A, B and β with respect to x, t and ξ. {A4}-{A6} are used to prove L^∞, L^1 and BV-estimates for approximate solutions. {A6} is also interpreted the regularity assumptions for the flux $A(x,t,\xi) - \nabla\beta(x,t,\xi)$ to (1.1). The condition {A7} fulfills to get a BV-estimate with respect to x and t for approximate solutions to (P).

Remark 2.1. By the assumption {A7}, it is deduced that

$$\partial_\xi \beta(x,t,\xi) \ge 0 \quad \text{for } (x,t,\xi) \in \mathbb{R}^N \times [0,T] \times \mathcal{U}. \tag{2.2}$$

More specifically, $\beta(x,t,\xi)$ degenerate on nondegenerate intervals with respect to ξ. In particular, if $\beta(x,t,\xi) \equiv \beta(\xi)$, then {A7} is equivalent to (2.2).

Remark 2.2. By the assumption {A7}, it follows that

$$\partial_{x_k}\partial_\xi\beta(x,t,\xi)=0$$

on the degenerate area $E\equiv\{(x,t,\xi)\in\mathbb{R}^N\times[0,T]\times\mathcal{U}\mid\partial_\xi\beta(x,t,\xi)=0\}$ for $k=1,\dots,N+1$. Therefore, the second term of extreme right-hand side in (2.1) vanishes on E. This is interpreted that the transport effect which is arose from diffusion term vanish on E.

By the assumption {A2}, there exists a constant $\alpha'>0$ such that

$$-\partial_\xi B(x,t,\xi)\le\alpha', \tag{2.3}$$

for $(x,t,\xi)\in\mathbb{R}_T^N\times\mathcal{U}$. We also impose an additional regularity assumption to prove the uniqueness of BV-entropy solution: for any $i,j=1,\dots,N$,

$$\textbf{\{A8\}}\quad\begin{cases}\partial_\xi\partial_{x_i}A^j,\ \partial_\xi\partial_{x_i}\partial_{x_j}\beta\in L^\infty(\mathbb{R}_T^N\times\mathcal{U}),\quad\sqrt{\partial_\xi\beta},\ \partial_{x_i}\sqrt{\partial_\xi\beta}\in L^1(\mathbb{R}_T^N\times\mathcal{U}),\\ \partial_{x_i}\partial_\xi\beta,\ \partial_{x_i}\sqrt{\partial_\xi\beta},\ \partial_{x_i}\partial_{x_j}\sqrt{\partial_\xi\beta}\in L^\infty(\mathbb{R}_T^N\times\mathcal{U}).\end{cases}$$

Remark 2.3. The framework of present paper has some applications. Typical example of the coefficients $A(x,t,\xi)$, $B(x,t,\xi)$, $\beta(x,t,\xi)$ is a separation variable type:

$$A(x,t,\xi)\equiv\tilde{A}(x,t)\widehat{A}(\xi),\quad B(x,t,\xi)\equiv\tilde{B}(x,t)\widehat{B}(\xi),\quad\beta(x,t,\xi)\equiv\tilde{\beta}(x,t)\widehat{\beta}(\xi).$$

Here the functions $\tilde{A}$, $\widehat{A}$, $\tilde{B}$, $\widehat{B}$, $\tilde{\beta}$, $\widehat{\beta}$ satisfy the assumptions {A1}-{A7}, respectively. For example, the following applications are given:

1. Viscous Burgers equations:

   $$\partial_t u+\nabla\cdot(\tilde{A}(x,t)u^2)+\tilde{B}(x,t)u^m=\Delta(\tilde{\beta}(x,t)u),$$

   where m is an even number and $\tilde{B}(x,t)\ge 0$ for $(x,t)\in\mathbb{R}_T^N$.

2. Porous medium equations with convection:

   $$\partial_t u+\nabla\cdot(\tilde{A}(x,t)u^{m_1})+\tilde{B}(x,t)u^{m_2}=\Delta(\tilde{\beta}(x,t)u^{m_3}).$$

   Here, m_1, m_2 and m_3 are nonnegative integers satisfy some properties associated with {A5}.

3. Stefan problems with convection:

   $$\partial_t u+\nabla\cdot(\tilde{A}(x,t)u^{m_1})+\tilde{B}(x,t)u^{m_2}=\Delta(\tilde{\beta}(x,t)\widehat{\beta}(u)),$$

   where $\widehat{\beta}(\xi)=c_2\xi$ on $(-\infty,0)$, $\widehat{\beta}(\xi)=0$ on $[0,c_1]$, $\widehat{\beta}(\xi)=c_2(\xi-c_1)$ on $[c_1,\infty)$ for some constants $c_1,c_2>0$. Moreover, the nonnegative integers m_1 and m_2 satisfy some properties associated with {A5}.

4. Models of sedimentation consolidation processes:

$$\partial_t u + \nabla \cdot \{u(C_1 u + C_2(1-u)) - C_3 u^2(1-u)^2\} = \Delta(\widehat{\beta}(u)u(1-u)^2),$$

where $\widehat{\beta}(\xi) = 0$ on $(-\infty, 0)$, $\widehat{\beta}(\xi) = (c_2/c_1)\xi$ on $[0, c_1]$, $\widehat{\beta}(\xi) = c_2$ on (c_1, ∞) for some constants $c_1, c_2 > 0$. Moreover, C_1, C_2 and C_3 are some constants.

Next, we introduce generalized solutions to (P). We first define weak solutions to (P) as follows:

Definition 2.4. Let $u_0 \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$. A function $u \in L^1(\mathbb{R}^N_T) \cap L^\infty(\mathbb{R}^N_T)$ is called a *weak solution* to (P), if it satisfies the two conditions below:

(I) $\beta(x,t,u) \in L^2(0,T; H^1(\mathbb{R}^N))$,

(II) For $\varphi \in C_0^\infty(\mathbb{R}^N \times [0,T))$,

$$\int_{\mathbb{R}^N_T} (u\partial_t\varphi + (A(x,t,u) - \nabla\beta(x,t,u)) \cdot \nabla\varphi - B(x,t,u)\varphi)dxdt + \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)dx = 0.$$

Usually, the weak solution is interpreted as a generalized solution to equations with divergence form. Then, the existence and uniqueness of it may be shown. However, we can not prove the uniqueness of weak solutions to (P) in general. Because, discontinuities break out from the nonlinear convective term $\nabla \cdot A(x,t,u)$ and the uniqueness of weak solutions are possibly broken because of it. Therefore, we formulate the weak solution satisfying a special condition. It is called by the name entropy solution. In this paper, we try to prove the existence and uniqueness of entropy solutions in the range of BV space to (P). To see this, we state the concept of entropy:

Definition 2.5. Let $\eta(\xi) \in C^2(\mathbb{R})$ and $q(x,t,\xi), r(x,t,\xi) \in L^1(\mathbb{R}^N_T \times \mathbb{R})^N \cap L^\infty(\mathbb{R}^N_T \times \mathbb{R})^N$ satisfying $q(x,t,\cdot), r(x,t,\cdot) \in C^2(\mathbb{R})^N$ for $(x,t) \in \mathbb{R}^N_T$. A triplet (η, q, r) is entropy triplet to (DP) if it satisfies

$$\partial_\xi q(x,t,\xi) = \eta'(\xi)\partial_\xi A(x,t,\xi), \quad \partial_\xi r(x,t,\xi) = \eta'(\xi)\partial_\xi \nabla\beta(x,t,\xi),$$

for a.e. $(x,t,\xi) \in \mathbb{R}^N_T \times \mathbb{R}$. Then, η is called entropy and (q,r) is called entropy flux.

Using the above entropy triplet, it may be derived an entropy inequality through a formal calculation as follows:

$$\begin{aligned}
&\partial_t\eta(u) + \nabla \cdot q(x,t,u) - \nabla \cdot r(x,t,u) \\
&= \eta'(u)(-\nabla \cdot A(x,t,u) - B(x,t,u) + \Delta\beta(x,t,u)) + [\nabla \cdot q](x,t,u) + [\partial_\xi q](x,t,u) \cdot \nabla u \\
&\quad -[\nabla \cdot r](x,t,u) - [\partial_\xi r](x,t,u) \cdot \nabla u \\
&= -\eta'(u)[\nabla \cdot A](x,t,u) - \eta'(u)B(x,t,u) + [\nabla \cdot q](x,t,u) - [\nabla \cdot r](x,t,u) \\
&\quad +\nabla \cdot (\eta'(u)(\nabla\beta(x,t,u) - [\nabla\beta](x,t,u))) + \eta'(u)[\Delta\beta](x,t,u) - \eta''(u)[\partial_\xi\beta](x,t,u)|\nabla u|^2.
\end{aligned}$$

Therefore, it is deduced that

$$\begin{aligned}
&\partial_t \eta(u) + \nabla \cdot q(x,t,u) - \nabla \cdot r(x,t,u) \\
&\quad -[\nabla \cdot q](x,t,u) + [\nabla \cdot r](x,t,u) - \nabla \cdot (\eta'(u)(\nabla \beta(x,t,u) - [\nabla \beta](x,t,u))) \\
&\quad +\eta'(u)([\nabla \cdot A](x,t,u) - [\Delta \beta](x,t,u) + B(x,t,u)) + \eta''(u)\partial_\xi \beta(x,t,u)|\nabla u|^2 = 0.
\end{aligned}$$

In fact, we justify the above calculation as an inequality by using the artificial viscosity term $\beta_\varepsilon(x,t,\xi) = \beta(x,t,\xi) + \varepsilon\xi$. Using the inequality, we define BV-entropy solutions to (P).

Definition 2.6. Let $u_0 \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$. A function $u \in L^\infty(\mathbb{R}^N_T) \cap BV(\mathbb{R}^N_T)$ is called an *BV-entropy solution* to (P), if it satisfies the two conditions below:

(I) $u \in C([0,T]; L^1(\mathbb{R}^N))$ and L^1-$\lim_{t\downarrow 0} u(t) = u_0$,

(II) $\beta(x,t,u) \in L^2(0,T; H^1(\mathbb{R}^N))$, and for all $\varphi \in C_0^\infty(\mathbb{R}^N_T)^+$, $k \in \mathbb{R}$,

$$\begin{aligned}
&\int_{\mathbb{R}^N_T} \{\eta(u)\partial_t \varphi + (q(x,t,u) - r(x,t,u)) \cdot \nabla\varphi + ([\nabla \cdot q](x,t,u) - [\nabla \cdot r](x,t,u))\varphi \\
&\qquad -\eta'(u)(\nabla \beta(x,t,u) - [\nabla\beta](x,t,u)) \cdot \nabla \varphi \\
&\qquad -\eta'(u)([\nabla \cdot A](x,t,u) - [\Delta\beta](x,t,u) + B(x,t,u))\varphi\} dxdt \\
&\geq \int_{\mathbb{R}^N_T} \eta''(u)|\sqrt{\partial_\xi \beta(x,t,u)} Du|^2 \varphi dxdt.
\end{aligned}$$

Remark 2.7. The above definition is inspired by [6]. Feature of this is to include the right-hand side which is called by entropy dissipation term. Since u is BV-function, the first derivative of u with respect to x is a Radon measure. Therefore, the term seems non-standard integral. However, we can get $\sqrt{\partial_\xi \beta(x,t,u)} Du \in L^2(\mathbb{R}^N_T)$ in Lemma 3.7. Therefore, this estimate give a meaning of the integral. Moreover, the regularity $\beta(x,t,u) \in L^2(0,T; H^1(\mathbb{R}^N))$ and the assumption {A3} imply that

$$[\partial_\xi \beta](x,t,u) Du = \nabla\beta(x,t,u) - [\nabla\beta](x,t,u) \in L^2(0,T; L^2(\mathbb{R}^N))^N.$$

Remark 2.8. Usually, we do not use the entropy dissipation term in the definition of entropy solutions. However, the term is used to prove the uniqueness of entropy solutions to degenerate parabolic equations. This method was established by Carrillo [5]. He did not use the term in the definition of entropy solutions. Before proving the uniqueness of it, he derived the term using test functions [5, Lemma 5]. After, Chen-Karlsen [6] introduced the above definition to prove the uniqueness of entropy solutions directly.

In the regular coefficient case, we may construct the entropy solutions in the space $BV(\mathbb{R}^N)$. By the advantage of the setting, we may use the Fréchet-Kolmogorov compactness theorem and the Kružkov doubling variable method. Under the above statements, we obtain the following result:

Theorem 2.9. *We assume the conditions {A0}-{A7}. Then, the following statements hold:*

(I) There exists a BV-entropy solution u to (P).

(II) We additionally impose the assumption {A8}. Let u, v be a pair of BV-entropy solutions to (P) with initial values u_0 and v_0, respectively. Then, there exist positive constants α', C such that

$$\int_{\mathbb{R}^N} |u(x,t) - v(x,t)|dx \le e^{(\alpha'+C)t} \int_{\mathbb{R}^N} |u_0(x) - v_0(x)|dx,$$

for $t \in (0,T)$. In particular, for each initial value u_0, a BV-entropy solution is uniquely determined.

3 Existence of the BV-entropy solutions

In this section, we prove several estimates for the approximate solutions u_ε to obtain the existence of BV-entropy solutions to (P). To see this, we formulate the regularized problem for (P) as follows:

$$\begin{cases} \partial_t u_\varepsilon^\nu + \nabla \cdot A^\nu(x,t,u_\varepsilon^\nu) + B^\nu(x,t,u_\varepsilon^\nu) = \Delta\beta_\varepsilon^\nu(x,t,u_\varepsilon^\nu),\ (x,t) \in \mathbb{R}_T^N, \\ u_\varepsilon^\nu(x,0) = u_0(x),\ x \in \mathbb{R}^N, \end{cases} \qquad (\mathrm{P})_\varepsilon^\nu$$

where $\beta_\varepsilon^\nu(x,t,\xi) = \beta^\nu(x,t,\xi) + \varepsilon\xi$ for $\varepsilon > 0$. The functions $A^\nu(x,t,\xi)$, $B^\nu(x,t,\xi)$ and $\beta^\nu(x,t,\xi)$ are smooth regularizations with respect to (x,t,ξ) satisfying

$$A^\nu(x,t,\xi) \to A(x,t,\xi),\ B^\nu(x,t,\xi) \to B(x,t,\xi),\ \beta^\nu(x,t,\xi) \to \beta(x,t,\xi),$$

as $\nu \to 0$ and $\nabla\beta^\nu(x,t,0) - A^\nu(x,t,0) = \vec{0}$, $B^\nu(x,t,0) = 0$ for all $(x,t,\xi) \in \mathbb{R}_T^N \times \mathbb{R}$. For simplifying the notation, we write $A(x,t,\xi) := A^\nu(x,t,\xi)$, $B(x,t,\xi) := B^\nu(x,t,\xi)$ and $\beta(x,t,\xi) := \beta^\nu(x,t,\xi)$ and $u_\varepsilon := u_\varepsilon^\nu$ in this section. These regularizations are used in the proof of Lemma 3.2, 3.4 and 3.6. In fact, we use the boundedness of coefficients and the mean value theorem with respect to ξ.

Remark 3.1. Under the assumptions {A3} and {A7}, it follows that

$$g_1(|\xi|) \le [\partial_\xi\beta_\varepsilon](x,t,\xi) \le g_2(|\xi|),$$

for $(x,t,\xi) \in \mathbb{R}_T^N \times \mathcal{U}$. Here, $g_1(|\xi|)$ and $g_2(|\xi|)$ are positive valued nonincreasing, nondecreasing continuous functions defined for $\xi \in \mathcal{U}$. Therefore, $(P)_\varepsilon^\nu$ is a uniformly parabolic problem.

By the theory of second order uniformly parabolic equations ([12, Chapter V, §8, Theorem 8.1], [14, §4, Theorem 14]), there exists a unique classical solution $u_\varepsilon(x,t)$ to $(\mathrm{P})_\varepsilon^\nu$. Moreover, we use the approximated signum function $\mathrm{sgn}_\rho(\xi)$ for $\rho > 0$ by

$$\mathrm{sgn}_\rho(\xi) = \begin{cases} \mathrm{sgn}(\xi) & \text{for } |\xi| \ge \rho, \\ \sin\left(\dfrac{\pi}{2\rho}\xi\right) & \text{for } |\xi| < \rho. \end{cases}$$

We start with checking L^∞ and L^1-estimates for the approximate solutions u_ε.

Lemma 3.2 (L^∞ estimate)**.** *There exist positive constants c_0, c_1, independent of ε and ν, such that*

$$||u_\varepsilon(\cdot,t)||_{L^\infty(\mathbb{R}^N)} \le (||u_0||_{L^\infty(\mathbb{R}^N)} + c_0 t)e^{c_1 t},$$

for $0 \le t \le T$.

Proof. The approximate equation into $(\mathrm{P})^\nu_\varepsilon$ can be translated by the following form:

$$\partial_t u_\varepsilon + ([\partial_\xi A(x,t,u_\varepsilon) - 2[\partial_\xi \nabla\beta](x,t,u_\varepsilon)]) \cdot \nabla u_\varepsilon - [\partial_\xi^2 \beta](x,t,u_\varepsilon)|\nabla u_\varepsilon|^2$$
$$+([\nabla\cdot A](x,t,u_\varepsilon) - [\Delta\beta](x,t,u_\varepsilon) + B(x,t,u_\varepsilon)) = [\partial_\xi \beta_\varepsilon](x,t,u_\varepsilon)\Delta u_\varepsilon.$$

We put $\Psi(x,t,u_\varepsilon) \equiv [\nabla\cdot A](x,t,u_\varepsilon) - [\Delta\beta](x,t,u_\varepsilon) + B(x,t,u_\varepsilon)$. Then, there exist positive constants c_0 and c_1 such that

$$\sup_{(x,t)\in\mathbb{R}^N\times[0,T]} |\Psi(x,t,0)| \le c_0, \qquad \sup_{(x,t,\xi)\in\mathbb{R}^N\times[0,T]\times\mathbb{R}} (-\partial_\xi \Psi(x,t,\xi)) \le c_1,$$

by {A5}. Using the maximum principle [11, Section 4 (4.6)], we have

$$||u_\varepsilon(\cdot,t)||_{L^\infty(\mathbb{R}^N)} \le (||u_0||_{L^\infty(\mathbb{R}^N)} + c_0 t)e^{c_1 t},$$

for $0 \le t \le T$. □

Let $T > 0$. According to Lemma 3.2, we may set a constant $U > 0$ satisfying

$$(||u_0||_{L^\infty(\mathbb{R}^N)} + c_0 T)e^{c_1 T} < U.$$

Remark 3.3. We may replace the assumption {A5} by the following conditions:

{A5}' For some constants U_1 and U_2 with $U_1 \le 0 \le U_2$, the following conditions hold:

$$\begin{cases} U_1 \le u_0(x) \le U_2 \ \text{ for } x \in \mathbb{R}^N, \\ \nabla\cdot A(x,t,U_1) + B(x,t,U_1) - \Delta\beta(x,t,U_1) \le 0 \ \text{ for } (x,t) \in \mathbb{R}^N \times [0,T], \\ \nabla\cdot A(x,t,U_2) + B(x,t,U_2) - \Delta\beta(x,t,U_2) \ge 0 \ \text{ for } (x,t) \in \mathbb{R}^N \times [0,T]. \end{cases}$$

Then, the argument of the maximum principle implies that $U_1 \le u(x,t) \le U_2$ for $(x,t) \in \mathbb{R}^N_T$. Under this type of L^∞-estimate, we may expect the global existence of L^∞-entropy solutions (which do not included in BV) in the case that $B(x,t,\xi) \equiv 0$. However, {A5}' is sometimes strong restrictions to apply some mathematical models. In concern with physical setting, we necessitate to choice assumptions appropriately.

To prove L^1 estimate, we use a cut-off function. Let γ is a non-increasing C^2 function from $\mathbb{R}^+$ to $\mathbb{R}$ satisfying

$$\gamma(x) = 1 \ \text{ for } x \in [0, 2^{-1}], \quad e^{-1} \le \gamma(x) \le 1 \ \text{ for } x \in [2^{-1}, 1], \quad \gamma(x) = e^{-x} \ \text{ for } x \ge 1.$$

Moreover, we set $\gamma_R(x) = \gamma(|x|/R)$ for $R > 0$, $x \in \mathbb{R}^N$. Then, there exists a positive constant c such that

$$|\nabla\gamma_R(x)| \le \frac{c}{R}\gamma_R(x), \quad |\Delta\gamma_R(x)| \le \frac{c}{R^2}\gamma_R(x), \tag{3.1}$$

for $R > 0$ and $x \in \mathbb{R}^N$ (ref. [13, Theorem 2.38]).

Lemma 3.4 (L^1 estimate)**.** *For $0 \le s \le t \le T$, the following L^1-estimate holds:*

$$||u_\varepsilon(\cdot,t)||_{L^1(\mathbb{R}^N)} \le e^{\alpha'(t-s)}||u_\varepsilon(\cdot,s)||_{L^1(\mathbb{R}^N)},$$

where α' is a positive constant in (2.3).

Proof. Let us give the following approximate equation posed in $(\mathrm{P})_\varepsilon^\nu$:

$$\partial_t u_\varepsilon + \nabla\cdot A(x,t,u_\varepsilon) + B(x,t,u_\varepsilon) = \Delta\beta_\varepsilon(x,t,u_\varepsilon). \tag{3.2}$$

Multiplying both side on (3.2) by $\mathrm{sgn}_\rho(u_\varepsilon)\gamma_R(x)$ for $\rho, R > 0$ and integrating the result with respect to $(x,\tau) \in \mathbb{R}^N \times [s,t]$ implies

$$\begin{aligned}
&\int_s^t\int_{\mathbb{R}^N} \partial_\tau u_\varepsilon\, \mathrm{sgn}_\rho(u_\varepsilon)\gamma_R(x)dxd\tau \\
&= -\int_s^t\int_{\mathbb{R}^N} \mathrm{sgn}_\rho'(u_\varepsilon)\nabla u_\varepsilon\cdot([\nabla\beta](x,\tau,u_\varepsilon) - A(x,\tau,u_\varepsilon))\gamma_R(x)dxd\tau \\
&\quad -\int_s^t\int_{\mathbb{R}^N} \mathrm{sgn}_\rho'(u_\varepsilon)[\partial_\xi\beta_\varepsilon](x,\tau,u_\varepsilon)|\nabla u_\varepsilon|^2\gamma_R(x)dxd\tau \\
&\quad -\int_s^t\int_{\mathbb{R}^N} \mathrm{sgn}_\rho(u_\varepsilon)([\nabla\beta](x,\tau,u_\varepsilon) - A(x,\tau,u_\varepsilon))\cdot\nabla\gamma_R(x)dxd\tau \\
&\quad -\int_s^t\int_{\mathbb{R}^N} \mathrm{sgn}_\rho(u_\varepsilon)B(x,\tau,u_\varepsilon)\gamma_R(x)dxd\tau,
\end{aligned} \tag{3.3}$$

by (2.1) and (3.1). We consider about the first term of right-hand side on (3.3). By the assumption $[\nabla\beta](x,t,0) - A(x,t,0) = \vec{0}$, the continuity $[\nabla\beta](x,t,\cdot) - A(x,t,\cdot) \in C(\mathcal{U})^N$ for $(x,t) \in \mathbb{R}^N_T$ and the definition of $\mathrm{sgn}_\rho(\cdot)$, there exists a positive constant c such that

$$\mathrm{sgn}_\rho'(u_\varepsilon)\nabla u_\varepsilon\cdot([\nabla\beta](x,t,u_\varepsilon) - A(x,t,u_\varepsilon))\gamma_R(x) \le c\sum_{i=1}^N|\partial_{x_i}u_\varepsilon|,$$

for $(x,t) \in \mathbb{R}^N_T$. Moreover, the above boundedness implies that

$$\mathrm{sgn}_\rho'(u_\varepsilon)\nabla u_\varepsilon\cdot([\nabla\beta](x,t,u_\varepsilon) - A(x,t,u_\varepsilon))\gamma_R(x)\chi_{\{(x,t)\in\mathbb{R}^N_T;|u_\varepsilon(x,t)|<\rho\}}(x,t) \to 0,$$

as $\rho \downarrow 0$ for a.e. $(x,t) \in \mathbb{R}^N_T$. Here, $\chi_\Omega(x,t)$ is a characteristic function on $\Omega \subset \mathbb{R}^N_T$. By the Lebesgue dominated convergence theorem, it is deduced that

$$\int_s^t\int_{\mathbb{R}^N} \mathrm{sgn}_\rho'(u_\varepsilon)\nabla u_\varepsilon\cdot([\nabla\beta](x,t,u_\varepsilon) - A(x,t,u_\varepsilon))\gamma_R(x)dxdt \to 0,$$

as $\rho \to 0$. The second term of right-hand side on (3.3) is nonnegative by the property $\mathrm{sgn}_\rho'(\xi) \ge 0$ for all $\xi \in \mathbb{R}$ and (2.2). As $\rho \to 0$, it is deduced that

$$\int_s^t\int_{\mathbb{R}^N} \partial_t|u_\varepsilon|\gamma_R(x)dxdt \le (||\partial_\xi(\nabla\beta - A)||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})^N}\frac{c}{R} + \alpha')\int_s^t\int_{\mathbb{R}^N} |u_\varepsilon|\gamma_R(x)dxdt,$$

by {A4} and (3.1). Using the Gronwall inequality and letting $R \to \infty$, we have

$$||u_\varepsilon(\cdot,t)||_{L^1(\mathbb{R}^N)} \le e^{\alpha'(t-s)}||u_\varepsilon(\cdot,s)||_{L^1(\mathbb{R}^N)},$$

for all $0 \le s \le t \le T$.

□

Remark 3.5. The assumption $B(x,t,0) = 0$ for $(x,t) \in \mathbb{R}^N \times [0,T]$ in {A4} can be removed. In this case, we obtain the following estimate: for $0 \le s \le t \le T$,

$$||u_\varepsilon(\cdot,t)||_{L^1(\mathbb{R}^N)} \le e^{\alpha'(t-s)}||u_\varepsilon(\cdot,s)||_{L^1(\mathbb{R}^N)} + \int_s^t e^{\alpha'(t-\tau)}||B(x,\tau,0)||_{L^1(\mathbb{R}^N)}d\tau.$$

Next, we prove a BV-estimate. We begin by introducing a family of functions

$$I_\rho(\Lambda) = \int_0^{|\Lambda|} \operatorname{sgn}_\rho(\tau)d\tau,$$

for $\rho > 0$ and $\Lambda = (\Lambda_1, \ldots, \Lambda_N) \in \mathbb{R}^N$. These functions approximate the Euclidean norm $|\Lambda|$ and have the properties below:

$$\frac{\partial I_\rho}{\partial \Lambda_i} = \operatorname{sgn}_\rho(|\Lambda|)\frac{\Lambda_i}{|\Lambda|}, \quad \frac{\partial^2 I_\rho}{\partial\Lambda_i\partial\Lambda_j} = \frac{\operatorname{sgn}_\rho(|\Lambda|)'|\Lambda|\Lambda_i\Lambda_j - \operatorname{sgn}_\rho(|\Lambda|)\Lambda_i\Lambda_j}{|\Lambda|^3} + \frac{\operatorname{sgn}_\rho(|\Lambda|)}{|\Lambda|}\delta_{ij}.$$

Therefore, there exists a constant $M_0 > 0$ such that

$$\sqrt{\left|\frac{\partial^2 I_\rho}{\partial\Lambda_i\partial\Lambda_j}\right|} \le \frac{M_0}{|\Lambda|^{\frac{1}{2}}}. \tag{3.4}$$

Moreover, we denote that $\partial_{x_{N+1}} := \partial_t$ and $\widehat{\nabla} := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N}, \partial_{x_{N+1}})$. The following lemma is a BV-estimate and a key-estimate to get the convergence of approximate solutions u_ε.

Lemma 3.6 (BV-estimate)**.** *There exist positive constants M_0, M_1, M_2 and κ, independent of ε and ν, such that for $0 \le t \le T$,*

$$TV(u_\varepsilon) \le \int_{\mathbb{R}^N} |\widehat{\nabla} u_\varepsilon|dx \le e^{(M_1 + \frac{M_0^2(N+1)^2}{4\kappa})t}\left(\int_{\mathbb{R}^N} |\nabla u_0|dx + M_2\right).$$

Proof. Differentiating both side on (3.2) with respect to x_i, it follows that

$$\partial_{x_i}\partial_t u_\varepsilon = -\partial_{x_i}\nabla\cdot A(x,t,u_\varepsilon) - \partial_{x_i}B(x,t,u_\varepsilon) + \partial_{x_i}\Delta\beta_\varepsilon(x,t,u_\varepsilon).$$

Multiplying both side of the above identity by $\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}\gamma_R(x)$, then it is calculated that

$$
\begin{aligned}
&\sum_{i=1}^{N+1}\int_{\mathbb{R}_T^N}\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}\partial_{x_i}\partial_t u_\varepsilon\gamma_R(x)dxdt\\
&=\sum_{i=1}^{N+1}\int_{\mathbb{R}_T^N}\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}\{-\nabla\cdot([\partial_{x_i}A](x,t,u_\varepsilon)+[\partial_\xi A](x,t,u_\varepsilon)\partial_{x_i}u_\varepsilon)\\
&\qquad-[\partial_{x_i}B](x,t,u_\varepsilon)-[\partial_\xi B](x,t,u_\varepsilon)\partial_{x_i}u_\varepsilon\\
&\qquad+\nabla\cdot([\partial_{x_i}\partial_\xi\beta](x,t,u_\varepsilon)\nabla u_\varepsilon+[\partial_\xi\beta_\varepsilon](x,t,u_\varepsilon)\partial_{x_i}\nabla u_\varepsilon)\\
&\qquad+\nabla\cdot([\partial_{x_i}\nabla\beta](x,t,u_\varepsilon)+[\partial_\xi\nabla\beta](x,t,u_\varepsilon)\partial_{x_i}u_\varepsilon+[\partial_\xi^2\beta](x,t,u_\varepsilon)\nabla u_\varepsilon\partial_{x_i}u_\varepsilon)\}\gamma_R(x)dxdt\\
&\equiv\sum_{i=1}^{N+1}\left(\sum_{k=1}^{2}I_{A,i}^k+\sum_{k=1}^{2}I_{B,i}^k+\sum_{k=1}^{5}I_{\beta,i}^k\right).
\end{aligned}
$$

We estimate each term of the above equality respectively. First, $\sum_{i=1}^{N+1}I_{A,i}^2$ is equal to

$$
\begin{aligned}
&-\sum_{i=1}^{N+1}\int_{\mathbb{R}_T^N}\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}(\nabla\cdot[\partial_\xi A](x,t,u_\varepsilon)\partial_{x_i}u_\varepsilon+[\partial_\xi A](x,t,u_\varepsilon)\cdot\nabla\partial_{x_i}u_\varepsilon)\gamma_R(x)dxdt\\
&\quad=-\int_{\mathbb{R}_T^N}(\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)\nabla\cdot[\partial_\xi A](x,t,u_\varepsilon)|\widehat{\nabla}u_\varepsilon|+[\partial_\xi A](x,t,u_\varepsilon)\cdot\nabla I_\rho(\widehat{\nabla}u_\varepsilon))\gamma_R(x)dxdt\\
&\quad=-\int_{\mathbb{R}_T^N}\nabla\cdot[\partial_\xi A](x,t,u_\varepsilon)(|\widehat{\nabla}u_\varepsilon|\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)-I_\rho(\widehat{\nabla}u_\varepsilon))\gamma_R(x)dxdt\\
&\qquad+\int_{\mathbb{R}_T^N}[\partial_\xi A](x,t,u_\varepsilon)\cdot\nabla\gamma_R(x)I_\rho(\widehat{\nabla}u_\varepsilon)dxdt.
\end{aligned}
$$

Here, we use the following property:

$$
\frac{\partial}{\partial x_j}I_\rho(\widehat{\nabla}u_\varepsilon)=\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}\sum_{i=1}^{N+1}\partial_{x_i}u_\varepsilon\partial_{x_j}\partial_{x_i}u_\varepsilon,
$$

and the Gauss divergence theorem. Similarly, $\sum_{i=1}^{N+1}I_{\beta,i}^4$ is equal to

$$
\begin{aligned}
&\sum_{i=1}^{N+1}\int_{\mathbb{R}_T^N}\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}(\nabla\cdot[\partial_\xi\nabla\beta](x,t,u_\varepsilon)\partial_{x_i}u_\varepsilon+[\partial_\xi\nabla\beta](x,t,u_\varepsilon)\cdot\nabla\partial_{x_i}u_\varepsilon)\gamma_R(x)dxdt\\
&=\int_{\mathbb{R}_T^N}\nabla\cdot[\partial_\xi\nabla\beta](x,t,u_\varepsilon)(|\widehat{\nabla}u_\varepsilon|\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)-I_\rho(\widehat{\nabla}u_\varepsilon))\gamma_R(x)dxdt\\
&\quad-\int_{\mathbb{R}_T^N}[\partial_\xi\nabla\beta](x,t,u_\varepsilon)\cdot\nabla\gamma_R(x)I_\rho(\widehat{\nabla}u_\varepsilon)dxdt.
\end{aligned}
$$

Moreover, $\sum_{i=1}^{N+1} I^5_{\beta,i}$ is equal to

$$\sum_{i=1}^{N+1}\int_{\mathbb{R}^N_T}\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}(\nabla[\partial^2_\xi\beta](x,t,u_\varepsilon)\cdot\nabla u_\varepsilon\partial_{x_i}u_\varepsilon$$
$$+[\partial^2_\xi\beta](x,t,u_\varepsilon)\Delta u_\varepsilon\partial_{x_i}u_\varepsilon+[\partial^2_\xi\beta](x,t,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla\partial_{x_i}u_\varepsilon)\gamma_R(x)dxdt$$
$$=\int_{\mathbb{R}^N_T}(\nabla[\partial^2_\xi\beta](x,t,u_\varepsilon)\cdot\nabla u_\varepsilon+[\partial^2_\xi\beta](x,t,u_\varepsilon)\Delta u_\varepsilon)(|\widehat{\nabla}u_\varepsilon|\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)-I_\rho(\widehat{\nabla}u_\varepsilon))\gamma_R(x)dxdt$$
$$-\int_{\mathbb{R}^N_T}[\partial^2_\xi\beta](x,t,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla\gamma_R(x)I_\rho(\widehat{\nabla}u_\varepsilon)dxdt,$$

by the regularity of β_ε. Next, we estimate the remaining terms as inequalities below:

$$\sum_{i=1}^{N+1}(I^1_{A,i}+I^3_{\beta,i})=\sum_{i=1}^{N+1}\int_{\mathbb{R}^N_T}\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}\{([\partial_{x_i}\Delta\beta](x,t,u_\varepsilon)-[\nabla\cdot\partial_{x_i}A](x,t,u_\varepsilon))$$
$$+([\partial_\xi\partial_{x_i}\nabla\beta](x,t,u_\varepsilon)-[\partial_\xi\partial_{x_i}A](x,t,u_\varepsilon))\cdot\nabla u_\varepsilon\}\gamma_R(x)dxdt$$
$$\le M_3+M_4(N+1)\int_{\mathbb{R}^N_T}|\nabla u_\varepsilon|\gamma_R(x)dxdt,$$

by {A6}. Here, M_3 and M_4 are positive constants satisfying

$$\int_{\mathbb{R}^N_T}\left|\sum_{i=1}^{N+1}(\partial_{x_i}\Delta\beta(x,t,u_\varepsilon)-[\nabla\cdot\partial_{x_i}A](x,t,u_\varepsilon))^2\right|^{\frac{1}{2}}dxdt\le M_3,$$

and

$$||\partial_\xi\partial_{x_i}\nabla\beta-\partial_\xi\partial_{x_i}A||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})^N}\le M_4,$$

for $i=1,\dots,N+1$. Moreover, it can be checked that

$$\sum_{i=1}^{N+1}(I^1_{B,i}+I^2_{B,i})$$
$$\le\int_{\mathbb{R}^N_T}\left|[\widehat{\nabla}B](x,t,u_\varepsilon)\right|\gamma_R(x)dxdt-\int_{\mathbb{R}^N_T}|\widehat{\nabla}u_\varepsilon|\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)[\partial_\xi B](x,t,u_\varepsilon)\gamma_R(x)dxdt$$
$$\le M_5+\alpha'\int_{\mathbb{R}^N_T}|\widehat{\nabla}u_\varepsilon|\gamma_R(x)dxdt,$$

where M_5 is a positive constant by {A2}. Here, we deal with the terms $I^1_{\beta,i}$ and $I^2_{\beta,i}$. It is crucial to treat these terms appropriately in this proof. In this paper, we overcome this difficulty by using the assumption {A7}. In fact, $\sum_{i=1}^{N+1}(I^1_{\beta,i}+I^2_{\beta,i})$ is equal to

$$-\sum_{i,k=1}^{N+1}\int_{\mathbb{R}^N_T}\frac{\partial^2 I_\rho(\widehat{\nabla}u_\varepsilon)}{\partial\Lambda_i\partial\Lambda_k}\nabla\partial_{x_k}u_\varepsilon\cdot([\partial_\xi\partial_{x_i}\beta](x,t,u_\varepsilon)\nabla u_\varepsilon+[\partial_\xi\beta_\varepsilon](x,t,u_\varepsilon)\nabla\partial_{x_i}u_\varepsilon)\gamma_R(x)dxdt$$
$$-\sum_{i=1}^{N+1}\int_{\mathbb{R}^N_T}\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}([\partial_\xi\partial_{x_i}\beta](x,t,u_\varepsilon)\nabla u_\varepsilon+[\partial_\xi\beta_\varepsilon](x,t,u_\varepsilon)\nabla\partial_{x_i}u_\varepsilon)\cdot\nabla\gamma_R(x)dxdt.$$

The first term of the above equation is estimated as follows

$$\sum_{i,k=1}^{N+1}\sum_{j=1}^{N}\int_{\mathbb{R}_T^N}\left\{\frac{1}{4\kappa}\left(\sqrt{\left|\frac{\partial^2 I_\rho(\widehat{\nabla}u_\varepsilon)}{\partial\Lambda_i\partial\Lambda_k}\right|}\partial_{x_j}u_\varepsilon\right)^2\right.$$

$$+\kappa\left([\partial_\xi\partial_{x_i}\beta](x,t,u_\varepsilon)\sqrt{\left|\frac{\partial^2 I_\rho(\widehat{\nabla}u_\varepsilon)}{\partial\Lambda_i\partial\Lambda_k}\right|}\partial_{x_j}\partial_{x_k}u_\varepsilon\right)^2$$

$$\left.-[\partial_\xi\beta](x,t,u_\varepsilon)\left(\sqrt{\left|\frac{\partial^2 I_\rho(\widehat{\nabla}u_\varepsilon)}{\partial\Lambda_i\partial\Lambda_k}\right|}\partial_{x_j}\partial_{x_k}u_\varepsilon\right)\left(\sqrt{\left|\frac{\partial^2 I_\rho(\widehat{\nabla}u_\varepsilon)}{\partial\Lambda_i\partial\Lambda_k}\right|}\partial_{x_j}\partial_{x_i}u_\varepsilon\right)\right\}\gamma_R(x)dxdt$$

$$\le\sum_{i,k=1}^{N+1}\sum_{j=1}^{N}\frac{1}{4\kappa}\int_{\mathbb{R}_T^N}\left(\sqrt{\left|\frac{\partial^2 I_\rho(\widehat{\nabla}u_\varepsilon)}{\partial\Lambda_i\partial\Lambda_k}\right|}\partial_{x_j}u_\varepsilon\right)^2\gamma_R(x)dxdt$$

$$\le\frac{M_0^2(N+1)^2}{4\kappa}\int_{\mathbb{R}_T^N}|\widehat{\nabla}u_\varepsilon|\gamma_R(x)dxdt,$$

by using the Gauss divergence theorem, {A7} and (3.4). On the other hand, it can be seen that

$$\sum_{i=1}^{N+1}\partial_{x_i}u\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}\partial_{x_i}\partial_t u_\varepsilon=\sum_{i=1}^{N+1}\frac{\partial I_\rho(\widehat{\nabla}u_\varepsilon)}{\partial\Lambda_i}\partial_t\partial_{x_i}u_\varepsilon=\frac{d}{dt}I_\rho(\widehat{\nabla}u_\varepsilon).$$

By the above calculations, we obtain the following inequality:

$$\int_{\mathbb{R}_t^N}\frac{d}{ds}I_\rho(\widehat{\nabla}u_\varepsilon)\gamma_R(x)dxds$$

$$\le M_3+M_5+(M_4(N+1)+\alpha'+\frac{M_0^2(N+1)^2}{4\kappa})\int_{\mathbb{R}_t^N}|\widehat{\nabla}u_\varepsilon|dxds$$

$$+\int_{\mathbb{R}_t^N}(-\nabla\cdot[\partial_\xi A](x,s,u_\varepsilon)+\nabla\cdot[\partial_\xi\nabla\beta](x,s,u_\varepsilon)+\nabla[\partial_\xi^2\beta](x,s,u_\varepsilon)\cdot\nabla u_\varepsilon$$

$$+[\partial_\xi^2\beta](x,s,u_\varepsilon)\mathit{\Delta}u_\varepsilon)(|\widehat{\nabla}u_\varepsilon|\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)-I_\rho(\widehat{\nabla}u_\varepsilon))dxds$$

$$+\int_{\mathbb{R}_t^N}\{([\partial_\xi A](x,s,u_\varepsilon)-[\partial_\xi\nabla\beta](x,s,u_\varepsilon)-[\partial_\xi^2\beta](x,s,u_\varepsilon)\nabla u_\varepsilon)I_\rho(\widehat{\nabla}u_\varepsilon)$$

$$-\sum_{i=1}^{N+1}\partial_{x_i}u_\varepsilon\frac{\operatorname{sgn}_\rho(|\widehat{\nabla}u_\varepsilon|)}{|\widehat{\nabla}u_\varepsilon|}([\partial_{x_i}\partial_\xi\beta](x,s,u_\varepsilon)\nabla u_\varepsilon+[\partial_\xi\beta_\varepsilon](x,s,u_\varepsilon)\partial_{x_i}\nabla u_\varepsilon)\}\cdot\nabla\gamma_R(x)dxds,$$

for any $t \in (0,T]$. Letting $\rho \to 0$, it is deduced that

$$\int_{\mathbb{R}^N} |\widehat{\nabla} u_\varepsilon| dx \le \int_{\mathbb{R}^N} |\nabla u_0| dx + M_2 + (M_1 + \frac{M_0^2(N+1)^2}{4\kappa}) \int_{\mathbb{R}^N_t} |\widehat{\nabla} u_\varepsilon| dxds$$
$$+\frac{c}{R}\int_{\mathbb{R}^N_t} \{|[\partial_\xi A](x,s,u_\varepsilon)| + |[\partial_\xi \nabla\beta](x,s,u_\varepsilon)| + |[\partial_\xi^2 \beta](x,s,u_\varepsilon)\nabla u_\varepsilon|$$
$$+\sqrt{2}\sum_{i=1}^{N+1}(|[\partial_{x_i}\partial_\xi\beta](x,s,u_\varepsilon)\nabla u_\varepsilon| + |[\partial_\xi\beta_\varepsilon](x,s,u_\varepsilon)\partial_{x_i}\nabla u_\varepsilon|)\}|\widehat{\nabla} u_\varepsilon|\gamma_R(x)dxds,$$

where $M_1 = M_4(N+1) + \alpha'$ and $M_2 = M_3 + M_5$. Using the Gronwall inequality and letting $R \to \infty$, we get the desired estimate. □

Moreover, we can obtain the following regularity estimate for the diffusion term along similar lines as in [17, Lemma 3.2].

Lemma 3.7. *There exists a positive constant c, independent of ε and ν, such that*

$$||\nabla\beta(x,t,u_\varepsilon)||_{L^2(\mathbb{R}^N_T)^N} \le c,$$

and

$$||\beta(x,\cdot+\tau,u_\varepsilon(x,\cdot+\tau)) - \beta(x,\cdot,u_\varepsilon(x,\cdot))||_{L^2(\mathbb{R}^N_{T-\tau})} \le c\sqrt{\tau},$$

for all $0 \le \tau < T$. In particular, $\{\beta(x,t,u_\varepsilon)\}_{\varepsilon>0}$ is strongly compact in $L^2_{loc}(\mathbb{R}^N_T)$.

Proof. Multiplying both side on the equality (3.2) by u_ε and integrating the equality with respect to $(x,t) \in \mathbb{R}^N_T$, then it follows that

$$\int_{\mathbb{R}^N_T} \left(\frac{1}{2}\partial_t(u_\varepsilon)^2 + u_\varepsilon\nabla\cdot A(x,t,u_\varepsilon) + u_\varepsilon B(x,t,u_\varepsilon)\right) dxdt = \int_{\mathbb{R}^N_T} u_\varepsilon \Delta\beta_\varepsilon(x,t,u_\varepsilon)dxdt.$$

The first term of the left-hand side on the above equality is calculated by

$$\int_{\mathbb{R}^N_T} \frac{1}{2}\partial_t(u_\varepsilon)^2 dxdt = \frac{1}{2}||u_\varepsilon(\cdot,T)||^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}||u_\varepsilon(\cdot,0)||^2_{L^2(\mathbb{R}^N)}.$$

Secondly, we consider the convective and diffusion terms. By the Gauss divergence theorem, the chain rule formula, {A1}, {A3} and {A6}, the following equality holds:

$$\int_{\mathbb{R}^N_T} (u_\varepsilon\nabla\cdot A(x,t,u_\varepsilon) - u_\varepsilon\Delta\beta_\varepsilon(x,t,u_\varepsilon))dxdt$$
$$= -\int_{\mathbb{R}^N_T} \nabla\cdot\left(\int_0^{u_\varepsilon}(A(x,t,\xi) - [\nabla\beta](x,t,\xi))d\xi\right)dxdt - \int_{\mathbb{R}^N_T}[\partial_\xi\beta_\varepsilon](x,t,u_\varepsilon)|\nabla u_\varepsilon|^2 dxdt$$
$$+\int_{\mathbb{R}^N_T}\int_0^{u_\varepsilon}([\nabla\cdot A](x,t,\xi) - [\Delta\beta](x,t,\xi))d\xi dxdt.$$

By the Gauss divergence theorem, the integrability of $A(\cdot,t,\xi)$, $\nabla\beta(\cdot,t,\xi)$ for $(t,\xi)\in[0,T]\times\mathcal{U}$ and the boundedness of u_ε, the first term of the above equality is equal to zero. Therefore, it is deduced that

$$\begin{aligned}\int_{\mathbb{R}^N_T}[\partial_\xi\beta_\varepsilon](x,t,u_\varepsilon)|\nabla u_\varepsilon|^2dxdt &\le \frac{1}{2}||u_\varepsilon(\cdot,0)||^2_{L^2(\mathbb{R}^N)}-\int_{\mathbb{R}^N_T}u_\varepsilon B(x,t,u_\varepsilon)dxdt\\ &-\int_{\mathbb{R}^N_T}\int_0^{u_\varepsilon}([\nabla\cdot A](x,t,\xi)-[\varDelta\beta](x,t,\xi))d\xi dxdt.\end{aligned}\tag{3.5}$$

The second and third terms of the right-hand side in (3.5) are bounded by the assumptions {A2} and {A6}. Hence, there exists a positive constant c such that

$$\int_{\mathbb{R}^N_T}[\partial_\xi\beta_\varepsilon](x,t,u_\varepsilon)|\nabla u_\varepsilon|^2dxdt\le\frac{1}{2}||u_\varepsilon(\cdot,0)||^2_{L^2(\mathbb{R}^N)}+c.\tag{3.6}$$

On this basis, it is obtained that

$$\begin{aligned}\int_{\mathbb{R}^N_T}|\nabla\beta(x,t,u_\varepsilon)|^2dxdt\le 2\int_{\mathbb{R}^N_T}&|[\nabla\beta](x,t,u_\varepsilon)|^2dxdt\\ &+2||\partial_\xi\beta||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}\int_{\mathbb{R}^N_T}[\partial_\xi\beta](x,t,u_\varepsilon)|\nabla u_\varepsilon|^2dxdt.\end{aligned}$$

By (3.6) and {A3}, the right-hand side on the above inequality is bounded. Therefore, we obtain the first assertion.

On the other hand, we verify the estimate with respect to the time variable t. Using the equation (3.2), it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N_{T-\tau}}(\beta(x,t+\tau,u_\varepsilon(x,t+\tau))-\beta(x,t,u_\varepsilon(x,t)))^2dxdt\\ &\le 2\tau\int_{\mathbb{R}^N_{T-\tau}}\left(\int_0^1[\partial_t\beta](x,t+\sigma\tau,u_\varepsilon(x,t+\tau))d\sigma\right)\\ &\qquad\times(\beta(x,t+\tau,u_\varepsilon(x,t+\tau))-\beta(x,t,u_\varepsilon(x,t+\tau)))dxdt\\ &\quad+2||[\partial_\xi\beta](x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}\int_{\mathbb{R}^N_{T-\tau}}\left(\int_t^{t+\tau}\partial_t u_\varepsilon(x,\zeta)d\zeta\right)\\ &\qquad\times(\beta(x,t,u_\varepsilon(x,t+\tau))-\beta(x,t,u_\varepsilon(x,t)))dxdt\\ &\le 4\tau||\beta(x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}||[\partial_t\beta](x,t,\xi)||_{L^\infty(\mathcal{U};L^1(\mathbb{R}^N_T))}+2||[\partial_\xi\beta](x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}\\ &\quad\times\int_{\mathbb{R}^N_{T-\tau}}\left(\int_t^{t+\tau}(-\nabla\cdot A(x,\zeta,u_\varepsilon)-B(x,\zeta,u_\varepsilon)+\varDelta\beta_\varepsilon(x,\zeta,u_\varepsilon))d\zeta\right)\\ &\qquad\times(\beta(x,t,u_\varepsilon(x,t+\tau))-\beta(x,t,u_\varepsilon(x,t)))dxdt.\end{aligned}$$

Here, by the boundedness of u_ε and {A3}, $||\beta(x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}$, $||[\partial_t\beta](x,t,\xi)||_{L^\infty(\mathcal{U};L^1(\mathbb{R}^N_T))}$, $||[\partial_\xi\beta](x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}$ are bounded. Changing the variable $\zeta=t+s$, integrating by parts with respect to x, then there exists a positive constant c such that

$$
\begin{aligned}
&\int_{\mathbb{R}^N_{T-\tau}} (\beta(x,t+\tau,u_\varepsilon(x,t+\tau)) - \beta(x,t,u_\varepsilon(x,t)))^2 dxdt \\
&\leq 4\tau||\beta(x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}||[\partial_t\beta](x,t,\xi)||_{L^\infty(\mathcal{U};L^1(\mathbb{R}^N_T))} + 2||[\partial_\xi\beta](x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)} \\
&\quad \times \int_0^\tau \int_{\mathbb{R}^N_{T-\tau}} \{-\nabla\cdot A(x,t+s,u_\varepsilon(x,t+s)) - B(x,t+s,u_\varepsilon(x,t+s)) \\
&\qquad + \Delta\beta_\varepsilon(x,t+s,u_\varepsilon(x,t+s))\}\{\beta(x,t,u_\varepsilon(x,t+\tau)) - \beta(x,t,u_\varepsilon(x,t))\} dxdtds \\
&= 4\tau||\beta(x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}||[\partial_t\beta](x,t,\xi)||_{L^\infty(\mathcal{U};L^1(\mathbb{R}^N_T))} + 2||[\partial_\xi\beta](x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)} \\
&\quad \times \int_0^\tau \int_{\mathbb{R}^N_{T-\tau}} (A(x,t+s,u_\varepsilon(x,t+s))\cdot\{\nabla\beta(x,t,u_\varepsilon(x,t+\tau)) - \nabla\beta(x,t,u_\varepsilon(x,t))\} \\
&\qquad - B(x,t+s,u_\varepsilon(x,t+s))\{\beta(x,t,u_\varepsilon(x,t+\tau)) - \beta(x,t,u_\varepsilon(x,t))\} \\
&\qquad - \nabla\beta_\varepsilon(x,t+s,u_\varepsilon(x,t+s))\cdot\{\nabla\beta(x,t,u_\varepsilon(x,t+\tau)) - \nabla\beta(x,t,u_\varepsilon(x,t))\}) dxdtds \\
&\leq 4\tau||\beta(x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}||[\partial_t\beta](x,t,\xi)||_{L^\infty(\mathcal{U};L^1(\mathbb{R}^N_T))} \\
&\quad + 4\tau||[\partial_\xi\beta](x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)}(||B(x,t,\xi)||_{L^\infty(\mathcal{U};L^1(\mathbb{R}^N_T))}||\beta(x,t,u_\varepsilon)||_{L^\infty(\mathbb{R}^N_T)} \\
&\qquad\quad + (||A(x,t,\xi)||_{L^\infty(\mathcal{U};L^2(\mathbb{R}^N_T))^N} + ||\nabla\beta_\varepsilon(x,t,u_\varepsilon)||_{L^2(\mathbb{R}^N_T)^N})||\nabla\beta(x,t,u_\varepsilon)||_{L^2(\mathbb{R}^N_T)^N}) \\
&\leq c\tau,
\end{aligned}
$$

by {A1}-{A3} and the first assertion of this lemma. Consequently, we obtain the desired estimates. Using the Fréchet-Kolmogorov compactness criterion, we conclude the proof of this lemma. □

3.1 Proof of Theorem 2.9 (I)

Let $\nu = C\varepsilon$, for a positive constant C. We set the family $\{u^\nu_\varepsilon\}_{\varepsilon,\nu>0} \equiv \{u_\varepsilon\}_{\varepsilon>0}$. To prove the convergence of $\{u_\varepsilon\}_{\varepsilon>0}$, we should prove the decay property of $u_\varepsilon(x)$ as $|x| \to \infty$. To see this, we prepare the sequence of functions $\{\tilde{\gamma}^{(1)}_{r,R}\ ;\ 0 < r < R\} \subset C^\infty(\mathbb{R})$ such that

$$
\tilde{\gamma}^{(1)}_{r,R}(s) = \begin{cases} 0 & \text{for } |s| < r, \\ \sin(\frac{\pi}{2(R-r)}|s-r|) & \text{for } r \leq |s| \leq R, \\ 1 & \text{for } |s| > R. \end{cases}
$$

Then, it follows that $0 \leq \tilde{\gamma}^{(1)}_{r,R} \leq 1$ and $|(\tilde{\gamma}^{(1)}_{r,R})'| \leq \frac{\pi}{2(R-r)}$ in $\mathbb{R}$. In addition, we define $\tilde{\gamma}_{r,R}(x) \equiv \prod_{i=1}^N \tilde{\gamma}^{(1)}_{r,R}(x_i)$ for $x \in \mathbb{R}^N$.

In the proof of Lemma 3.4, we use $\tilde{\gamma}_{r,R}$ instead of γ_R. Then, it is deduced that

$$\int_0^t \int_{\mathbb{R}^N} \partial_t |u_\varepsilon| \tilde{\gamma}_{r,R}(x) dxdt$$
$$\leq \int_0^t \int_{\mathbb{R}^N} ||\partial_\xi (\nabla \beta_\varepsilon - A)||_{L^\infty(\mathbb{R}_T^N \times \mathcal{U})^N} \frac{\pi}{2(R-r)} |u_\varepsilon| dxdt + \alpha' \int_0^t \int_{\mathbb{R}^N} |u_\varepsilon| \tilde{\gamma}_{r,R}(x) dxdt.$$

Using the Gronwall inequality, it can be seen that

$$\lim_{R\to\infty} \int_{|x|>R} |u_\varepsilon(\cdot,t)| dx \leq e^{\alpha' t} \int_{|x|>r} |u_0| dx,$$

for each $r > 0$ and $t \in [0,T]$. Since u_0 belongs to $L^1(\mathbb{R}^N)$, the right-hand side of the above inequality goes to zero as $r \to \infty$. Therefore, the equi-integrability of u_ε holds. By Lemma 3.2, Lemma 3.4, Lemma 3.6 and the above estimate, the Fréchet-Kolmogorov theorem implies that there exists a subsequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ and

$$u_{\varepsilon_n} \to u \quad \text{in } L^1(\mathbb{R}_T^N),$$

as $n \to \infty$. Moreover, it is also deduced that $u \in C([0,T]; L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}_T^N) \cap BV(\mathbb{R}_T^N)$.

Finally, we prove the entropy inequality in Definition 2.6. By the L^1-convergence of u_ε, the boundedness u_ε, $u \in L^\infty(\mathbb{R}_T^N)$ and the L^2_{loc}-weak convergence of $\nabla\beta_\varepsilon(x,t,u_\varepsilon)$, the left-hand side of the entropy inequality in Definition 2.6 holds. Moreover, the lower semicontinuity of L^2-norm implies that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_T^N} \eta''(u_\varepsilon) |\sqrt{\partial_\xi \beta_\varepsilon(x,t,u_\varepsilon)} \nabla u_\varepsilon|^2 \varphi dxdt \geq \int_{\mathbb{R}_T^N} \eta''(u) |\sqrt{\partial_\xi \beta(x,t,u)} Du|^2 \varphi dxdt.$$

Hence, Theorem 2.9 (I) is shown.

4 Uniqueness of the BV-entropy solutions

Let $\varphi \in C_0^\infty(\mathbb{R}_T^N)^+$. In addition, we introduce a symmetric function $\theta \in C_0^\infty(\mathbb{R})^+$ satisfying $\int_{\mathbb{R}} \theta(t) dt = 1$ and $\mathrm{supp}[\theta(t)] \subset \{|t| \leq 1\}$. Similarly, we use a spherically symmetric function $\omega \in C_0^\infty(\mathbb{R}^N)^+$ satisfying $\int_{\mathbb{R}^N} \omega(x) dx = 1$ and $\mathrm{supp}[\omega(x)] \subset \{|x| \leq 1\}$. Let $\delta_0, \delta > 0$ and define $\theta_{\delta_0}(t) = (1/\delta_0)\theta(t/\delta_0)$ and $\omega_\delta(x) = (1/\delta^N)\omega(x/\delta)$. These are smooth functions on $\mathbb{R}$ and $\mathbb{R}^N$, respectively, and satisfy

$$\lim_{\delta_0 \downarrow 0} \int_0^T \theta_{\delta_0}(t) \varphi(x,t) dt = \varphi(x,0), \quad \lim_{\delta\downarrow 0} \int_{\mathbb{R}^N} \omega_\delta(x) \varphi(x,t) dx = \varphi(0,t),$$

for $(x,t) \in \mathbb{R}_T^N$. We now employ the test function $\phi_\delta^{\delta_0}$ defined by

$$\phi_\delta^{\delta_0}(x,y,t,s) = \varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \omega_\delta\left(\frac{x-y}{2}\right) \theta_{\delta_0}\left(\frac{t-s}{2}\right), \tag{4.1}$$

for $(x,t,y,s) \in \mathbb{R}^N_T \times \mathbb{R}^N_T$. Then, the following property holds

$$\lim_{\delta \downarrow 0} \int_{\mathbb{R}^N \times \mathbb{R}^N} |x-y| \left| \omega_\delta \left(\frac{x-y}{2} \right) \right| dxdy = 0,$$

and there exists a positive constant C such that

$$\begin{aligned}
&\lim_{\delta \downarrow 0} \int_{\mathbb{R}^N_T \times \mathbb{R}^N} |x-y| \left| \partial_{x_i} \omega_\delta \left(\frac{x-y}{2} \right) \right| \varphi \left(\frac{x+y}{2}, t \right) dxdydt \leq C \int_{\mathbb{R}^N_T} \varphi(x,t) dxdt, \\
&\lim_{\delta \downarrow 0} \int_{\mathbb{R}^N_T \times \mathbb{R}^N} |x-y|^2 \left| \partial_{x_i} \partial_{x_j} \omega_\delta \left(\frac{x-y}{2} \right) \right| \varphi \left(\frac{x+y}{2}, t \right) dxdydt \leq C \int_{\mathbb{R}^N_T} \varphi(x,t) dxdt,
\end{aligned} \tag{4.2}$$

for $1 \leq i, j \leq N$. Moreover, we have

$$\nabla_x \omega_\delta \left(\frac{x-y}{2} \right) + \nabla_y \omega_\delta \left(\frac{x-y}{2} \right) = 0, \quad (\nabla_x + \nabla_y) \phi^{\delta_0}_\delta = (\nabla_x + \nabla_y) \varphi \omega_\delta \theta_{\delta_0}.$$

In this section, the proof of Theorem 2.9 (II) is presented. Hereafter, we give the entropy triplet in the following concrete form:

$$\begin{aligned}
\eta(u) = \eta_\rho(u) &\equiv \int_k^u \operatorname{sgn}_\rho(\xi - k) d\xi, \\
q(x,t,u) = q_\rho(x,t,u) &\equiv \int_k^u \operatorname{sgn}_\rho(\xi - k) [\partial_\xi A](x,t,\xi) d\xi, \\
r(x,t,u) = r_\rho(x,t,u) &\equiv \int_k^u \operatorname{sgn}_\rho(\xi - k) [\partial_\xi \nabla \beta](x,t,\xi) d\xi,
\end{aligned}$$

for $k \in \mathbb{R}$. Then, it can be seen that

$$\begin{aligned}
\eta_\rho(u) \to |u-k|, \quad q_\rho(x,t,u) &\to \operatorname{sgn}(u-k)(A(x,t,u) - A(x,t,k)), \\
r_\rho(x,t,u) &\to \operatorname{sgn}(u-k)([\nabla \beta](x,t,u) - [\nabla \beta](x,t,k)),
\end{aligned}$$

as $\rho \to 0$. Moreover, we obtain that

$$\begin{aligned}
[\nabla \cdot q_\rho](x,t,u) &= \int_k^u \operatorname{sgn}_\rho(\xi - k) [\partial_\xi \nabla \cdot A](x,t,\xi) d\xi, \\
[\nabla \cdot r_\rho](x,t,u) &= \int_k^u \operatorname{sgn}_\rho(\xi - k) [\partial_\xi \varDelta \beta](x,t,\xi) d\xi.
\end{aligned}$$

Then, it can be also seen that

$$\begin{aligned}
[\nabla \cdot q_\rho](x,t,u) &\to \operatorname{sgn}(u-k)([\nabla \cdot A](x,t,u) - [\nabla \cdot A](x,t,k)), \\
[\nabla \cdot r_\rho](x,t,u) &\to \operatorname{sgn}(u-k)([\varDelta \beta](x,t,u) - [\varDelta \beta](x,t,k)),
\end{aligned}$$

as $\rho \to 0$. Then, the entropy inequality in the definition of BV-entropy solutions implies that

$$\int_{\mathbb{R}^N_T} \{\eta_\rho(u)\partial_t\varphi + (q_\rho(x,t,u) - r_\rho(x,t,u)) \cdot \nabla\varphi + ([\nabla \cdot q_\rho](x,t,u) - [\nabla \cdot r_\rho](x,t,u))\varphi$$
$$- \operatorname{sgn}_\rho(u-k)(\nabla\beta(x,t,u) - [\nabla\beta](x,t,u)) \cdot \nabla\varphi$$
$$- \operatorname{sgn}_\rho(u-k)([\nabla \cdot A](x,t,u) - [\Delta\beta](x,t,u) + B(x,t,u))\varphi\} dxdt$$
$$\geq \int_{\mathbb{R}^N_T} \operatorname{sgn}'_\rho(u-k)|\sqrt{[\partial_\xi\beta](x,t,u)}Du|^2\varphi dxdt.$$

Hereafter, we omit the subscription ρ of η_ρ, q_ρ and r_ρ. Using the above entropy inequality, we prove the uniqueness of BV-entropy solutions. To see this, we show the following inequality which is crucial to prove the uniqueness result.

Proposition 4.1. *We assume the conditions* $\{A0\}$-$\{A8\}$. *For a pair of* BV*-entropy solutions* u *and* v *to* (P)*, the following inequality holds:*

$$\int_{\mathbb{R}^N_T} \operatorname{sgn}(u-v)\{(u-v)\partial_t\varphi - (B(x,t,u) - B(x,t,v))\varphi + (\beta(x,t,u) - \beta(x,t,v))\Delta\varphi$$
$$+2([\nabla\beta](x,t,u) - [\nabla\beta](x,t,v)) \cdot \nabla\varphi + (A(x,t,u) - A(x,t,v)) \cdot \nabla\varphi\} dxdt$$
$$\geq -C(||\varphi||_{L^\infty(\mathbb{R}^N_T)} + \max_{1\leq i\leq N} ||\partial_{x_i}\varphi||_{L^\infty(\mathbb{R}^N_T)}) \int_{\mathbb{R}^N_T \cap \operatorname{supp}(\varphi)} |u-v| dxdt,$$

for $\varphi \in C_0^\infty(\mathbb{R}^N_T)^+$ *and a positive constant* C.

Proof. *Step 1.* Let u, v be a pair of BV-entropy solutions. We put $k = v(y,s)$ and $\varphi = \phi_\delta^{\delta_0}(x,y,t,s)$ (see (4.1)) in the definition of BV-entropy solution u. Integrating the inequality on $\mathbb{R}^N_T$ with respect to (y,s), then

$$\int_{(\mathbb{R}^N_T)^2} \{\eta(u)\partial_t\phi_\delta^{\delta_0} + (q(x,t,u) - r(x,t,u)) \cdot \nabla_x\phi_\delta^{\delta_0} + ([\nabla_x \cdot q](x,t,u) - [\nabla_x \cdot r](x,t,u))\phi_\delta^{\delta_0}$$
$$- \operatorname{sgn}_\rho(u(x,t) - v(y,s))(\nabla_x\beta(x,t,u) - [\nabla_x\beta](x,t,u)) \cdot \nabla_x\phi_\delta^{\delta_0}$$
$$- \operatorname{sgn}_\rho(u(x,t) - v(y,s))([\nabla_x \cdot A](x,t,u) - [\Delta_x\beta](x,t,u) + B(x,t,u))\phi_\delta^{\delta_0}\} d\mathbf{x}d\mathbf{y}$$
$$\geq \int_{(\mathbb{R}^N_T)^2} \operatorname{sgn}'_\rho(u(x,t) - v(y,s))|\sqrt{[\partial_\xi\beta](x,t,u)}D_xu|^2\phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y}.$$

Here, we write that $d\mathbf{x} = dxdt$ and $d\mathbf{y} = dyds$. We denote the left-hand side on the above inequality by $J_1(\rho)$. Similarly, we have the following inequality:

$$\begin{aligned}
&\int_{(\mathbb{R}^N_T)^2}\{\eta(v)\partial_s\phi^{\delta_0}_\delta+(q(y,s,v)-r(y,s,v))\cdot\nabla_y\phi^{\delta_0}_\delta+([\nabla_y\cdot q](y,s,v)-[\nabla_y\cdot r](y,s,v))\phi^{\delta_0}_\delta\\
&\qquad-\operatorname{sgn}_\rho(v(y,s)-u(x,t))(\nabla_y\beta(y,s,v)-[\nabla_y\beta](y,s,v))\cdot\nabla_y\phi^{\delta_0}_\delta\\
&\qquad-\operatorname{sgn}_\rho(v(y,s)-u(x,t))([\nabla_y\cdot A](y,s,v)-[\varDelta_y\beta](y,s,v)+B(y,s,v))\phi^{\delta_0}_\delta\}d\mathbf{x}d\mathbf{y}\\
&\quad\geq\int_{(\mathbb{R}^N_T)^2}\operatorname{sgn}'_\rho(v(y,s)-u(x,t))|\sqrt{[\partial_\xi\beta](y,s,v)}D_yv|^2\phi^{\delta_0}_\delta d\mathbf{x}d\mathbf{y}.
\end{aligned}$$

We also denote the left-hand side on the above inequality by $J_2(\rho)$. Moreover, we then set $J(\rho)\equiv J_1(\rho)+J_2(\rho)$ in what follows. The desired result is obtained by combining the estimates for $J(\rho)$.

Step 2. In this step, we consider the diffusion terms in $J(\rho)$. We calculate that

$$\begin{aligned}
&-(r(x,t,u)\cdot\nabla_x\phi^{\delta_0}_\delta+r(y,s,v)\cdot\nabla_y\phi^{\delta_0}_\delta)-([\nabla_x\cdot r](x,t,u)+[\nabla_y\cdot r](y,s,v))\phi^{\delta_0}_\delta\\
&-\operatorname{sgn}_\rho(u-v)\{(\nabla_x\beta(x,t,u)-[\nabla_x\beta](x,t,u))\cdot\nabla_x\phi^{\delta_0}_\delta\\
&\qquad-(\nabla_y\beta(y,s,v)-[\nabla_y\beta](y,s,v))\cdot\nabla_y\phi^{\delta_0}_\delta-([\varDelta_x\beta](x,t,u)-[\varDelta_y\beta](y,s,v))\phi^{\delta_0}_\delta\}\\
&=\operatorname{sgn}_\rho(u-v)\{-(\nabla_x\beta(x,t,u)-[\nabla_x\beta](x,t,u))\cdot(\nabla_x\phi^{\delta_0}_\delta+\nabla_y\phi^{\delta_0}_\delta)\\
&\qquad+(\nabla_y\beta(y,s,v)-[\nabla_y\beta](y,s,v))\cdot(\nabla_x\phi^{\delta_0}_\delta+\nabla_y\phi^{\delta_0}_\delta)\\
&\qquad+(\nabla_x\beta(x,t,u)-[\nabla_x\beta](x,t,u))\cdot\nabla_y\phi^{\delta_0}_\delta\\
&\qquad-(\nabla_y\beta(y,s,v)-[\nabla_y\beta](y,s,v))\cdot\nabla_x\phi^{\delta_0}_\delta+([\varDelta_x\beta](x,t,u)-[\varDelta_y\beta](y,s,v))\phi^{\delta_0}_\delta\}\\
&\quad-(r(x,t,u)\cdot\nabla_x\phi^{\delta_0}_\delta+r(y,s,v)\cdot\nabla_y\phi^{\delta_0}_\delta)-([\nabla_x\cdot r](x,t,u)+[\nabla_y\cdot r](y,s,v))\phi^{\delta_0}_\delta.
\end{aligned}$$

The right-hand side of the above equality denote by $\sum_{k=1}^7 I^k_\beta(\rho)$. At first, we see that

$$\begin{aligned}
I^6_\beta(\rho)\to&-\operatorname{sgn}(u-v)\{([\nabla_x\beta](x,t,u)-[\nabla_x\beta](x,t,v))\cdot\nabla_x\varphi\\
&\qquad\qquad+([\nabla_y\beta](y,s,u)-[\nabla_y\beta](y,s,v))\cdot\nabla_y\varphi\}\omega_\delta\theta_{\delta_0}\\
&-\operatorname{sgn}(u-v)\{([\nabla_x\beta](x,t,u)-[\nabla_y\beta](y,s,u))\\
&\qquad\qquad-([\nabla_x\beta](x,t,v)-[\nabla_y\beta](y,s,v))\}\cdot(\nabla_x\omega_\delta)\varphi\theta_{\delta_0}\equiv I^6_\beta,
\end{aligned}$$

as $\rho\to0$. This implies that

$$\begin{aligned}
&\left|\lim_{\delta\downarrow0}\lim_{\delta_0\downarrow0}\int_{(\mathbb{R}^N_T)^2}I^6_\beta d\mathbf{x}d\mathbf{y}\right|\\
&\leq2||\partial_\xi\nabla_x\beta(x,t,\xi)||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})^N}\max_{1\leq i\leq N}||\partial_{x_i}\varphi||_{L^\infty(\mathbb{R}^N_T)}\int_{\mathbb{R}^N_{T,\varphi}}|u-v|d\mathbf{x}\\
&\quad+|\lim_{\delta\downarrow0}\int_{\mathbb{R}^N_T\times\mathbb{R}^N}\operatorname{sgn}(u-v)\{([\nabla_x\beta](x,t,u)-[\nabla_y\beta](y,t,u))\\
&\qquad\qquad-([\nabla_x\beta](x,t,v)-[\nabla_y\beta](y,t,v))\}\cdot(\nabla_x\omega_\delta)\varphi d\mathbf{x}dy|.
\end{aligned}$$

Here, we write that $\mathbb{R}^N_{T,\varphi} \equiv \mathbb{R}^N_T \cap \mathrm{supp}(\varphi)$. Using (4.2), the second term of right-hand side on the above inequality is less than

$$\left|\lim_{\delta\downarrow 0}\int_{\mathbb{R}^N_T\times\mathbb{R}^N}\sum_{i=1}^N \max_{1\le j\le N}||\partial_\xi\partial_{x_i}\partial_{x_j}\beta||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}\left(\sum_{j=1}^N |x_j-y_j|\right)|u-v||\partial_{x_i}\omega_\delta|\varphi d\mathbf{x}dy\right|$$

$$\le C\sum_{i=1}^N \max_{1\le j\le N}||\partial_\xi\partial_{x_i}\partial_{x_j}\beta||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}||\varphi||_{L^\infty(\mathbb{R}^N_T)}\int_{\mathbb{R}^N_{T,\varphi}}|u-v|d\mathbf{x},$$

for some positive constant C. Next, the term $I^5_\beta(\rho)+I^7_\beta(\rho)$ is estimated below:

$$\left|\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0}\lim_{\rho\downarrow 0}\int_{(\mathbb{R}^N_T)^2}(I^5_\beta(\rho)+I^7_\beta(\rho))d\mathbf{x}d\mathbf{y}\right|$$

$$= |\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0}\int_{(\mathbb{R}^N_T)^2}\mathrm{sgn}(u-v)\{([\varDelta_x\beta](x,t,u)-[\varDelta_y\beta](y,s,v))$$
$$-([\varDelta_x\beta](x,t,u)-[\varDelta_x\beta](x,t,v))-([\varDelta_y\beta](y,s,v)-[\varDelta_y\beta](y,s,u))\}\phi^{\delta_0}_\delta d\mathbf{x}d\mathbf{y}|$$

$$=\left|\int_{\mathbb{R}^N_T}\mathrm{sgn}(u-v)([\varDelta_x\beta](x,t,u)-[\varDelta_x\beta](x,t,v))\varphi d\mathbf{x}\right|$$

$$\le ||\partial_\xi\varDelta\beta||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}||\varphi||_{L^\infty(\mathbb{R}^N_T)}\int_{\mathbb{R}^N_{T,\varphi}}|u-v|d\mathbf{x}.$$

Thirdly, the term $I^3_\beta(\rho)$ is calculated that

$$\lim_{\rho\downarrow 0}\int_{(\mathbb{R}^N_T)^2}I^3_\beta(\rho)d\mathbf{x}d\mathbf{y}=\lim_{\rho\downarrow 0}\int_{(\mathbb{R}^N_T)^2}\left(\nabla_x\int_v^u \mathrm{sgn}_\rho(\xi-v)[\partial_\xi\beta](x,t,\xi)d\xi\right.$$
$$\left.-\int_v^u\mathrm{sgn}_\rho(\xi-v)[\nabla_x\partial_\xi\beta](x,t,\xi)d\xi\right)\cdot\nabla_y\phi^{\delta_0}_\delta d\mathbf{x}d\mathbf{y}$$

$$=-\int_{(\mathbb{R}^N_T)^2}\int_v^u\mathrm{sgn}(\xi-v)([\partial_\xi\beta](x,t,\xi)\nabla_x\cdot\nabla_y\phi^{\delta_0}_\delta+[\nabla_x\partial_\xi\beta](x,t,\xi)\cdot\nabla_y\phi^{\delta_0}_\delta)d\xi d\mathbf{x}d\mathbf{y}\equiv\widehat{I}^3_\beta,$$

by using the chain rule formula in [7] and the Gauss divergence theorem. Similarly, the symmetricity of the Kružkov entropy:

$$\int_u^v\mathrm{sgn}(\xi-u)\beta(x,t,\xi)d\xi=\int_v^u\mathrm{sgn}(\xi-v)\beta(x,t,\xi)d\xi$$

implies that

$$\lim_{\rho\downarrow 0}\int_{(\mathbb{R}^N_T)^2}I^4_\beta(\rho)d\mathbf{x}d\mathbf{y}$$

$$=-\int_{(\mathbb{R}^N_T)^2}\int_u^v\mathrm{sgn}(\xi-u)([\partial_\xi\beta](y,s,\xi)\nabla_y\cdot\nabla_x\phi^{\delta_0}_\delta+[\nabla_y\partial_\xi\beta](y,s,\xi)\cdot\nabla_x\phi^{\delta_0}_\delta)d\xi d\mathbf{x}d\mathbf{y}$$

$$=-\int_{(\mathbb{R}^N_T)^2}\int_v^u\mathrm{sgn}(\xi-v)([\partial_\xi\beta](y,s,\xi)\nabla_y\cdot\nabla_x\phi^{\delta_0}_\delta+[\nabla_y\partial_\xi\beta](y,s,\xi)\cdot\nabla_x\phi^{\delta_0}_\delta)d\xi d\mathbf{x}d\mathbf{y}\equiv\widehat{I}^4_\beta.$$

On the other hand, $J(\rho)$ is greater than

$$\int_{(\mathbb{R}_T^N)^2} \operatorname{sgn}'_\rho(u-v)([\partial_\xi \beta](x,t,u)|D_x u|^2 + [\partial_\xi \beta](y,s,v)|D_y v|^2)\phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y},$$

by the definitions of $J_1(\rho)$ and $J_2(\rho)$. We denote the above integral by $E(\rho)$. Then, it can be seen that

$$\begin{aligned}
E(\rho) &\geq \int_{(\mathbb{R}_T^N)^2} 2\operatorname{sgn}'_\rho(u-v)(\sqrt{[\partial_\xi \beta](x,t,u)}D_x u)\cdot(\sqrt{[\partial_\xi \beta](y,s,v)}D_y v)\phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \\
&= 2\int_{(\mathbb{R}_T^N)^2} (\sqrt{[\partial_\xi \beta](x,t,u)}D_x u)\cdot \left(\nabla_y \int_u^v \operatorname{sgn}'_\rho(u-\zeta)\sqrt{[\partial_\xi \beta](y,s,\zeta)}d\zeta \right. \\
&\qquad\qquad \left. - \int_u^v \operatorname{sgn}'_\rho(u-\zeta)\nabla_y\sqrt{[\partial_\xi \beta](y,s,\zeta)}d\zeta\right)\phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \\
&= -2\int_{(\mathbb{R}_T^N)^2} (\sqrt{[\partial_\xi \beta](x,t,u)}D_x u)\cdot \nabla_y \phi_\delta^{\delta_0}\left(\int_u^v \operatorname{sgn}'_\rho(u-\zeta)\sqrt{[\partial_\xi \beta](y,s,\zeta)}d\zeta\right) d\mathbf{x}d\mathbf{y} \\
&\quad -2\int_{(\mathbb{R}_T^N)^2} (\sqrt{[\partial_\xi \beta](x,t,u)}D_x u)\cdot \left(\int_u^v \operatorname{sgn}'_\rho(u-\zeta)\nabla_y\sqrt{[\partial_\xi \beta](y,s,\zeta)}d\zeta\right)\phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \\
&\equiv E_1(\rho) + E_2(\rho),
\end{aligned}$$

by using the chain rule formula and the Gauss divergence theorem. By the boundedness of $\partial_\xi \beta$ and the assumption $\{A8\}$, it follows that

$$\sqrt{\partial_\xi \beta},\quad \partial_{x_i}\sqrt{\partial_\xi \beta} \in L^1(\mathbb{R}_T^N \times \mathcal{U}) \cap L^\infty(\mathbb{R}_T^N \times \mathcal{U}),$$

for $i = 1, \ldots, N$. Then, these imply that

$$\int_\xi^v \operatorname{sgn}'_\rho(\xi-\zeta)\sqrt{[\partial_\xi \beta](y,s,\zeta)}d\zeta,\quad \int_\xi^v \operatorname{sgn}'_\rho(\xi-\zeta)\nabla_y\sqrt{[\partial_\xi \beta](y,s,\zeta)}d\zeta \in C(\mathcal{U}),$$

for $(y,s) \in \mathbb{R}_T^N$. Therefore, we may use the chain rule formula with respect to x. Using also the Gauss divergence theorem, the term $E_1(\rho)$ is calculated that

$$\begin{aligned}
&2\int_{(\mathbb{R}_T^N)^2} \left(\int_v^u \sqrt{[\partial_\xi \beta](x,t,\xi)}\int_\xi^v \operatorname{sgn}'_\rho(\xi-\zeta)\sqrt{[\partial_\xi \beta](y,s,\zeta)}d\zeta d\xi\right)\nabla_x\cdot\nabla_y \phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \\
&\quad +2\int_{(\mathbb{R}_T^N)^2} \left(\int_v^u \nabla_x\sqrt{[\partial_\xi \beta](x,t,\xi)}\int_\xi^v \operatorname{sgn}'_\rho(\xi-\zeta)\sqrt{[\partial_\xi \beta](y,s,\zeta)}d\zeta d\xi\right)\cdot\nabla_y \phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \\
&\to 2\int_{(\mathbb{R}_T^N)^2} \left(\int_v^u (-\operatorname{sgn}(\xi-v))\sqrt{[\partial_\xi \beta](y,s,\xi)}\sqrt{[\partial_\xi \beta](x,t,\xi)}d\xi\right)\nabla_x\cdot\nabla_y \phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \\
&\quad +2\int_{(\mathbb{R}_T^N)^2} \left(\int_v^u (-\operatorname{sgn}(\xi-v))\sqrt{[\partial_\xi \beta](y,s,\xi)}\nabla_x\sqrt{[\partial_\xi \beta](x,t,\xi)}d\xi\right)\cdot\nabla_y \phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y},
\end{aligned}$$

as $\rho \to 0$. Here, we use the formula below:

$$\int_{\xi}^{v} \operatorname{sgn}'_{\rho}(\xi-\zeta)\sqrt{\partial_\xi\beta(y,s,\zeta)}d\zeta \to \operatorname{sgn}(v-\xi)\sqrt{\partial_\xi\beta(y,s,\xi)} = -\operatorname{sgn}(\xi-v)\sqrt{\partial_\xi\beta(y,s,\xi)},$$

as $\rho \downarrow 0$. Similarly, we also have

$$\begin{aligned} E_2(\rho) \to 2\int_{(\mathbb{R}_T^N)^2} & \left(\int_v^u (-\operatorname{sgn}(\xi-v))\nabla_y\sqrt{[\partial_\xi\beta](y,s,\xi)}\sqrt{[\partial_\xi\beta](x,t,\xi)}d\xi\right)\cdot\nabla_x\phi_\delta^{\delta_0}d\mathbf{x}d\mathbf{y} \\ & +2\int_{(\mathbb{R}_T^N)^2}\left(\int_v^u(-\operatorname{sgn}(\xi-v))\nabla_y\sqrt{[\partial_\xi\beta](y,s,\xi)}\cdot\nabla_x\sqrt{[\partial_\xi\beta](x,t,\xi)}d\xi\right)\phi_\delta^{\delta_0}d\mathbf{x}d\mathbf{y}, \end{aligned}$$

as $\rho \downarrow 0$. Therefore, it is deduced that

$$\begin{aligned} & \lim_{\rho\downarrow 0} E(\rho) - (\widehat{I}_\beta^3 + \widehat{I}_\beta^4) \\ & \geq \int_{(\mathbb{R}_T^N)^2}\left(\int_v^u \operatorname{sgn}(\xi-v)([\partial_\xi\beta](x,t,\xi)\right. \\ & \qquad\left. -2\sqrt{[\partial_\xi\beta](x,t,\xi)}\sqrt{[\partial_\xi\beta](y,s,\xi)}+[\partial_\xi\beta](y,s,\xi))d\xi\right)\nabla_x\cdot\nabla_y\phi_\delta^{\delta_0}d\mathbf{x}d\mathbf{y} \\ & \quad+\int_{(\mathbb{R}_T^N)^2}\left(\int_v^u \operatorname{sgn}(\xi-v)([\nabla_y\partial_\xi\beta](y,s,\xi)\right. \\ & \qquad\left. -2\sqrt{[\partial_\xi\beta](x,t,\xi)}\nabla_y\sqrt{[\partial_\xi\beta](y,s,\xi)})d\xi\right)\cdot\nabla_x\phi_\delta^{\delta_0}d\mathbf{x}d\mathbf{y} \\ & \quad+\int_{(\mathbb{R}_T^N)^2}\left(\int_v^u \operatorname{sgn}(\xi-v)([\nabla_x\partial_\xi\beta](x,t,\xi)\right. \\ & \qquad\left. -2\nabla_x\sqrt{[\partial_\xi\beta](x,t,\xi)}\sqrt{[\partial_\xi\beta](y,s,\xi)})d\xi\right)\cdot\nabla_y\phi_\delta^{\delta_0}d\mathbf{x}d\mathbf{y} \\ & \quad-2\int_{(\mathbb{R}_T^N)^2}\left(\int_v^u \operatorname{sgn}(\xi-v)\nabla_x\sqrt{[\partial_\xi\beta](x,t,\xi)}\cdot\nabla_y\sqrt{[\partial_\xi\beta](y,s,\xi)})d\xi\right)\phi_\delta^{\delta_0}d\mathbf{x}d\mathbf{y}. \end{aligned} \tag{4.3}$$

Here, we put

$$\varepsilon(x,t,y,s,\xi) \equiv [\partial_\xi\beta](x,t,\xi) - 2\sqrt{[\partial_\xi\beta](x,t,\xi)}\sqrt{[\partial_\xi\beta](y,s,\xi)} + [\partial_\xi\beta](y,s,\xi). \tag{4.4}$$

According to (4.3) and (4.4), it is inferred that

$$\begin{aligned}
&\lim_{\rho\downarrow 0} E(\rho) - (\widehat{I}^3_\beta + \widehat{I}^4_\beta) \\
&\geq \int_{(\mathbb{R}^N_T)^2} \left(\int_v^u \operatorname{sgn}(\xi - v)\varepsilon(x,t,y,s,\xi)d\xi \right) \nabla_x \cdot \nabla_y \phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \\
&\quad + \int_{(\mathbb{R}^N_T)^2} \left(\int_v^u \operatorname{sgn}(\xi - v)\nabla_y\varepsilon(x,t,y,s,\xi)d\xi \right) \cdot [\nabla_x \phi_\delta^{\delta_0} + \nabla_y \phi_\delta^{\delta_0}] d\mathbf{x}d\mathbf{y} \\
&\quad + \int_{(\mathbb{R}^N_T)^2} \left(\int_v^u \operatorname{sgn}(\xi - v)(\nabla_x\varepsilon(x,t,y,s,\xi) - \nabla_y\varepsilon(x,t,y,s,\xi))d\xi \right) \cdot \nabla_y \phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \\
&\quad + \int_{(\mathbb{R}^N_T)^2} \left(\int_v^u \operatorname{sgn}(\xi - v)\nabla_x \cdot \nabla_y\varepsilon(x,t,y,s,\xi)d\xi \right) \phi_\delta^{\delta_0} d\mathbf{x}d\mathbf{y} \equiv \sum_{k=1}^4 R_k.
\end{aligned}$$

Here, the definition of $\phi_\delta^{\delta_0}(x,t,y,s)$ implies that

$$\nabla_x \cdot \nabla_y \phi_\delta^{\delta_0} = \nabla_x \cdot \nabla_y \varphi \theta_{\delta_0} \omega_\delta + \nabla_y \varphi \cdot \nabla_x \omega_\delta \theta_{\delta_0} + \nabla_x \varphi \cdot \nabla_y \omega_\delta \theta_{\delta_0} + \varphi \theta_{\delta_0} \nabla_x \cdot \nabla_y \omega_\delta.$$

Using the above identity, we divide R_1 by four parts. More specifically, we write $R_1 = \sum_{i=1}^4 R_{1,i}$. Then, it follows that

$$\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0} R_{1,1} = \int_{\mathbb{R}^N_T} \left(\int_v^u \operatorname{sgn}(\xi - v)\varepsilon(x,t,x,t,\xi)d\xi \right) \varDelta_x \varphi d\mathbf{x} = 0,$$

and

$$\begin{aligned}
&|\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0} R_{1,2}| \leq 2\sqrt{2} ||\sqrt{\partial_\xi \beta}||_{L^\infty(\mathbb{R}^N_T \times \mathcal{U})} \max_{1\leq i\leq N} ||\partial_{x_i}\sqrt{\partial_\xi \beta}||_{L^\infty(\mathbb{R}^N_T \times \mathcal{U})} \\
&\qquad\qquad \times \left| \lim_{\delta\downarrow 0} \int_{\mathbb{R}^N_T \times \mathbb{R}^N} \left(\int_v^u \operatorname{sgn}(\xi - v)\, |x-y|\, d\xi \right) \nabla_y \varphi \cdot \nabla_x \omega_\delta d\mathbf{x}dy \right| \\
&\leq 2\sqrt{2}C ||\sqrt{\partial_\xi \beta}||_{L^\infty(\mathbb{R}^N_T \times \mathcal{U})} \max_{1\leq i\leq N} ||\partial_{x_i}\sqrt{\partial_\xi \beta}||_{L^\infty(\mathbb{R}^N_T \times \mathcal{U})} \sum_{i=1}^N ||\partial_{x_i}\varphi||_{L^\infty(\mathbb{R}^N_T)} \int_{\mathbb{R}^N_{T,\varphi}} |u - v| d\mathbf{x},
\end{aligned}$$

for some positive constant C. Similarly, it can be checked that

$$\begin{aligned}
|\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0} R_{1,3}| &\leq 2\sqrt{2}C ||\sqrt{\partial_\xi \beta}||_{L^\infty(\mathbb{R}^N_T \times \mathcal{U})} \max_{1\leq i\leq N} ||\partial_{x_i}\sqrt{\partial_\xi \beta}||_{L^\infty(\mathbb{R}^N_T \times \mathcal{U})} \\
&\quad \times \sum_{i=1}^N ||\partial_{x_i}\varphi||_{L^\infty(\mathbb{R}^N_T)} \int_{\mathbb{R}^N_{T,\varphi}} |u - v| d\mathbf{x}.
\end{aligned}$$

Moreover, we also have

$$
\begin{aligned}
&|\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0} R_{1,4}| \\
&\le \max_{1\le i\le N} ||\partial_{x_i}\sqrt{\partial_\xi\beta}||^2_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})} \lim_{\delta\downarrow 0}\int_{\mathbb{R}^N_T\times\mathbb{R}^N}\left|\sum_{i=1}^N (x_i-y_i)\right|^2 |\nabla_x\cdot\nabla_y\omega_\delta||u-v||\varphi|d\mathbf{x}dy \\
&\le C\max_{1\le i\le N} ||\partial_{x_i}\sqrt{\partial_\xi\beta}||^2_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}||\varphi||_{L^\infty(\mathbb{R}^N_T)}\int_{\mathbb{R}^N_{T,\varphi}}|u-v|d\mathbf{x},
\end{aligned}
$$

by (4.2). Secondly, we calculate that

$$
|\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0}(R_2+R_4)| \le C(||\nabla\varphi||_{L^\infty(\mathbb{R}^N_T)} + ||\varphi||_{L^\infty(\mathbb{R}^N_T)})\int_{\mathbb{R}^N_{T,\varphi}}|u-v|d\mathbf{x}.
$$

Here, we use the following facts. There exists a positive constant C such that

$$
\begin{aligned}
&|\nabla_y\varepsilon(x,t,y,s,\xi)| \le 2||\sqrt{\partial_\xi\beta}||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}||\nabla\sqrt{\partial_\xi\beta}||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})^N} + ||\nabla\partial_\xi\beta||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})^N} \le C, \\
&|\nabla_x\cdot\nabla_y\varepsilon(x,t,y,s,\xi)| \le 2||\nabla\sqrt{\partial_\xi\beta}||^2_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})^N} \le C,
\end{aligned}
$$

for $(x,t,y,s,\xi)\in\mathbb{R}^N_T\times\mathbb{R}^N_T\times\mathcal{U}$ by the assumption {A8}. Moreover, it follows that

$$
\begin{aligned}
R_3 = &\int_{(\mathbb{R}^N_T)^2}\left(\int_v^u \operatorname{sgn}(\xi-v)\{([\nabla_x\partial_\xi\beta](x,t,\xi) - [\nabla_y\partial_\xi\beta](y,s,\xi))\right. \\
&-2\sqrt{[\partial_\xi\beta](y,s,\xi)}(\nabla_x\sqrt{[\partial_\xi\beta](x,t,\xi)} - \nabla_y\sqrt{[\partial_\xi\beta](y,s,\xi)}) \\
&\left.+2\nabla_y\sqrt{[\partial_\xi\beta](y,s,\xi)}(\sqrt{[\partial_\xi\beta](x,t,\xi)} - \sqrt{[\partial_\xi\beta](y,s,\xi)})\}d\xi\right)\cdot(\nabla_y\varphi\omega_\delta + \varphi\nabla_y\omega_\delta)\theta_{\delta_0}d\mathbf{x}d\mathbf{y},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0} R_3| \le &\left|\sum_{i=1}^N\left(\max_{1\le j\le N}||\partial_{x_i}\partial_{x_j}\partial_\xi\beta||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}\right.\right. \\
&+2||\sqrt{\partial_\xi\beta}||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}\max_{1\le j\le N}||\partial_{x_j}\partial_{x_i}\sqrt{\partial_\xi\beta}||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})} \\
&\left.+2||\partial_{y_i}\sqrt{\partial_\xi\beta}||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}\max_{1\le j\le N}||\partial_{x_j}\sqrt{\partial_\xi\beta}||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}\right) \\
&\left.\times\lim_{\delta\downarrow 0}\int_{\mathbb{R}^N_T\times\mathbb{R}^N}|u-v|\left(\sum_{j=1}^N|x_j-y_j|\right)(\partial_{y_i}\omega_\delta)\varphi d\mathbf{x}dy\right| \\
\le &\sqrt{2}C||\varphi||_{L^\infty(\mathbb{R}^N_T)}\int_{\mathbb{R}^N_{T,\varphi}}|u-v|d\mathbf{x}.
\end{aligned}
$$

Additionally, we calculate that

$$\begin{aligned}
&\int_{(\mathbb{R}^N_T)^2} (I^1_\beta(\rho) + I^2_\beta(\rho)) d\mathbf{x}d\mathbf{y} \\
&= -\int_{(\mathbb{R}^N_T)^2} \left(\nabla_x \int_v^u \mathrm{sgn}_\rho(\xi - v)[\partial_\xi \beta](x,t,\xi) d\xi \right. \\
&\qquad\qquad \left. - \int_v^u \mathrm{sgn}_\rho(\xi - v)[\nabla_x \partial_\xi \beta](x,t,\xi) d\xi \right) \cdot (\nabla_x \varphi + \nabla_y \varphi) \omega_\delta \theta_{\delta_0} d\mathbf{x}d\mathbf{y} \\
&\quad + \int_{(\mathbb{R}^N_T)^2} \left(\nabla_y \int_u^v \mathrm{sgn}_\rho(u - \xi)[\partial_\xi \beta](y,s,\xi) d\xi \right. \\
&\qquad\qquad \left. - \int_u^v \mathrm{sgn}_\rho(u - \xi)[\nabla_y \partial_\xi \beta](y,s,\xi) d\xi \right) \cdot (\nabla_x \varphi + \nabla_y \varphi) \omega_\delta \theta_{\delta_0} d\mathbf{x}d\mathbf{y},
\end{aligned}$$

by the chain rule formula. Using the Gauss divergence theorem for the first and third terms of the right-hand side on the above equality, it follows that

$$\begin{aligned}
&\int_{(\mathbb{R}^N_T)^2} (I^1_\beta(\rho) + I^2_\beta(\rho)) d\mathbf{x}d\mathbf{y} \\
&= \int_{(\mathbb{R}^N_T)^2} \left\{ \left(\int_v^u \mathrm{sgn}_\rho(\xi - v)[\partial_\xi \beta](x,t,\xi) d\xi \right) (\varDelta_x \varphi + \nabla_x \cdot \nabla_y \varphi) \right. \\
&\qquad\qquad \left. - \left(\int_u^v \mathrm{sgn}_\rho(u - \xi)[\partial_\xi \beta](y,s,\xi) d\xi \right) (\nabla_y \cdot \nabla_x \varphi + \varDelta_y \varphi) \right\} \omega_\rho \theta_{\rho_0} d\mathbf{x}d\mathbf{y} \\
&\quad + \int_{(\mathbb{R}^N_T)^2} \left(\int_v^u \mathrm{sgn}_\rho(\xi - v)[\partial_\xi \beta](x,t,\xi) d\xi \right. \\
&\qquad\qquad \left. + \int_u^v \mathrm{sgn}_\rho(u - \xi)[\partial_\xi \beta](y,s,\xi) d\xi \right) (\nabla_x \varphi + \nabla_y \varphi) \cdot (\nabla_x \omega_\delta) \theta_{\delta_0} d\mathbf{x}d\mathbf{y} \\
&\quad + \int_{(\mathbb{R}^N_T)^2} \left(\int_v^u \mathrm{sgn}_\rho(\xi - v)[\nabla_x \partial_\xi \beta](x,t,\xi) d\xi \right. \\
&\qquad\qquad \left. - \int_u^v \mathrm{sgn}_\rho(u - \xi)[\nabla_y \partial_\xi \beta](y,s,\xi) d\xi \right) \cdot (\nabla_x \varphi + \nabla_y \varphi) \omega_\delta \theta_{\delta_0} d\mathbf{x}d\mathbf{y}.
\end{aligned}$$

The above equality yields

$$\begin{aligned}
&\lim_{\delta \downarrow 0} \lim_{\delta_0 \downarrow 0} \lim_{\rho \downarrow 0} \int_{(\mathbb{R}^N_T)^2} (I^1_\beta(\rho) + I^2_\beta(\rho)) d\mathbf{x}d\mathbf{y} = \int_{\mathbb{R}^N_T} \mathrm{sgn}(u - v)(\beta(x,t,u) - \beta(x,t,v)) \varDelta \varphi d\mathbf{x} \\
&+ \lim_{\delta \downarrow 0} \int_{\mathbb{R}^N_T \times \mathbb{R}} \int_v^u \mathrm{sgn}(\xi - v)([\partial_\xi \beta](x,t,\xi) - [\partial_\xi \beta](y,t,\xi)) d\xi (\nabla_x \varphi + \nabla_y \varphi) \cdot \nabla \omega_\delta d\mathbf{x}dy \\
&+2 \int_{\mathbb{R}^N_T} \mathrm{sgn}(u - v)([\nabla_x \beta](x,t,u) - [\nabla_x \beta](x,t,v)) \cdot \nabla_x \varphi d\mathbf{x} = \sum_{i=5}^7 R_i.
\end{aligned}$$

Then, the term R_6 is estimated that

$$|R_6| \le 2C \max_{1\le i\le N} ||\partial_{x_i}\partial_\xi\beta||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})} \sum_{i=1}^N ||\partial_{x_i}\varphi||_{L^\infty(\mathbb{R}^N_T)} \int_{\mathbb{R}^N_{T,\varphi}} |u-v|d\mathbf{x}.$$

Step 3. We next consider the convection and the remaining integral terms in $J(\rho)$ respectively. We estimate the convective term:

$$\int_{(\mathbb{R}^N_T)^2} \{(q(x,t,u)\cdot\nabla_x\phi^{\delta_0}_\delta + q(y,s,v)\cdot\nabla_y\phi^{\delta_0}_\delta) + ([\nabla_x\cdot q](x,t,u) + [\nabla_y\cdot q](y,s,v))\phi^{\delta_0}_\delta$$
$$-\operatorname{sgn}_\rho(u-v)([\nabla_x\cdot A](x,t,u) - [\nabla_y\cdot A](y,s,v))\phi^{\delta_0}_\delta\}d\mathbf{x}d\mathbf{y} \equiv K(\rho).$$

Then, it is obtained that

$$\begin{aligned}
\lim_{\rho\downarrow 0} K(\rho) &= \int_{(\mathbb{R}^N_T)^2} \operatorname{sgn}(u-v)(A(x,t,u)-A(y,s,v))\cdot(\nabla_x\phi^{\delta_0}_\delta + \nabla_y\phi^{\delta_0}_\delta)d\mathbf{x}d\mathbf{y} \\
&\quad - \int_{(\mathbb{R}^N_T)^2} \operatorname{sgn}(u-v)(A(x,t,u)\cdot\nabla_y\phi^{\delta_0}_\delta - A(y,s,v)\cdot\nabla_x\phi^{\delta_0}_\delta)d\mathbf{x}d\mathbf{y} \\
&\quad + \int_{(\mathbb{R}^N_T)^2} \operatorname{sgn}(u-v)(A(y,s,u)\cdot\nabla_y\phi^{\delta_0}_\delta - A(x,t,v)\cdot\nabla_x\phi^{\delta_0}_\delta)d\mathbf{x}d\mathbf{y} \\
&\quad - \int_{(\mathbb{R}^N_T)^2} \operatorname{sgn}(u-v)([\nabla_x\cdot A](x,t,v) - [\nabla_y\cdot A](y,s,u))\phi^{\delta_0}_\delta d\mathbf{x}d\mathbf{y} \equiv \sum_{i=1}^4 K_i.
\end{aligned}$$

Here, we deal with the terms K_2 and K_3 as follows:

$$\begin{aligned}
\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0}(K_2+K_3) &= \lim_{\delta\downarrow 0}\int_{\mathbb{R}^N_T\times\mathbb{R}^N} \operatorname{sgn}(u-v)\{(A(x,t,u)-A(y,t,u)) \\
&\qquad -(A(x,t,v)-A(y,t,v))\}\cdot(\nabla_x\omega_\delta)\varphi d\mathbf{x}dy \\
&\le C\max_{1\le i,j\le N} ||\partial_\xi\partial_{x_i}A^j||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}||\varphi||_{L^\infty(\mathbb{R}^N_T)} \int_{\mathbb{R}^N_{T,\varphi}} |u-v|d\mathbf{x},
\end{aligned}$$

for some positive constant C. Moreover, we have

$$\lim_{\delta\downarrow 0}\lim_{\delta_0\downarrow 0} K_4 \le \max_{1\le i\le N} ||\partial_\xi\partial_{x_i}A^i||_{L^\infty(\mathbb{R}^N_T\times\mathcal{U})}||\varphi||_{L^\infty(\mathbb{R}^N_T)} \int_{\mathbb{R}^N_{T,\varphi}} |u-v|d\mathbf{x}.$$

Step 4. By the step 1-3, we obtain the next result

$$\begin{aligned}
&\int_{\mathbb{R}^N_T} \operatorname{sgn}(u-v)\{(u-v)\partial_t\varphi - (B(x,t,u)-B(x,t,v))\varphi\}d\mathbf{x} \\
&\quad + \int_{\mathbb{R}^N_T} \operatorname{sgn}(u-v)\{(\beta(x,t,u)-\beta(x,t,v))\Delta\varphi + 2([\nabla\beta](x,t,u)-[\nabla\beta](x,t,v))\cdot\nabla\varphi \\
&\qquad +[A(x,t,u)-A(x,t,v)]\cdot\nabla\varphi\}d\mathbf{x} \\
&\ge -C(||\varphi||_{L^\infty(\mathbb{R}^N_T)} + \max_{1\le i\le N}||\partial_{x_i}\varphi||_{L^\infty(\mathbb{R}^N_T)}) \int_{\mathbb{R}^N_{T,\varphi}} |u-v|d\mathbf{x}.
\end{aligned}$$

Then, it is obtained the desired result. □

4.1 Proof of Theorem 2.9 (II)

By Proposition 4.1, we can show the uniqueness of BV-entropy solutions to (P). In fact, we take functions $\chi_r(x) \in C_0^\infty(\mathbb{R}^N)^+$ and $\widehat{\chi}_r(s) \in C_0^\infty((0,t))^+$ such that

$$\chi_r(x) = \begin{cases} 1 & (|x| \leq r), \\ 0 & (|x| \geq r+1), \end{cases} \qquad \widehat{\chi}_r(s) = \begin{cases} 1 & (r \leq s \leq t-r), \\ 0 & (t \leq s), \end{cases}$$

for $t \in (0,T]$ as the function φ in Proposition 4.1. For a sufficiently large $r > 0$, we get the following inequality:

$$\int_0^t \frac{d}{ds}||u(\cdot,s) - v(\cdot,s)||_{L^1(\mathbb{R}^N)}ds \leq (\alpha' + C)\int_0^t ||u(\cdot,s) - v(\cdot,s)||_{L^1(\mathbb{R}^N)}ds,$$

for $t \in (0,T]$. Using the Gronwall inequality, we get the claim of Theorem 2.9 (II).

Remark 4.2. For simplicity, we estimate the diffusion terms and the convection terms, separately. If we together estimate the second term of I_β^6 and $K_2 + K_3$, then we can substitute the boundedness assumptions of first line in $\{$A8$\}$ for $\partial_\xi\partial_{x_i}(\nabla\beta - A) \in L^\infty(\mathbb{R}_T^N \times \mathcal{U})^N$ for $i = 1, \ldots, N$ in $\{$A6$\}$.

Remark 4.3. In [10], if the convection term form separation variable type (e.g. $A(x,t,\xi) \equiv \tilde{A}(x,t)\widehat{A}(\xi)$), then it follows that $\lim_{\delta,\delta_0 \to 0}(K_2 + K_3 + K_4) = 0$. Therefore, if $B(x,t,\xi) \equiv 0$ and $\beta(x,t,\xi) \equiv \beta(\xi)$, then it follows that $\alpha' = C = 0$. In this case, the continuous dependence result in Theorem 2.9 (II) become an L^1-contraction principle. This suggests the applicability of the contraction semigroup theory. On the other hand, when $A(x,t,\xi)$ is not separation variable type, it should be applied to the theory of quasi-contraction semigroup by the result of present paper.

Acknowledgments

The author would like to express my hearty gratitude to the anonymous referees for reviewing the original manuscript and for their valuable comments and suggestions that helped clarify and refine this paper.

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