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WEAK SOLVABILITY FOR PARABOLIC VARIATIONAL INCLUSIONS AND APPLICATION TO QUASI-VARIATIONAL PROBLEMS

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Abstract. We study a class of variational evolution inclusions of the form $(*)$ in a Hilbert space H:

$$(*) \quad u'(t) + \partial\varphi^t(p; u(t)) \ni f(t) \text{ in } H,\ 0 < t < T, \quad u(0) = u_0,$$

including nonlocal parameter p, where $\partial\varphi^t(p;\cdot)$ is the subdifferential of a time-dependent convex function $z \mapsto \varphi^t(p; z)$ with non-local dependence upon p. It is well known that many of interesting free-boundary problems are formally described in the variational form $(*)$ under an additional requirement that p is determined as a non-local function of u by the free boundary conditions. In this paper we establish a general theory on the weak solvability of parabolic evolution inclusions of the form $(*)$, specifying a class $\{\varphi^t(p;\cdot)\}$ of convex functions, and prove an existence result for quasi-variational evolution inclusions coupling $(*)$ with a feedback system $p = \Lambda_{p_0} u$, p_0 being the initial datum for p.

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1 Introduction

Throughout this paper, let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$. Given $f \in L^2(0,T;H),\ 0<T<\infty$, we consider a parabolic variational evolution inclusion

$$u'(t) + \partial\varphi^t(p;u(t)) \ni f(t) \text{ in } H,\ t \in (0,T). \tag{1.1}$$

Here p is a parameter given in an admissible set $\mathcal{X}_0$ which is a bounded and closed subset of $C([0,T];X)$, X being a (real) Banach space. In this formulation (1.1) we are given a family $\{\varphi^t(p;\cdot)\}$ of non-negative convex functions $z \mapsto \varphi^t(p;z)$ with parameters $t \in [0,T]$ and p in $\mathcal{X}_0$.

As is easily understood, some regularity assumptions on the mapping $(t,p) \mapsto \varphi^t(p;\cdot)$ are necessary in order for problem (1.1) to possess a strong solution $u \in W^{1,2}(0,T;H)$ or $u \in W^{1,2}_{loc}((0,T];H) \cap C([0,T];H)$; for instance, in [17, 29] we required that for a given $p \in \mathcal{X}_0$ and each positive number r there are functions $a_{p,r}(\cdot) \in W^{1,2}(0,T)$ and $b_{p,r}(\cdot) \in W^{1,1}(0,T)$ satisfying the following properties: for any $s,t \in [0,T]$ and $z \in H$ with $|z|_H \le r$ and $\varphi^s(p;z) < \infty$, there exists an element $\tilde{z} \in H$ with $\varphi^t(p;\tilde{z}) < \infty$ such that

$$|\tilde{z} - z|_H \le |a_{p,r}(t) - a_{p,r}(s)|(1 + \varphi^s(p;z)^{\frac{1}{2}}), \tag{1.2}$$

$$\varphi^t(p;\tilde{z}) \le \varphi^s(p;z) + |b_{p,r}(t) - b_{p,r}(s)|(1 + \varphi^s(p;z)), \tag{1.3}$$

and the mapping $(t,p,z) \mapsto \varphi^t(p;z)$ is lower semicontinuous in $(t,p,z) \in [0,T] \times \mathcal{X}_0 \times H$, namely, if $t_n \in [0,T],\ t_n \to t,\ p_n \in \mathcal{X}_0, p_n \to p$ in $C([0,T];X)$ and $z_n \to z$ weakly in H (as $n \to \infty$), then

$$\liminf_{n\to\infty} \varphi^{t_n}(p_n;z_n) \ge \varphi^t(p;z). \tag{1.4}$$

However, the regularity assumptions (1.2)-(1.3) are sometimes too restrictive, for instance, in the application of the result on (1.1) to variational inclusions arising from free boundary problems, such as superconductivity (cf. [2, 19, 25]). Therefore, it is worthwhile to discuss the solvability of (1.1) generated by $\partial\varphi^t(p;\cdot)$ under a weaker dependence of $(t,p) \mapsto \varphi^t(p;\cdot)$. We also note from (1.2)-(1.4) that $(t,p) \to \varphi^t(p;\cdot)$ is continuous with respect to (t,p) in the sense of Mosco [21] as a mapping from $[0,T] \times \mathcal{X}_0$ into the space of all proper l.s.c. convex functions on H.

One of the objectives of this paper is to specify a class of convex functions $\varphi^t(p;\cdot)$ on H with weak dependence upon (t,p) in order for (1.1) to possess a weak solution in an appropriate variational sense. Another one is to discuss quasi-variational evolution problems in the same framework:

$$\begin{cases} u'(t) + \partial\varphi^t(p;u(t)) \ni f(t) \text{ in } H,\ 0<t<T, \quad u(0) = u_0, \\ p = \Lambda_{p_0} u \text{ in } \mathcal{X}_0, \end{cases} \tag{1.5}$$

where u_0 and p_0 are prescribed as the initial data for u and p as well as f in $L^2(0,T;H)$, and Λ_{p_0} is a feedback system which is an operator from a subset of $C([0,T];H)$ into $\mathcal{X}_0$. One can imagine as Λ_{p_0} a sort of integral operators acting on $C([0,T];H)$. For instance, considering a phase transition system in $\Omega \times (0,T)$ with a smooth bounded domain Ω in $\mathbf{R}^3$ in which $u = u(x,t)$ is the phase parameter yielding a heat source $h(x,t,u)$, we see that temperature $\theta(x,t)$ is determined through a heat equation with source $h(x,t,u)$ and in this case we take as the class $\mathcal{X}_0$ the set of all temperature distributions $p(t) := \theta(\cdot,t)$ or more generally of all couples $p := [\theta, d]$ of temperature θ and some additional constraint parameter d. This example suggests that the mapping Λ_{p_0} assigning $p := \theta$ or $[\theta, d]$ to u is a compact operator between suitable function spaces. In most cases handled in this paper, Λ_{p_0} possesses somewhat compactness property.

(Notation)

For a real Banach space V we denote the norm by $|\cdot|_V$, the dual space V^* with its norm $|\cdot|_{V^*}$ and the duality pairing between V^* and V by $\langle\cdot,\cdot\rangle_{V^*,V}$. For a proper, l.s.c. (lower semicontinuous) and convex function φ, the effective domain $D(\varphi)$ of φ is the set $\{z \in V | \varphi(z) < \infty\}$ and the subdifferential of φ, denoted by $\partial_*\varphi$, from V into V^* is defined by

$$\partial_*\varphi(z) := \begin{cases} \left\{z^* \in V^*; \langle z^*, v - z\rangle_{V^*,V} \le \varphi(v) - \varphi(z) \text{ for all } v \in V\right\} & \text{if } z \in D(\varphi), \\ \emptyset & \text{otherwise,} \end{cases}$$

and the domain $D(\partial_*\varphi)$ is the set $\{z \in V | \partial_*\varphi(z) \neq \emptyset\}$. In particular, if V is a Hilbert space H, and H is identified with its dual space, i.e. $H = H^*$ and φ is a proper, l.s.c. and convex function on H, then we simply denote $\partial_*\varphi$ and $\langle\cdot,\cdot\rangle_{H,H}$ by $\partial\varphi$ and $(\cdot,\cdot)_H$, respectively. We refer the detail study of subdifferentials of convex functions to the text books [1, 5] and [18; Chapter 4].

2 Formulation and main theorems

In this section let us give more precisely the formulation of evolution inclusion (1.1). Throughout this paper, let φ_0 be a proper, l.s.c. and convex function on H and let X be a Banach space; X is not necessarily reflexive.

For simplicity we assume that

$$\varphi_0(\cdot) \ge 0 \quad \text{on } H, \quad \varphi_0(0) = 0; \quad \text{hence } 0 \in \partial\varphi_0(0), \tag{2.1}$$

and for a fixed positive number T put

$$\Phi_0(u) := \int_0^T \varphi_0(u(t))dt, \quad u \in L^2(0,T;H), \tag{2.2}$$

which is a proper, l.s.c and convex function on $L^2(0,T;H)$ and its effective domain $D(\Phi_0)$ is the set $\{u \in L^2(0,T;H) \mid \Phi_0(u) < \infty\}$. Let $\mathcal{X}$ be a Banach space, $\mathcal{X} \subset C([0,T];X)$ with continuous embedding and $\mathcal{X}_0$ be a closed and bounded subset of $\mathcal{X}$. Now, we assume that

(Φ) (i) to each $p \in \mathcal{X}_0$ and each $t \in [0,T]$ a proper, l.s.c. convex function $\varphi^t(p;\cdot)$ on H is assigned, satisfying

$$\varphi^t(p;z) \ge \varphi_0(z), \quad \forall z \in H, \tag{2.3}$$

(ii) if $p,\ \tilde{p} \in \mathcal{X}_0$ and $p = \tilde{p}$ on $[0,t],\ 0 \le t \le T$, then $\varphi^t(p;\cdot) = \varphi^t(\tilde{p};\cdot)$ on H,

(iii) if $\{p_n\} \subset \mathcal{X}_0$, $p_n \to p$ in $\mathcal{X}$, $t_n \in [0,T]$ with $t_n \to t$ and $z_n \to z$ weakly in H, then

$$\liminf_{n\to\infty} \varphi^{t_n}(p_n; z_n) \ge \varphi^t(p;z), \tag{2.4}$$

Now, we introduce a subset $\mathcal{X}_S$ of $\mathcal{X}_0$, which consists of all parameters $p \in \mathcal{X}_0$ such that $\{\varphi^t(p;\cdot)\}_{0\le t\le T}$ satisfies:

(Φ_S) for each positive number r there are real-valued functions $a_{p,r} \in W^{1,2}(0,T)$ and $b_{p,r} \in W^{1,1}(0,T)$ having the property that for any $s,t \in [0,T]$ and any $z \in D(\varphi^s(p;\cdot))$ with $|z|_H \le r$ there is $\tilde{z} \in D(\varphi^t(p;\cdot))$ such that

$$|\tilde{z} - z|_H \le |a_{p,r}(t) - a_{p,r}(s)|(1 + \varphi^s(p;z)^{\frac{1}{2}}),$$

and

$$\varphi^t(p;\tilde{z}) - \varphi^s(p;z) \le |b_{p,r}(t) - b_{p,r}(s)|(1 + \varphi^s(p;z)).$$

Throughout this paper we suppose that $\mathcal{X}_S \ne \emptyset$. Now, given $p \in \mathcal{X}_S$, consider the evolution inclusion:

$$u'(t) + \partial\varphi^t(p;u(t)) \ni f(t) \quad \text{in } H, \quad \text{a.e. } t \in (0,T), \tag{2.5}$$

where f is given in $L^2(0,T;H)$. By the definition of subdifferential, (2.5) is equivalent to the following variational inequality

$$(u'(t), u(t) - z)_H + \varphi^t(p;u(t)) \le \varphi^t(p;z) + (f(t), u(t) - z)_H, \tag{2.6}$$

$$\forall z \in D(\varphi^t(p;\cdot)), \quad \text{a.e. } t \in (0,T).$$

In the sequel, we denote by $P(\varphi^t(p;\cdot); f)$ the evolution inclusion (2.5) or variational inequality (2.6) and by $CP(\varphi^t(p;\cdot); f, u_0)$ its Cauchy problem associated with initial condition $u(0) = u_0$.

By virtue of the abstract theory on nonlinear evolution inclusions governed by time-dependent subdifferentials (cf. [16, 17, 29]), if $p \in \mathcal{X}_S$, $f \in L^2(0,T;H)$ and $u_0 \in D(\varphi^0(p;\cdot))$, then the Cauchy problem $CP(\varphi^t(p;\cdot); f, u_0)$ has one and only one solution u in $W^{1,2}(0,T;H)$ such that $t \mapsto \varphi^t(p;u(t))$ is absolutely continuous on $[0,T]$. Also, if u_0 is in $\overline{D(\varphi^0(p;\cdot))}$, the closure of $D(\varphi^0(p;\cdot))$ in H, then it has one and only one solution u in $C([0,T];H) \cap W^{1,2}_{\text{loc}}((0,T];H)$ such that $t \mapsto \varphi^t(p;u(t))$ is integrable on $(0,T)$ and is absolutely continuous on each compact interval in $(0,T]$. These are called strong solutions of $CP(\varphi^t(p;\cdot); f, u_0)$. It is easy to derive from (2.5) or (2.6) by using the integration by parts that the strong solution u satisfies

$$\int_0^t (\eta'(\tau), u(\tau) - \eta(\tau))_H d\tau + \int_0^t \varphi^\tau(p;u(\tau))d\tau + \frac{1}{2}|u(t) - \eta(t)|_H^2$$

$$\le \int_0^t \varphi^\tau(p;\eta(\tau))d\tau + \int_0^t (f(\tau), u(\tau) - \eta(\tau))_H d\tau + \frac{1}{2}\big|u_0 - \eta(0)\big|_H^2, \tag{2.7}$$

$$\forall \eta \in \mathcal{K}_0(p), \quad \forall t \in (0,T],$$

where

$$\mathcal{K}_0(p) := \{\eta \in W^{1,2}(0,T;H) \mid \varphi^{(\cdot)}(p;\eta(\cdot)) \in L^1(0,T)\}. \tag{2.8}$$

Moreover, under the regularity $u \in C([0,T];H) \cap W^{1,2}_{loc}((0,T];H)$ we see that (2.7) is equivalent to (2.5) (and hence (2.6)) with $u(0) = u_0$. However, without this regularity, the variational inequality (2.7) is a more general concept than (2.5). This is a reason why we set up a wider class $\mathcal{X}_W$ of parameters p than $\mathcal{X}_S$, which solves the variational inequality (2.7)-(2.8).

Definition 2.1. Let $p \in \mathcal{X}_0$, $f \in L^2(0,T;H)$ and $u_0 \in H$. We say that a function $u : [0,T] \to H$ is a weak solution of the Cauchy problem $CP(\varphi^t(p;\cdot); f, u_0)$, if $u \in C([0,T];H)$ with $u(0) = u_0$, $\varphi^{(\cdot)}(p;u(\cdot)) \in L^1(0,T)$ and the variational inequality (2.7)-(2.8) holds.

In the sequel, assuming that $\mathcal{X}_S$ is non-empty, we define the weak class $\mathcal{X}_W$ $(\subset \mathcal{X}_0)$ as the closure of $\mathcal{X}_S$ in $\mathcal{X}$ and suppose that the dependence of $\varphi^t(p;\cdot)$ upon $(t,p) \in [0,T] \times \mathcal{X}_W$ is characterized by using a family $\{L_{p\bar{p}}(t) \in B(H) \mid p,\ \bar{p} \in \mathcal{X}_W,\ t \in [0,T]\}$ of bounded, linear and invertible operators in H, where $B(H)$ stands for the space of all bounded linear operators in H, and a family $\{\sigma_{p\bar{p},\varepsilon} \in W^{1,2}(0,T;H) \mid p,\ \bar{p} \in \mathcal{X},\ 0 < \varepsilon \le 1\}$ as follows:

(A1) There is a positive constant L_0 such that

$$|L_{p\bar{p}} - I|_{C([0,T];B(H))} + |L'_{p\bar{p}}|_{L^2(0,T;B(H))} \le L_0 |p - \bar{p}|_{\mathcal{X}}, \forall p,\ \bar{p} \in \mathcal{X}_W, \tag{2.9}$$

where $L'_{p\bar{p}}(t) = \frac{d}{dt} L_{p\bar{p}}(t)$ in $B(H)$. Moreover, for any elements $p, \bar{p} \in \mathcal{X}_W$ we have

$$L^*_{p\bar{p}}(t) = L^{-1}_{p\bar{p}}(t) = L_{\bar{p}p}(t),\ \forall p, \bar{p} \in \mathcal{X}_W,\ \forall t \in [0,T], \tag{2.10}$$

namely the adjoint $L^*_{p\bar{p}}(t)$ of $L_{p\bar{p}}(t)$ coincides with the inverse of $L_{p\bar{p}}(t)$ and it is $L_{\bar{p}p}(t)$ for all $t \in [0,T]$.

(A2) There is a positive constant σ_0 such that

$$\sup_{t \in [0,T]} \varphi_0(\sigma_{p\bar{p},\varepsilon}(t)) \le \sigma_0, \quad |\sigma_{p\bar{p},\varepsilon}|_{W^{1,2}(0,T;H)} \le \sigma_0(|p - \bar{p}|_{\mathcal{X}} + \varepsilon), \tag{2.11}$$

$$\forall p,\ \bar{p} \in \mathcal{X}_W,\ \forall \varepsilon \in (0,1].$$

(A3) There are a continuous function $c_0(\cdot)$ on $[0,1]$ with $c_0(0) = 0$ and a non-negative continuous function $c_1(\cdot)$ on $[0,1]$ with $c_1(0) = 0$ satisfying the following property that for any $\varepsilon \in (0,1]$ there is a positive number δ_ε $(< \varepsilon)$ such that for any $p, \bar{p} \in \mathcal{X}_W$ with $|\bar{p} - p|_{\mathcal{X}} \le \delta_\varepsilon$, any $t \in [0,T]$ and any $z \in D(\varphi^t(p;\cdot))$ the element $\bar{z} := (1 + c_0(\varepsilon)) L_{p\bar{p}}(t) z + \sigma_{p\bar{p},\varepsilon}(t)$ belongs to $D(\varphi^t(\bar{p};\cdot))$ and

$$\varphi^t(\bar{p};\bar{z}) \le \varphi^t(p;z) + c_1(\varepsilon)(1 + \varphi^t(p;z)). \tag{2.12}$$

Remark 2.1. From the assumption $\mathcal{X}_S \neq \emptyset$ we easily check $\mathcal{K}_0(p) \neq \emptyset$ for any $p \in \mathcal{X}_W$. In fact, for any $\tilde{p} \in \mathcal{X}_S$ the evolution inclusion $\tilde{u}'(t)+\varphi^t(\tilde{p};\tilde{u}(t)) \ni 0,\ \tilde{u}(0) = \tilde{u}_0 \in D(\varphi^0(\tilde{p};\cdot))$, has one and only one solution $\tilde{u} \in W^{1,2}(0,T;H)$. Therefore, for any $p \in \mathcal{X}_W$ and $\varepsilon \in (0,1]$ we choose $\tilde{p} \in \mathcal{X}_S$ such that $|p-\tilde{p}|_{\mathcal{X}} \leq \delta_\varepsilon$, where δ_ε is as in condition (A3) and put

$$u(t) = (1+c_0(\varepsilon))L_{\tilde{p}p}(t)\tilde{u}(t) + \sigma_{\tilde{p}p,\varepsilon}(t), \quad t \in [0,T].$$

We see from (A1)-(A3) that $t \to \varphi^t(p;u(t))$ is bounded on $[0,T]$ and $u \in W^{1,2}(0,T;H)$. Thus $u \in \mathcal{K}_0(p)$.

The first main result is concerned with the existence and uniqueness of a weak solution to $CP(\varphi^t(p;\cdot);f,u_0)$.

Theorem 2.1. *Suppose that the family $\{\varphi^t(p,\cdot)|p \in \mathcal{X}_W, t \in [0,T]\}$ satisfies conditions (Φ), (Φ_S) and (A1)-(A3). Let $p \in \mathcal{X}_W$, $f \in L^2(0,T;H)$ and $u_0 \in \overline{D(\varphi^0(p;\cdot))}$ (the closure of $D(\varphi^0(p;\cdot))$ in H). Then:*

(i) Let $\{p_n\}$, $\{f_n\}$ and $\{u_{0n}\}$ with $u_{0n} \in D(\varphi^0(p_n;\cdot))$ be any sequences in $\mathcal{X}_S$, $L^2(0,T;H)$ and H, respectively, such that $p_n \to p$ in $\mathcal{X}$, $f_n \to f$ in $L^2(0T;H)$ and $u_{0n} \to u_0$ in H as $n \to \infty$. Then the strong solution u_n of $CP(\varphi^t(p_n;\cdot);f_n,u_{0n})$ converges in $C([0,T];H)$ as $n \to \infty$. Moreover, the limit u is a weak solution of $CP(\varphi^t(p;\cdot);f,u_0)$.

(ii) $CP(\varphi^t(p;\cdot);f,u_0)$ admits one and only one weak solution.

The second theorem is concerned with the continuous dependence of the weak solution of $CP(\varphi^t(p;\cdot);f,u_0)$ upon the data p, f and u_0.

Theorem 2.2. *Suppose that the family $\{\varphi^t(p,\cdot) \mid p \in \mathcal{X}_W, t \in [0,T]\}$ satisfies conditions (Φ), (Φ_S) and (A1)-(A3). Then:*

(i) Let $p \in \mathcal{X}_W$, f, $\bar{f} \in L^2(0,T;H)$ and u_0, $\bar{u}_0 \in \overline{D(\varphi^t(p;\cdot))}$. Then, for the weak solution u of $CP(\varphi^t(p;\cdot);f,u_0)$ and the weak solution $\bar{u}$ of $CP(\varphi^t(p;\cdot);\bar{f},\bar{u}_0)$ the following inequality holds:

$$\frac{1}{2}|u(t)-\bar{u}(t)|_H^2 \leq \frac{1}{2}|u_0-\bar{u}_0|_H^2+\int_0^t (f(\tau)-\bar{f}(\tau),u(\tau)-\bar{u}(\tau))_H d\tau, \quad \forall t \in [0,T]. \tag{2.13}$$

(ii) Let $\{p_n\}$, $\{f_n\}$ and $\{u_{0n}\}$ with $u_{0n} \in \overline{D(\varphi^0(p_n;\cdot))}$ be sequences in $\mathcal{X}_W$, $L^2(0,T;H)$ and H, respectively, such that

$$p_n \to p \text{ in } \mathcal{X},\ f_n \to f \text{ in } L^2(0,T;H),\ u_{0n} \to u_0 \text{ in } H,$$

as $n \to \infty$. Then the weak solution u_n of $CP(\varphi^t(p_n;\cdot);f_n,u_{0n})$ converges to the weak solution u of $CP(\varphi^t(p;\cdot);f,u_0)$ in $C([0,T];H)$.

The original idea of the weak solvability for nonlinear evolution inclusions is found in the paper [7] and a set of conditions $\{(A1),(A2),(A3)\}$ is its generalization which enables us to deal with a much wider class of nonlinear evolution inclusions generated by time-dependent subdifferetials in a weak sense.

We shall prove the above theorems in section 3. Prior to it, conditions (A1)-(A3) are illustrated by simple examples of $\varphi^t(p;\cdot)$ in the finite dimensional cases of H and X.

Example 2.1. We simply consider the space $\mathbf{R}^3$ as H and $\mathbf{R}^4$ as X, and a continuous convex function $z \mapsto \varphi_0(z) := \frac{1}{2}|z|^2$ on $\mathbf{R}^3$ as $\varphi_0(\cdot)$; we denote simply by $|z|$ the norm $|z|_{\mathbf{R}^3}$. Now, let $\mathcal{X} := W^{1,2}(0,T;\mathbf{R}^3) \times C([0,T])$ $(\subset C([0,T];\mathbf{R}^4))$. For positive constants m^*, d_* and d^* with $d_* < d^*$ we define $\mathcal{X}_0$ by

$$\mathcal{X}_0 := \{p := [v,d] \in \mathcal{X} \mid |v|_{W^{1,2}(0,T;\mathbf{R}^3)} \le m^*,\ d_* \le d(t) \le d^*,\ \forall t \in [0,T]\}, \tag{2.14}$$

which is a bounded, closed and convex subset of $\mathcal{X}$.

Now, for each $p := [v,d] \in \mathcal{X}_0$ and $t \in [0,T]$ we put

$$K^t(p) := \{z \in \mathbf{R}^3 \mid |z - v(t)| \le d(t)\}, \tag{2.15}$$

which is non-empty, compact and convex in $\mathbf{R}^3$, and define a proper, l.s.c. convex function $\varphi^t(p;\cdot)$ on $\mathbf{R}^3$ by

$$\varphi^t(p;z) := \begin{cases} \dfrac{1}{2}|z|^2, & \text{if } z \in K^t(p), \\ \infty, & \text{otherwise}, \end{cases} \tag{2.16}$$

namely

$$\varphi^t(p;z) := \frac{1}{2}|z|^2 + I_{K^t(p)}(z), \quad \forall z \in \mathbf{R}^3, \tag{2.17}$$

where $I_{K^t(p)}(\cdot)$ is the indicator function of $K^t(p)$, i.e.

$$I_{K^t(p)}(z) := \begin{cases} 0, & \text{if } z \in K^t(p), \\ \infty, & \text{otherwise} \end{cases}$$

It is easy to see that the family $\{\varphi^t(p;\cdot)\}$ satisfies condition (Φ). Also, for any functions $p := [v,d], \bar{p} := [\bar{v},\bar{d}] \in \mathcal{X}_0$ and $\varepsilon \in (0,1]$ we take as $L_{p\bar{p}}(t)$ the identity I and as $\sigma_{p\bar{p},\varepsilon}$ the function $\bar{v} - v + \varepsilon v$. We take also a positive constant m_1^* such that

$$|w|_{C([0,T];\mathbf{R}^3)} \le m_1^*, \quad \forall w \text{ with } |w|_{W^{1,2}(0,T;\mathbf{R}^3)} \le m^*.$$

We begin to show that $\mathcal{X}_1 := \{p := [v,d] \in \mathcal{X}_0 \mid v \in C^1([0,T];\mathbf{R}^3),\ d \in C^1([0,T])\} \subset \mathcal{X}_S$. To this end, let $p := [v,d] \in \mathcal{X}_1$, $s,t \in [0,T]$ and $z \in K^s(p)$. Then we have $|z - v(s)| \le d(s) = d(t) \cdot \frac{d(s)}{d(t)}$ which is rewritten in the form

$$|\tilde{z} - v(t)| \le d(t), \quad \text{with } \tilde{z} := \frac{d(t)}{d(s)}z - \frac{d(t)}{d(s)}v(s) + v(t).$$

Therefore $\tilde{z} \in K^t(p)$ and

$$|\tilde{z} - z| \le \frac{|d(t) - d(s)|}{d_*}(|z| + |v(s)|) + |v(t) - v(s)| \le C|t-s|(2m_1^* + d^* + 1),$$

where $C = \frac{1}{d_*}(|d|_{C^1([0,T])} + |v|_{C^1([0,T])})$. It follows immediately from this inequality that condition $(\Phi)_S$ holds for

$$a_{p,r}(t) = C(2m_1^* + d^* + 1)t, \quad b_{p,r}(t) = C(m_1^* + d^*)(2m_1^* + d^* + 1)t.$$

This implies that $\mathcal{X}_1 \subset \mathcal{X}_S \subset \mathcal{X}_0$. Since $\mathcal{X}_1$ is dense in $\mathcal{X}_0$ with respect to the topology of $\mathcal{X}$, we have $\mathcal{X}_W = \mathcal{X}_0$.

As $(A1)$ is easily checked, we show $(A2)$ and $(A3)$ below. Given $\varepsilon \in (0,1)$, assume that $p := [v, d] \in \mathcal{X}_0$, $\bar{p} := [\bar{v}, \bar{d}] \in \mathcal{X}_0$, $|\bar{v} - v|_{W^{1,2}(0,T;\mathbf{R}^3)} \le \varepsilon$ and $|\bar{d}(t) - d(t)| \le \varepsilon d_*$ for all $t \in [0,T]$. If $|z - v(t)| \le d(t)$, then we have by $\bar{d}(t) \ge d_*$

$$(1-\varepsilon)|z - v(t)| \le (1-\varepsilon)d(t) \le (1-\varepsilon)(\varepsilon d_* + \bar{d}(t)) = \bar{d}(t) - \varepsilon\bar{d}(t) + (1-\varepsilon)\varepsilon d_* \le \bar{d}(t).$$

Therefore, putting

$$\bar{z} := (1-\varepsilon)(z - v(t)) + \bar{v}(t) = (1-\varepsilon)z + \sigma_{p\bar{p},\varepsilon}(t),$$

we see that

$$|\bar{z} - \bar{v}(t)| \le \bar{d}(t), \quad \text{hence } \bar{z} \in K^t(\bar{p}),$$

$$\varphi^t(\bar{p}; \bar{z}) - \varphi^t(p; z) = \frac{1}{2}|\bar{z}|^2 - \frac{1}{2}|z|^2 \le c_2\varepsilon =: c_1(\varepsilon),$$

$$|\sigma'_{p\bar{p},\varepsilon}(t)| \le |\bar{v}'(t) - v'(t)| + \varepsilon|v'(t)|, \quad \text{hence } |\sigma_{p\bar{p},\varepsilon}|_{W^{1,2}(0,T;\mathbf{R}^3)} \le \sigma_0(|p - \bar{p}|_{\mathcal{X}} + \varepsilon),$$

for certain positive constants c_2 and σ_0 depending only on the class $\mathcal{X}_0$. Thus (A2) and (A3) hold.

Example 2.2. We take again $H = \mathbf{R}^3$, $X = \mathbf{R}^4$, $\varphi_0(z) = \frac{1}{2}|z|^2$ on $\mathbf{R}^3$ and $\mathcal{X} = W^{1,2}(0,T;\mathbf{R}^3) \times C([0,T])$, just as Example 2.1. Now, for positive constants R_0, R_1, R_2 $(> R_1)$, m^*, d_*, d^* $(> d_*)$ we put

$$V_{R_0,R_1,R_2} := \{v = (v^{(1)}, v^{(2)}, v^{(3)}) \mid |v^{(1)}| \le R_0, |v^{(2)}| \le R_0, R_1 \le v^{(3)} \le R_2\},$$

$$\mathcal{X}_0 := \left\{ p = [v, d] \in \mathcal{X} \;\middle|\; \begin{array}{l} |v|_{W^{1,2}(0,T;\mathbf{R}^3)} \le m^*, \\ v(t) \in V_{R_0,R_1,R_2},\ d_* \le d(t) \le d^*,\ \forall t \in [0,T] \end{array} \right\}$$

and for $p := [v, d] \in \mathcal{X}_0$ and $t \in [0,T]$

$$\varphi^t(p; z) = \frac{1}{2}|z|^2 + I_{K^t(p)}, \quad \forall z \in \mathbf{R}^3,$$

with

$$K^t(p) = \{z \in \mathbf{R}^3 \mid (v(t), z - v(t)) = 0,\ |z - v(t)| \le d(t)\}. \tag{2.18}$$

Next, we define the families $\{L_{p\bar{p}}\}$ and $\{\sigma_{p\bar{p},\varepsilon}\}$. To do so, given any non-zero vector $v = (v^{(1)}, v^{(2)}, v^{(3)}) \in \mathbf{R}^3$, let us consider the hyperplane which is orthogonal to the vector v at the point $(v^{(1)}, v^{(2)}, v^{(3)})$. Denote it by $\Gamma(v)$; note that $v \in \Gamma(v)$.

Also, given $p := [v, d]$, $\bar{p} := [\bar{v}, \bar{d}] \in \mathcal{X}_0$, we denote by $\alpha_{v\bar{v}}(t)$ the angle which is made by the vectors $v(t)$ and $\bar{v}(t)$, namely $\alpha_{v\bar{v}}(t)$ is given by the formula

$$\cos\alpha_{v\bar{v}}(t) = \frac{(v(t), \bar{v}(t))}{|v(t)|\ |\bar{v}(t)|}, \quad 0 \le \alpha_{v\bar{v}}(t) \le \pi - \mu \tag{2.19}$$

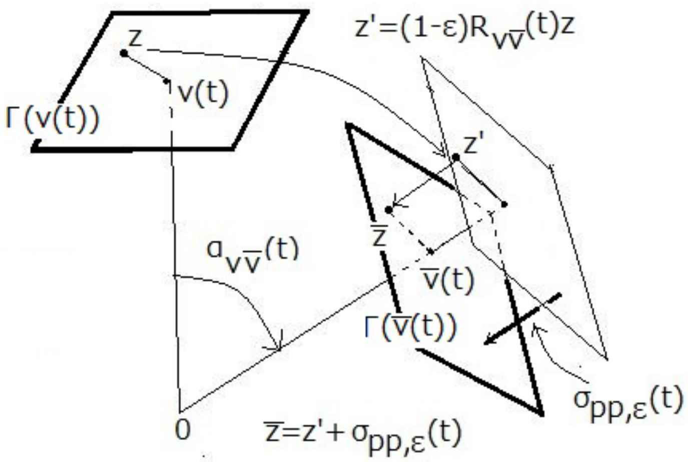
for a small positive constant μ depending only on $\mathcal{X}_0$. In the case when $v(t)$ and $\bar{v}(t)$ are linearly independent, denote by $\hat{\ell}_{v\bar{v}}(t) = (\hat{\ell}^{(1)}_{v\bar{v}}(t), \hat{\ell}^{(2)}_{v\bar{v}}(t), \hat{\ell}^{(3)}_{v\bar{v}}(t))$ the unit vector through the origin which is orthogonal to two vectors $v(t)$ and $\bar{v}(t)$, namely $\hat{\ell}_{v\bar{v}}(t) = \frac{v(t)\times\bar{v}(t)}{|v(t)\times\bar{v}(t)|}$, and consider the rotation $R_{v\bar{v}}(t)$ with angle $\alpha_{v\bar{v}}(t)$ around the axis $\hat{\ell}_{v\bar{v}}(t)$. In the linearly dependent case of $v(t)$ and $\bar{v}(t)$ we put $R_{v\bar{v}}(t) = I$. By using this rotation we define

$$L_{p\bar{p}}(t) = R_{v\bar{v}}(t), \quad \sigma_{p\bar{p},\varepsilon}(t) = \left(1 - \frac{|v(t)|}{|\bar{v}(t)|}\right)\bar{v}(t) + \varepsilon\frac{|v(t)|}{|\bar{v}(t)|}\bar{v}(t), \quad 0 \le t \le T, \tag{2.20}$$

for any $\varepsilon \in (0,1]$. We note that the mapping $z \to \bar{z}$ given by

$$\bar{z} - \bar{v}(t) = (1-\varepsilon)R_{v\bar{v}}(t)(z - v(t)), \text{ namely } \bar{z} = (1-\varepsilon)R_{v\bar{v}}(t)z + \sigma_{p\bar{p},\varepsilon}(t), \tag{2.21}$$

maps $v(t)$ to $\bar{v}(t)$ and the plane $\Gamma(v(t))$ onto $\Gamma(\bar{v}(t))$.



(Properties of $R_{v\bar{v}}(t)$)

Let $p = [v,d]$, $\bar{p} = [\bar{v},\bar{d}] \in \mathcal{X}_0$. When $v(t)$ and $\bar{v}(t)$ are linearly independent, we have the expression of $R_{v\bar{v}}(t)$ in the form, which is called Rodrigues' rotation formula,

$$R_{v\bar{v}}(t) = I + \sin\alpha_{v\bar{v}}(t)M_{v\bar{v}}(t) + (1-\cos\alpha_{v\bar{v}}(t))M_{v\bar{v}}(t)^2 \tag{2.22}$$

with

$$M_{v\bar{v}}(t) := \begin{pmatrix} 0 & -\hat{\ell}^{(3)}_{v\bar{v}}(t) & \hat{\ell}^{(2)}_{v\bar{v}}(t) \\ \hat{\ell}^{(3)}_{v\bar{v}}(t) & 0 & -\hat{\ell}^{(1)}_{v\bar{v}}(t) \\ -\hat{\ell}^{(2)}_{v\bar{v}}(t) & \hat{\ell}_{v\bar{v}}(1)(t) & 0 \end{pmatrix}. \tag{2.23}$$

Also, the the third term of the right side of (2.22) is written in the form

$$(1-\cos\alpha_{v\bar{v}}(t))M_{v\bar{v}}(t)^2 = \frac{1}{2\cos^2\frac{\alpha_{v\bar{v}}(t)}{2}}((\sin\alpha_{v\bar{v}}(t))M_{v\bar{v}}(t))^2.$$

Here we observe from $|v(t)\times\bar{v}(t)|=(\sin\alpha_{v\bar{v}}(t))|v(t)||\bar{v}(t)|$ that

$$(\sin\alpha_{v\bar{v}}(t))\hat{\ell}_{v\bar{v}}(t)=\frac{v(t)\times\bar{v}(t)}{|v(t)||\bar{v}(t)|}=\frac{(v(t)-\bar{v}(t))\times\bar{v}(t)}{|v(t)||\bar{v}(t)|}. \tag{2.24}$$

The right side of (2.24) is differentiable a.e. in $t\in[0,T]$, so is the left side. Therefore, by putting the left side of (2.24) to be 0 for t at which $v(t)$ and $\bar{v}(t)$ are linearly dependent, (2.24) makes sense for a.e. $t\in[0,T]$ and $(\sin\alpha_{v\bar{v}})\hat{\ell}_{v\bar{v}}\in W^{1,2}(0,T;\mathbf{R}^3)$. Simultaneously this shows by the expression (2.23) that $R_{v\bar{v}}(t)$ is differentiable in t a.e. on $[0,T]$, and furthermore it follows from (2.22) and (2.24) that

$$|R_{v\bar{v}}(t)-I|_{B(\mathbf{R}^3)}\le c_{\mathcal{X}_0}|v(t)-\bar{v}(t)| \tag{2.25}$$

and

$$|R'_{v\bar{v}}(t)|_{B(\mathbf{R}^3)}\le c_{\mathcal{X}_0}(|v'(t)|+|\bar{v}'(t)|+1)(|v(t)-\bar{v}(t)|+|v'(t)-\bar{v}'(t)|) \tag{2.26}$$

for a.e. $t\in[0,T]$, where $c_{\mathcal{X}_0}$ is a positive constant depending only on the class $\mathcal{X}_0$; hereafter we denote such a generic positive constant by $c_{\mathcal{X}_0}$.

By the way we check the differentiability of $\alpha_{v\bar{v}}(t)$. For simplicity, put $\hat{v}(t)=\frac{v(t)}{|v(t)|}$ and $\hat{\bar{v}}(t)=\frac{\bar{v}(t)}{|\bar{v}(t)|}$. Since $|\hat{v}(t)-\hat{\bar{v}}(t)|=2\sin\frac{\alpha_{v\bar{v}}(t)}{2}$, it follows that

$$\delta_0\alpha_{v\bar{v}}(t)\le\frac{\sin\frac{\alpha_{v\bar{v}}(t)}{2}}{\frac{\alpha_{v\bar{v}}(t)}{2}}\cdot\alpha_{v\bar{v}}(t)=2\sin\frac{\alpha_{v\bar{v}}(t)}{2}=\left|\hat{v}(t)-\hat{\bar{v}}(t)\right|,$$

where $\delta_0=\min\{\frac{\sin x}{x}\mid 0\le x\le\frac{\pi}{2}-\frac{\mu}{2}\}$ (>0), so that

$$\alpha_{v\bar{v}}(t)\le c_{\mathcal{X}_0}|v(t)-\bar{v}(t)|,\quad\forall t\in[0,T]. \tag{2.27}$$

Furthermore it follows that

$$\begin{aligned}
&\lim_{\Delta t\to 0}\frac{2}{\Delta t}\left\{\sin\frac{\alpha_{v\bar{v}}(t+\Delta t)}{2}-\sin\frac{\alpha_{v\bar{v}}(t)}{2}\right\}\\
=\;&\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\left|\hat{v}(t+\Delta t)-\hat{\bar{v}}(t+\Delta t)\right|-\left|\hat{v}(t)-\hat{\bar{v}}(t)\right|\right\}\\
=\;&\lim_{\Delta t\to 0}\frac{1}{\Delta t}\cdot\frac{|\hat{v}(t+\Delta t)-\hat{\bar{v}}(t+\Delta t)|^2-|\hat{v}(t)-\hat{\bar{v}}(t)|^2}{|\hat{v}(t+\Delta t)-\hat{\bar{v}}(t+\Delta t)|+|\hat{v}(t)-\hat{\bar{v}}(t)|}\\
=\;&\frac{(\hat{v}'(t)-\hat{\bar{v}}'(t),\hat{v}(t)-\hat{\bar{v}}(t))}{|\hat{v}(t)-\hat{\bar{v}}(t)|},
\end{aligned} \tag{2.28}$$

provided that $\hat{v}(t)\ne\hat{\bar{v}}(t)$, namely $v(t)$ and $\bar{v}(t)$ are linearly independent. The last equality of (2.28) is not larger than $|\hat{v}'(t)-\hat{\bar{v}}'(t)|$ and $\alpha_{v\bar{v}}(t)=0$ in the case of $\hat{v}(t)=\hat{\bar{v}}(t)$. Therefore, $\sin\frac{\alpha_{v\bar{v}}}{2}$ is differentiable a.e. on $[0,T]$ and $|\frac{d}{dt}\sin\frac{\alpha_{v\bar{v}}(t)}{2}|\le\frac{1}{2}|\hat{v}'(t)-\hat{\bar{v}}'(t)|$ for a.e. $t\in[0,T]$, so that $\alpha_{v\bar{v}}(t)$ is differentiable a.e. on $[0,T]$ and

$$|\alpha'_{v\bar{v}}(t)|\le\frac{1}{\delta_1}|\hat{v}'(t)-\hat{\bar{v}}'(t)|\le c_{\mathcal{X}_0}|v'(t)-\bar{v}'(t)|, \tag{2.29}$$

where $\delta_1 = \min\{\cos x \mid 0 \le x \le \frac{\pi}{2} - \frac{\mu}{2}\}$ (> 0).

(Verification of (A1)-(A3))

Condition (A1) is obtained by (2.25) and (2.26), and (A2) is easily checked from (2.20) and (2.21), since $v, \bar{v} \in W^{1,2}(0,T;\mathbf{R}^3)$. Next, we verify (A3). Let $\varepsilon \in (0,1]$ be any number, and let $p = [v,d]$ and $\bar{p} = [\bar{v},\bar{d}]$ be any elements in $\mathcal{X}_0$ so that $|\bar{d}(t) - d(t)| \le \varepsilon d_*$ for any $t \in [0,T]$ and $|\bar{v} - v|_{C([0,T];\mathbf{R}^3)} + |\bar{v}' - v'|_{L^2(0,T;\mathbf{R}^3)} \le \varepsilon$. Then it follows that for any $z \in D(\varphi^t(p;\cdot)) = K^t(p)$ the element $\bar{z}$ given by (2.21) satisfies

$$|\bar{z} - \bar{v}(t)| \le (1-\varepsilon)|z - v(t)| \le (1-\varepsilon)d(t) \le \bar{d}(t), \quad \text{hence } \bar{z} \in K^t(\bar{p}).$$

Therefore the mapping (2.21) maps $K^t(p)$ into $K^t(\bar{p})$ and by (2.25)

$$\begin{aligned} |\bar{z} - z| &\le (1-\varepsilon)|R_{v\bar{v}}(t) - I|_{B(\mathbf{R}^3)}|z| + \sigma_{p\bar{p},\varepsilon} + \varepsilon|z| \\ &\le (1-\varepsilon)c_{\mathcal{X}_0}|v(t) - \bar{v}(t)||z| + \sigma_{p\bar{p},\varepsilon} + \varepsilon|z|. \end{aligned}$$

It is easy to derive (A3) from the above inequalities with continuous functions $c_0(\varepsilon) = -\varepsilon$ and $c_1(\varepsilon) = c_2\varepsilon$ for a positive constant c_2.

(Verification of (Φ) *and* (Φ_S)*)*

The verification of (Φ) is easy for $\varphi_0(z) = \frac{1}{2}|z|^2$ on $\mathbf{R}^3$, and regarding (Φ_S) we observe that $\mathcal{X}_1 := \{p = [v,d] \in \mathcal{X}_0 \mid v \in C^1([0T];\mathbf{R}^3),\ d \in C^1([0,T])\} \subset \mathcal{X}_S$, whence $\mathcal{X}_W = \mathcal{X}_0$. In fact, to show this we use the rotation $R_v(s,t)$ with the angle $\alpha_v(s,t)$, given by $\cos\alpha_v(s,t) = \frac{(v(s),v(t))}{|v(s)||v(t)|}$, around the axis $\hat{\ell}_v(s,t) = (\hat{\ell}_v^{(1)}(s,t), \hat{\ell}_v^{(2)}(s,t), \hat{\ell}_v^{(3)}(s,t)) := \frac{v(s)\times v(t)}{|v(s)\times v(t)|}$ in the case of $\hat{v}(s) := \frac{v(s)}{|v(s)|} \ne \hat{v}(t) := \frac{v(t)}{|v(t)|}$ for each $p = [v,d] \in \mathcal{X}_1$ and $s,t \in [0,T]$; by definition $R_v(s,t) = I$ in the case of $\hat{v}(s) = \hat{v}(t)$. Now, given $z \in K^s(p)$, consider

$$\tilde{z} := \frac{d(t)}{d(s)}R_v(s,t)z + \left(1 - \frac{d(t)|v(s)|}{d(s)|v(t)|}\right)v(t) = \frac{d(t)}{d(s)}R_v(s,t)(z - v(s)) + v(t). \qquad (2.29)$$

Then $\tilde{z} \in K^t(p)$ and

$$|\tilde{z} - z| \le \frac{d(t)}{d(s)}|(R_v(s,t) - I)z| + \left|\frac{d(t)}{d(s)} - 1\right||z| + |v(t)| - \frac{d(t)}{d(s)}|v(s)|. \qquad (2.30)$$

Just as in the case of rotation $R_{v\bar{v}}(t)$, we have the expression of $R_v(s,t)$, of the form

$$R_v(s,t) = I + \sin\alpha_v(s,t)M_v(s,t) + (1 - \cos\alpha_v(s,t))M_v(s,t)^2, \text{ if } \hat{v}(s) \ne \hat{v}(t), \qquad (2.31)$$

with a matrix $M_v(s,t)$ of the form

$$M_v(s,t) := \begin{pmatrix} 0 & -\hat{\ell}_v^{(3)}(s,t) & \hat{\ell}_v^{(2)}(s,t) \\ \hat{\ell}_v^{(3)}(s,t) & 0 & -\hat{\ell}_v^{(1)}(s,t) \\ -\hat{\ell}_v^{(2)}(s,t) & \hat{\ell}_v^{(1)}(s,t) & 0 \end{pmatrix},$$

provided that $\hat{v}(s) \ne \hat{v}(t)$; the formula (2.31) makes sense in the case of $\hat{v}(s) = \hat{v}(t)$ by the same reason as before. Moreover we get the following estimates of $\alpha_v(s,t)$ and $R_v(s,t)$ similar to those of $\alpha_{v\bar{v}}(t)$ and $R_{v\bar{v}}(t)$:

$$\sin\alpha_v(s,t) \le \alpha_v(s,t) \le c_{\mathcal{X}_0}|v(t) - v(s)|,\ \forall s,t \in [0,T],\ s \le t, \qquad (2.32)$$

and

$$\lim_{t\downarrow s}\frac{2}{t-s}\sin\frac{\alpha_v(s,t)}{2}=\lim_{t\downarrow s}\frac{1}{t-s}|\hat{v}(t)-\hat{v}(s)|=|\hat{v}'(s)|,\quad \forall s\in[0,T],$$

whence

$$\lim_{t\downarrow s}\frac{\alpha_v(s,t)}{t-s}=|\hat{v}'(s)|\le c_{\mathcal{X}_0}|v'(s)|,\ \forall s\in[0,T] \tag{2.33}$$

As a consequence, we have by (2.31) with (2.32) and (2.33)

$$\begin{aligned}&\left|\lim_{t\downarrow s}\frac{R_v(s,t)-I}{t-s}\right|_{B(\mathbf{R}^3)}\le c_{\mathcal{X}_0}|v'(s)|,\ \forall s\in[0,T).\\ &|R_v(s,t)-I|_{B(\mathbf{R}^3)}\le c_{\mathcal{X}_0}\int_s^t|v'(\tau)|d\tau,\quad 0\le\forall s\le\forall t\le T,\end{aligned}\tag{2.34}$$

On account of estimates (2.34) we derive from (2.30) that

$$|\tilde{z}-z|\le c_{\mathcal{X}_0}(1+|z|)\int_s^t(|v'(\tau)|+|d'(\tau)|+1)d\tau. \tag{2.35}$$

This inequality implies that (Φ_S) holds for $p\in\mathcal{X}_1$. Thus $\mathcal{X}_1\subset\mathcal{X}_S$ and $\mathcal{X}_W=\mathcal{X}_0$, since $\mathcal{X}_1$ is dense in $\mathcal{X}_0$ with respect to the topology of $\mathcal{X}$.

Remark 2.2. The main difference between the above examples is that the dimension of convex constraint sets is three in Example 2.1, while that is two in Example 2.2. In the former case we do not use any rotation operator in order to verify conditions (A1)-(A3), but in the latter case we need it to do so.

Remark 2.3. In the set-up of weak formulation of the type (2.7) we need the non-empty class $\mathcal{K}_0(p)$ of test functions (cf. (2.8)) and any test function η must be in $W^{1,2}(0,T;H)$. This requires at least that the hyperplane including time-dependent convex constraint set must move smoothly in time. This is why we need the square-summable time-derivatives of rotation $R_{v\bar{v}}(t)$ and function $\sigma_{p\bar{p},\varepsilon}(t)$ in Example 2.2.

3 Proof of Theorems 2.1 and 2.2

In this section, we give the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1.

Let $p\in\mathcal{X}_W$, $f\in L^2(0,T;H)$, $u_0\in\overline{D(\varphi^0(p;\cdot))}$. Let $\{p_n\},\{f_n\}$ and $\{u_{0n}\}$ be sequences as in the statement (i). We use the following simple notation:

$$L_{nm}(t):=L_{p_np_m}(t),\quad \sigma_{nm,\varepsilon}(t):=\sigma_{p_np_m,\varepsilon}(t),$$

where $L_{p_np_m}$ and $\sigma_{p_np_m,\varepsilon}$ are the operators and functions given in $(A1)-(A3)$ for parameters $p_n,\ p_m\in\mathcal{X}_0$ and $\varepsilon\in(0,1]$.

Now, for each n, problem

$$u_n'(t) + \partial\varphi^t(p_n; u_n(t)) \ni f_n(t) \text{ in } H, \quad \text{a.e. } t \in (0,T), \quad u_n(0) = u_{0n}, \tag{3.1}$$

has a unique strong solution $u_n \in W^{1,2}(0,T;H)$ such that $t \to \varphi^t(p_n; u_n(t))$ is absolutely continuous on $[0,T]$. The evolution inclusion in (3.1) is equivalent to

$$(u_n'(t), u_n(t) - z)_H + \varphi^t(p_n; u_n(t)) \leq \varphi^t(p_n; z) + (f_n(t), u_n(t) - z)_H, \tag{3.2}$$

$$\forall z \in H, \text{ a.e. } t \in (0,T).$$

For any small $\varepsilon \in (0,1]$ it follows from $(A1) - (A3)$ that there is a positive integer N_ε such that for all integers $n,\ m \geq N_\varepsilon$

$$\tilde{u}_{mn}(t) := (1 + c_0(\varepsilon))L_{mn}(t)u_m(t) + \sigma_{mn,\varepsilon}(t) \in D(\varphi^t(p_n; \cdot)), \tag{3.3}$$

$$\tilde{u}_{mn}'(t) := (1 + c_0(\varepsilon))L_{mn}(t)u_m'(t) + \sigma_{mn,\varepsilon}'(t) + (1 + c_0(\varepsilon))L_{mn}'(t)u_m(t) \tag{3.4}$$

and

$$\varphi^t(p_n; \tilde{u}_{mn}(t)) \leq \varphi^t(p_m; u_m(t)) + c_1(\varepsilon)(1 + \varphi^t(p_m; u_m(t))). \tag{3.5}$$

Lemma 3.1. *There is a constant* $M_1 := M_1(p) \geq 0$ *(depending on p) such that*

$$|u_n|_{C([0,T];H)} + \int_0^T \varphi^t(p_n; u_n(t))dt \leq M_1, \quad \forall n \in \mathbf{N}. \tag{3.6}$$

Proof. Fix $m \geq N_\varepsilon$ and consider $\tilde{u}_{mn}$ as given by (3.3) for all $n \geq N_\varepsilon$. Then, by taking $z = \tilde{u}_{mn}(t)$ in (3.2) we obtain

$$(u_n'(t), u_n(t) - \tilde{u}_{mn}(t))_H + \varphi^t(p_n; u_n(t)) \leq \varphi^t(p_n; \tilde{u}_{mn}(t)) + (f_n(t), u_n(t) - \tilde{u}_{mn}(t))_H. \tag{3.7}$$

Using the integration by parts in (3.7), we see that

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}|u_n(t)|_H^2 - (u_n(t), \tilde{u}_{mn}(t))_H\right\} + \varphi^t(p_n; u_n(t)) \\ &\leq -(u_n(t), \tilde{u}_{mn}'(t))_H + \varphi^t(p_n; \tilde{u}_{mn}(t)) + (f_n(t), u_n(t) - \tilde{u}_{mn}(t))_H. \end{aligned} \tag{3.8}$$

Moreover it follows from (3.3), (3.4) and (3.5) that $\{\tilde{u}_{mn}\}_{n\geq N_\varepsilon}$ is bounded in $W^{1,2}(0,T;H)$ and $\{\varphi^{(\cdot)}(p_n; \tilde{u}_{mn}(t))dt\}_{n\geq N_\varepsilon}$ is bounded in $C([0,T])$, as long as m is fixed. Now, rearrange (3.8) in the form

$$\frac{d}{dt}G_n(t) \leq m_1 G_n(t) + m_2(1 + |f_n(t)|_H^2 + |\tilde{u}_{mn}(t)|_H^2 + |\tilde{u}_{mn}'(t)|_H^2)$$

where $G_n(t) = \frac{1}{2}|u_n(t)|_H^2 - (u_n(t), \tilde{u}_{mn}(t))_H$ and m_1 and m_2 are positive constants independent of n. Applying the Gronwall's inequality to it, we get that $G_n(t)$ is uniformly bounded on $[0,T]$ and hence $\{u_n\}$ is bounded in $C([0,T];H)$. As a consequence, by going back to (3.2), we see that $\int_0^T \varphi^t(v_n; u_n(t))dt$ is bounded, too. Thus (3.6) holds for a certain positive constant $M_1 \geq 0$. $\diamondsuit$

By taking $\tilde{u}_{mn}(t)$ as z in (3.2) to get (3.7) again and substituting (3.3) and (3.5) we obtain that

$$
\begin{aligned}
&(u_n'(t), u_n(t) - u_m(t))_H + \varphi^t(v_n; u_n(t)) \\
&\le \varphi^t(v_m; u_m(t)) + (f_n(t), u_n(t) - u_m(t))_H + \Gamma_{nm,\varepsilon}(t) + \tilde{\Gamma}_{nm,\varepsilon}(t),
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
\Gamma_{nm,\varepsilon}(t) := {} & (u_n'(t), (L_{mn}(t) - I)u_m(t))_H + c_0(\varepsilon)(u_n'(t), L_{mn}(t)u_m(t))_H \\
& + \frac{d}{dt}(u_n(t), \sigma_{mn,\varepsilon}(t))_H
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\tilde{\Gamma}_{nm,\varepsilon}(t) := {} & (f(t), (I - L_{mn}(t))u_m(t) - c_0(\varepsilon)L_{mn}(t)u_m(t) - \sigma_{mn,\varepsilon}(t))_H \\
& - (u_n(t), \sigma'_{mn,\varepsilon}(t))_H + c_1(\varepsilon)(1 + \varphi^t(p_m; u_m(t))).
\end{aligned} \tag{3.11}
$$

Now, exchanging n and m in (3.9) and (3.10)-(3.11), we similarly have

$$
\begin{aligned}
&(u_m'(t), u_m(t) - u_n(t))_H + \varphi^t(p_m; u_m(t)) \\
&\le \varphi^t(p_n; u_n(t)) + (f_m(t), u_m(t) - u_n(t))_H + \Gamma_{mn,\varepsilon}(t) + \tilde{\Gamma}_{mn,\varepsilon}(t).
\end{aligned} \tag{3.12}
$$

Now, adding (3.9) and (3.12), we obtain that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u_n(t) - u_m(t)|_H^2 \le {} & (f_n(t) - f_m(t), u_n(t) - u_m(t))_H \\
& + \Gamma_{mn,\varepsilon}(t) + \Gamma_{nm,\varepsilon}(t) + \tilde{\Gamma}_{mn,\varepsilon}(t) + \tilde{\Gamma}_{nm,\varepsilon}(t)
\end{aligned} \tag{3.13}
$$

for a.e. $t \in [0, T]$. As to the right hand side of (3.13) we prepare the following lemma.

Lemma 3.2. *(i) We have that*

$$
\begin{aligned}
&\Gamma_{mn,\varepsilon}(t) + \Gamma_{nm,\varepsilon}(t) \\
&= \frac{d}{dt}\{(u_n(t), (L_{mn}(t) - I)u_m(t))_H\} + c_0(\varepsilon)\frac{d}{dt}(u_n(t), L_{mn}(t)u_m(t))_H \\
&\quad + \frac{d}{dt}\{(u_n(t), \sigma_{mn,\varepsilon}(t))_H + (u_m(t), \sigma_{nm,\varepsilon}(t))_H\} \\
&\quad - (1 + c_0(\varepsilon))(u_n(t), L'_{mn}(t)u_m(t))_H
\end{aligned} \tag{3.14}
$$

and as $\varepsilon \to 0$ *(and hence* $n,\ m \to \infty$*)*

$$
\int_0^t \{\Gamma_{mn,\varepsilon}(\tau) + \Gamma_{nm,\varepsilon}(\tau)\}d\tau \to 0 \ \text{ uniformly in } t \in [0, T]. \tag{3.15}
$$

(ii) As $\varepsilon \to 0$ *(and hence* $n,\ m \to \infty$*) we have that*

$$
\int_0^T \left\{\left|\tilde{\Gamma}_{mn,\varepsilon}(\tau)\right| + \left|\tilde{\Gamma}_{nm,\varepsilon}(\tau)\right|\right\} d\tau \to 0. \tag{3.16}
$$

Proof. By (3.10) we have for a.e. $\tau \in [0,T]$

$$
\begin{aligned}
& \Gamma_{mn,\varepsilon}(\tau) + \Gamma_{nm,\varepsilon}(\tau) \\
= \;& (u_n'(\tau), (L_{mn}(\tau) - I)u_m(\tau))_H + (u_m'(\tau), (L_{nm}(\tau) - I)u_n(\tau))_H \\
& + c_0(\varepsilon)\{(u_n'(\tau), L_{mn}(\tau)u_m(\tau))_H + (u_m'(\tau), L_{nm}(\tau)u_n(\tau))_H\} \\
& + \frac{d}{d\tau}\{(u_n(\tau), \sigma_{mn,\varepsilon}(\tau))_H + (u_m(\tau), \sigma_{nm,\varepsilon}(\tau))_H\} \\
= \;& \frac{d}{d\tau}(u_n(\tau), (L_{mn}(\tau) - I)u_m(\tau))_H - (u_n(\tau), L_{mn}'(\tau)u_m(\tau))_H \\
& + c_0(\varepsilon)\frac{d}{d\tau}(u_n(\tau), L_{mn}(\tau)u_m(\tau))_H - c_0(\varepsilon)(u_n(\tau), L_{mn}'(\tau)u_m(\tau))_H \\
& + \frac{d}{d\tau}\{(u_n(\tau), \sigma_{mn,\varepsilon}(\tau))_H + (u_m(\tau), \sigma_{nm,\varepsilon}(\tau))_H\} \\
= \;& \frac{d}{d\tau}(u_n(\tau), (L_{mn}(\tau) - I)u_m(\tau))_H + c_0(\varepsilon)\frac{d}{d\tau}(u_n(\tau), L_{mn}(\tau)u_m(\tau))_H \\
& + \frac{d}{d\tau}\{(u_n(\tau), \sigma_{mn,\varepsilon}(\tau))_H + (u_m(\tau), \sigma_{nm,\varepsilon}(\tau))_H\} \\
& - (1 + c_0(\varepsilon))(u_n(\tau), L_{mn}'(\tau)u_m(\tau))_H.
\end{aligned}
$$

Thus (3.14) holds. Now, integrating it in τ over $[0,t]$ yields that

$$
\begin{aligned}
& \left| \int_0^t \{\Gamma_{mn,\varepsilon}(\tau) + \Gamma_{nm,\varepsilon}(\tau)\} d\tau \right| \\
\le \;& 2T\left\{ |L_{mn} - I|_{C([0,T];B(H))} + |c_0(\varepsilon)||L_{mn}|_{C([0,T];B(H))} \right\} |u_n|_{C([0,T];H)} |u_m|_{C([0,T];H)} \\
& + (1 + |c_0(\varepsilon)|)|u_n|_{C([0,T];H)}|u_m|_{C([0,T];H)} \int_0^T |L_{mn}'(\tau)|_{B(H)} d\tau \\
& + 2|u_n|_{C([0,T];H)}|\sigma_{mn,\varepsilon}|_{C([0,T];H)}|_H + 2|u_m|_{C([0,T];H)}|\sigma_{nm,\varepsilon}|_{C([0,T];H)}.
\end{aligned}
$$

Since $|p_n - p_m|_{\mathcal{X}} \to 0$ as $n,\ m \to \infty$, it follows from conditions (A1)-(A3) and estimate (3.6) of Lemma 3.1 that the right side of the above inequality tends to 0 as $\varepsilon \to 0$ and $n,\ m \to \infty$, so that (3.15) is obtained. Similarly (3.16) is proved. $\diamondsuit$

Finally, we have by integrating (3.13) in time that

$$
\begin{aligned}
\frac{1}{2}|u_n(t) - u_m(t)|_H^2 \le \frac{1}{2}|u_{0n} - u_{0m}|_H^2 + \int_0^t (f_n(\tau) - f_m(\tau), u_n(\tau) - u_m(\tau))_H d\tau \\
+ \int_0^t \{\Gamma_{mn,\varepsilon}(\tau) + \Gamma_{nm,\varepsilon}(\tau)\} d\tau + \int_0^T \left\{ \left|\tilde{\Gamma}_{mn,\varepsilon}(\tau)\right| + \left|\tilde{\Gamma}_{nm,\varepsilon}(\tau)\right| \right\} d\tau
\end{aligned}
$$

for all $t \in [0,T]$. By virtue of Lemma 3.2, we conclude that $\{u_n\}$ is a Cauchy sequence in $C([0,T];H)$. We denote by u its limit in $C([0,T];H)$ and it is clear that $u(0) = u_0$. Next we show that u is a weak solution of $CP(\varphi^t(p;\cdot); f, u_0)$. To see it we note (cf. (2.7)) that u_n satisfies

$$
\int_0^t (\eta_n'(\tau), u_n(\tau) - \eta_n(\tau))_H d\tau + \int_0^t \varphi^\tau(p_n; u_n(\tau)) d\tau + \frac{1}{2}|u_n(t) - \eta_n(t)|_H^2
$$

$$\leq \int_0^t \varphi^\tau(p_n;\eta_n(\tau))d\tau + \int_0^t (f_n(\tau), u_n(\tau)-\eta_n(\tau))_H d\tau + \frac{1}{2}|u_{0n}-\eta_n(0)|_H^2, \tag{3.17}$$
$$\forall \eta_n \in \mathcal{K}_0(p_n), \quad \forall t \in (0,T],$$

Let $\varepsilon > 0$ be a small number. Then, by conditions (A1)-(A3), there is a large integer N_ε such that for any $\eta \in \mathcal{K}_0(p)$ the function

$$\eta_{n\varepsilon}(t) := (1+c_0(\varepsilon))L_{pp_n}(t)\eta(t) + \sigma_{pp_n,\varepsilon}(t)$$

satisfies that $\eta_{n\varepsilon}(t) \in D(\varphi^t(p_n;\cdot))$ for any $n \geq N_\varepsilon$, and we see that

$$\eta_{n\varepsilon} \to \eta \text{ in } W^{1,2}(0,T;H), \quad \varphi^{(\cdot)}(p_n;\eta_{n\varepsilon}) \to \varphi^{(\cdot)}(p;\eta) \text{ in } L^1(0,T)$$

as $\varepsilon \to 0$ and $n \to \infty$. Also, by (iii) of (Φ),

$$\infty > \liminf_{n\to\infty} \int_0^t \varphi^\tau(p_n;u_n(\tau))d\tau \geq \int_0^t \varphi^\tau(p;u(\tau))d\tau, \quad \forall t \in [0,T].$$

Therefore, taking $\eta_{n\varepsilon}$ as η_n in (3.17) and letting $\varepsilon \to 0$ and $n \to \infty$, we have (2.7). This shows that u is a weak solution of $CP(\varphi^t(p;\cdot);f,u_0)$.

In order to accomplish the proof of Theorem 2.1 it remains to prove the uniqueness of weak solution of $CP(\varphi^t(p;\cdot);f,u_0)$. Let $\bar{u}$ be any weak solution of $CP(\varphi^t(p;\cdot);f,u_0)$. We shall show below that $\bar{u}$ coincides with the weak solution u which was constructed above as the limit of the strong solution u_n of $CP(\varphi^t(p_n;\cdot);f_n,u_{0n})$. For every large n and small $\varepsilon > 0$ we put

$$\bar{u}_{n\varepsilon}(\tau) := (1+c_0(\varepsilon))L_{pp_n}(\tau)\bar{u}(\tau) + \sigma_{pp_n,\varepsilon}(\tau) \in D(\varphi^\tau(p_n;\cdot)). \tag{3.18}$$

and

$$\tilde{u}_{n\varepsilon}(\tau) := (1+c_0(\varepsilon))L_{p_np}(\tau)u_n(\tau) + \sigma_{p_np,\varepsilon}(\tau) \in D(\varphi^\tau(p;\cdot)); \tag{3.19}$$

in this case we have by (2.12)

$$\varphi^\tau(p_n;\bar{u}_{n\varepsilon}(\tau)) \leq \varphi^\tau(p;\bar{u}(\tau)) + c_1(\varepsilon)(1+\varphi^\tau(p;\bar{u}(\tau))) \tag{3.20}$$

and

$$\varphi^\tau(p;\tilde{u}_{n\varepsilon}(\tau)) \leq \varphi^\tau(p_n;u_n(\tau)) + c_1(\varepsilon)(1+\varphi^\tau(p_n;u_n(\tau)). \tag{3.21}$$

Now we note that u_n satisfies

$$\begin{aligned} &\int_0^t (u_n'(\tau), u_n(\tau)-\bar{u}_{n\varepsilon}(\tau))_H d\tau + \int_0^t \varphi^\tau(p_n;u_n(\tau))d\tau \\ &\leq \int_0^t \varphi^\tau(p_n;\bar{u}_{n\varepsilon}(\tau))d\tau + \int_0^t (f_n(\tau), u_n(\tau)-\bar{u}_{n\varepsilon}(\tau))_H d\tau \end{aligned} \tag{3.22}$$

as well as $\bar{u}$ satisfies

$$\begin{aligned} &\int_0^t (\tilde{u}_{n\varepsilon}'(\tau), \bar{u}(\tau)-\tilde{u}_{n\varepsilon}(\tau))_H d\tau + \int_0^t \varphi^\tau(p;\bar{u}(\tau))d\tau + \frac{1}{2}|\bar{u}(t)-\tilde{u}_{n\varepsilon}(t)|_H^2 \\ &\leq \int_0^t \varphi^\tau(p;\tilde{u}_{n\varepsilon}(\tau))d\tau + \int_0^t (f(\tau), \bar{u}(\tau)-\tilde{u}_{n\varepsilon}(\tau))_H d\tau + \frac{1}{2}|u_0-\tilde{u}_{n\varepsilon}(0)|_H^2 \end{aligned} \tag{3.23}$$

for all $t \in (0, T]$. Substitute (3.20) and (3.21) in (3.22) and (3.23), respectively, and add these resultants to obtain that

$$\begin{aligned}\frac{1}{2}|\bar{u}(t) - \tilde{u}_{n\varepsilon}(t)|_H^2 \leq \frac{1}{2}|u_0 - \tilde{u}_{n\varepsilon}(0)|_H^2 + \int_0^t \{(f_n, u_n - \bar{u}_{n\varepsilon})_H + (f, \bar{u} - \tilde{u}_{n\varepsilon})_H\} d\tau \\ - \int_0^t (u'_n, u_n - \bar{u}_{n\varepsilon})_H d\tau - \int_0^t (\tilde{u}'_{n\varepsilon}, \bar{u} - \tilde{u}_{n\varepsilon})_H d\tau \qquad (3.24) \\ + c_1(\varepsilon) \int_0^t (2 + \varphi^\tau(p; \bar{u}) + \varphi^\tau(p_n; u_n)) d\tau\end{aligned}$$

for all $t \in [0, T]$. Here note from (3.18) and (3.19) that

$$\begin{aligned}& \int_0^t (u'_n, u_n - \bar{u}_{n\varepsilon})_H d\tau + \int_0^t (\tilde{u}'_{n\varepsilon}, \bar{u} - \tilde{u}_{n\varepsilon})_H d\tau \\ = \; & \int_0^t (u'_n, u_n)_H d\tau - (1 + c_0(\varepsilon)) \int_0^t (u'_n, L_{pp_n} \bar{u})_H d\tau - \int_0^t (u'_n, \sigma_{pp_n,\varepsilon})_H d\tau \\ & + (1 + c_0(\varepsilon)) \int_0^t (L_{p_n p} u'_n, \bar{u})_H d\tau - (1 + c_0(\varepsilon))^2 \int_0^t (L_{p_n p} u'_n, L_{p_n p} u_n)_H d\tau \\ & - (1 + c_0(\varepsilon)) \int_0^t (L_{p_n p} u'_n, \sigma_{pp_n,\varepsilon})_H d\tau + \int_0^t ((1 + c_0(\varepsilon)) L'_{p_n p} u_n + \sigma'_{p_n p,\varepsilon}, \bar{u} - \tilde{u}_{n\varepsilon})_H d\tau \\ = \; & -\frac{1}{2}(2c_0(\varepsilon) + c_0(\varepsilon)^2)(|u_n(t)|_H^2 - |u_n(0)|_H^2) - (u_n(t), \sigma_{pp_n,\varepsilon}(t))_H + (u_n(0), \sigma_{pp_n,\varepsilon}(0))_H \\ & + \int_0^t (u_n, \sigma'_{pp_n,\varepsilon})_H d\tau - (1 + c_0(\varepsilon))(L_{p_n p}(t) u_n(t), \sigma_{pp_n,\varepsilon}(t))_H \\ & + (1 + c_0(\varepsilon))(L_{p_n p}(0) u_n(0), \sigma_{pp_n,\varepsilon}(0))_H + (1 + c_0(\varepsilon)) \int_0^t \left(u_n, L'_{pp_n} \sigma_{pp_n,\varepsilon}\right)_H d\tau \\ & + \int_0^t ((1 + c_0(\varepsilon)) L'_{p_n p} u_n + \sigma'_{p_n p,\varepsilon}, \bar{u} - \tilde{u}_{n\varepsilon})_H d\tau,\end{aligned}$$

which tends to 0 unformly in $t \in [0, T]$ as $n, m \to \infty$ and $\varepsilon \to 0$ (cf. (2.9)-(2.11)). Similarly

$$\begin{aligned}\int_0^t \{(f_n, u_n - \bar{u}_{n\varepsilon})_H + (f, \bar{u} - \tilde{u}_{n\varepsilon})_H\} d\tau = \int_0^t (f, \bar{u} - \tilde{u}_{n\varepsilon} + u_n - \bar{u}_{n\varepsilon})_H d\tau \\ + \int_0^t (f_n - f, u_n - \bar{u}_{n\varepsilon})_H d\tau \to 0\end{aligned}$$

uniformly in $t \in [0, T]$ as $n, m \to \infty$ and $\varepsilon \to 0$. Therefore, from (3.24) we obtain an inequality of the form:

$$\limsup_{\varepsilon \to 0, n \to \infty} |\bar{u}(t) - \tilde{u}_{n\varepsilon}(t)|_H^2 \leq 0,$$

which shows that $\bar{u}(t) = u(t)$ for all $t \in [0, T]$, since $\tilde{u}_{n\varepsilon}(t) \to u(t)$ as $n \to \infty$ and $\varepsilon \to 0$. Thus the weak solution of $CP(\varphi^t(p; \cdot); u_0, f)$ is unique. $\diamondsuit$

Proof of Theorem 2.2.

We first prove (i). Let u be the weak solution of $CP(\varphi^t(p;\cdot);f,u_0)$ as well as $\bar{u}$ be the weak solution of $CP(\varphi^t(p;\cdot);\bar{f},\bar{u}_0)$ for $p \in \mathcal{X}_W$, $f,\bar{f} \in L^2(0,T:H)$ and $u_0,\bar{u}_0 \in \overline{D(\varphi^0(p;\cdot))}$. Now, choose sequences $\{p_n\} \subset \mathcal{X}_S$, $\{u_{0n}\} \subset H$ with $u_{0n} \in D(\varphi^0(p_n;\cdot))$ and $\{\bar{u}_{0n}\} \subset H$ with $\bar{u}_{0n} \in D(\varphi^0(p_n;\cdot))$ such that $p_n \to p$ in $\mathcal{X}$, $u_{0n} \to u_0$ in H and $\bar{u}_{0n} \to \bar{u}_0$ in H (as $n \to \infty$). Denoting by u_n and $\bar{u}_n$ the strong solutions of $CP(\varphi^t(p_n;\cdot);f,u_{0n})$ and $CP(\varphi^t(p_n;\cdot);\bar{f},\bar{u}_{0n})$, respectively, as was seen in Theorem 2.1, we see that $u_n \to u$ and $\bar{u}_n \to \bar{u}$ in $C([0,T];H)$. Also, by taking the difference of equations satisfied by strong solutions u_n and $\bar{u}_n$ we have

$$u_n'(\tau) - \bar{u}_n'(\tau) + \xi_n(\tau) - \bar{\xi}_n(\tau) = f(\tau) - \bar{f}(\tau), \quad \text{a.e. } \tau \in (0,T), \tag{3.25}$$

where $\xi_n(\tau) \in \partial\varphi^\tau(p_n;u_n(\tau))$ and $\bar{\xi}_n(\tau) \in \partial\varphi^\tau(p_n;\bar{u}_n(\tau))$ for a.e. $\tau \in (0,T)$. Now, taking the inner product between (3.25) and $u_n(\tau) - \bar{u}_n(\tau)$ and integrating the resultants in τ over $[0,t]$, we get

$$\frac{1}{2}|u_n(t) - \bar{u}_n(t)|_H^2 + \int_0^t (\xi_n(\tau) - \bar{\xi}_n(\tau), u_n(\tau) - \bar{u}_n(\tau))_H \tag{3.26}$$

$$= \frac{1}{2}|u_{0n} - \bar{u}_{0n}|_H^2 + \int_0^t (f(\tau) - \bar{f}(\tau), u_n(\tau) - \bar{u}_n(\tau))_H.$$

By the monotonicity of subdifferentials the second term of the left side of (3.26) is nonnegative. Hence, passing to the limit as $n \to \infty$, we obtain (2.13).

Next we prove (ii). We use the same notations as in the statement (ii) of Theorem 2.2. For each n we note that u_n and u_{0n} can be approximated by $\hat{p}_n \in \mathcal{X}_S$ and $\hat{u}_{0n} \in D(\varphi^0(\hat{p}_n;\cdot)$ so that $|p_n - \hat{p}_n|_{\mathcal{X}} \le \frac{1}{n}$ and $|u_{0n} - \hat{u}_{0n}|_H \le \frac{1}{n}$. Moreover, denoting by $\hat{u}_n$ the strong solution of $CP(\varphi^t(\hat{p}_n;\cdot);f_n,\hat{u}_{0n})$, we may assume on account of Theorem 2.1,(i) that $|u_n - \hat{u}_n|_{C([0,T];H)} < \frac{1}{n}$. Since

$$\hat{p}_n \to p \ \text{ in } \mathcal{X}, \quad \hat{u}_{0n} \to u_0 \ \text{ in } H,$$

it follows from (i) of Theorem 2.1 that $\hat{u}_n \to u$ and hence $u_n \to u$ in $C([0,T];H)$. $\diamondsuit$

4 Quasi-variational Problems

In this section, we use the same notation as in section 2; let Φ_0 be the convex function defined by (2.2). As an application of Theorems 2.1 and 2.2, we treat a quasi-variational problem with feedback system Λ_{p_0} which is an (single-valued) operator, associated with initial value $p_0 \in X$, from $C([0,T];H) \cap D(\Phi_0)$ into $\mathcal{X}_W$ such that

(Λ1) $[\Lambda_{p_0}u](0) = p_0, \ \forall u \in C([0,T];H) \cap D(\Phi_0)$,

(Λ2) if $\{u_n\} \subset D(\Phi_0)$, $u_n \to u$ in $C([0,T];H)$ (as $n \to \infty$) and $\sup_{n\in\mathbf{N}} \Phi_0(u_n) < \infty$, then $\Lambda_{p_0}u_n \to \Lambda_{p_0}u$ in $\mathcal{X}$,

($\Lambda 3$) for each $M > 0$, Λ_{p_0} maps the set $\{u \in C([0,T];H) \mid |u|_{C([0,T];H)} + \Phi_0(u) \le M\}$ into a relatively compact subset of $\mathcal{X}_W$.

Now, our quasi-variational problem is formulated by the following system:

$$u'(t) + \partial\varphi^t(p;u(t)) \ni f(t) \text{ in } H, \quad 0 < t < T, \quad u(0) = u_0, \tag{4.1}$$

$$p = \Lambda_{p_0} u \quad \text{in } \mathcal{X}_0. \tag{4.2}$$

We denote system (4.1)-(4.2) by $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$.

Definition 4.1. We call $\{u, p\}$ a weak solution of $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$, if $u \in C([0,T];H) \cap D(\Phi_0)$, $\Lambda_{p_0} u = p$ and u is the weak solution of $CP(\varphi^t(p;\cdot); f, u_0)$, or equivalently if u is the weak solution of $CP(\varphi^t(\Lambda_{p_0} u;\cdot); f, u_0)$.

Our existence result for $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$ is stated as follows.

Theorem 4.1. *Suppose that the family $\{\varphi^t(p;\cdot) | p \in \mathcal{X}_W, t \in [0,T]\}$ satisfies conditions (Φ), (Φ_S) and $(A1) - (A3)$ and that $\mathcal{X}_W$ is convex in $\mathcal{X}$. Let p_0 be an element in X such that $\mathcal{X}_W(p_0) := \{p \in \mathcal{X}_W \mid p(0) = p_0\}$ is non-empty in $\mathcal{X}_W$ and Λ_{p_0} be a feedback system such that $(\Lambda 1) - (\Lambda 3)$ hold. Suppose that there is a positive constant M^* such that for any $p \in \mathcal{X}_W$ there is a function η in $\mathcal{K}_0(p)$ satisfying*

$$|\eta|_{W^{1,2}(0,T;H)} + \int_0^T \varphi^t(p;\eta(t))dt \le M^*. \tag{4.3}$$

Moreover, let $f \in L^2(0,T;H)$ and $u_0 \in H$ such that $u_0 \in D(\varphi^0(\tilde{p};\cdot))$ for some $\tilde{p} \in \mathcal{X}_W(p_0)$ (hence for all $p \in \mathcal{X}_W(p_0)$). Then, there exists at least one weak solution of $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$.

Proof. We note that $\mathcal{X}_W(p_0)$ is non-empty, closed and convex in $\mathcal{X}$. According to Theorem 2.1, for each $p \in \mathcal{X}_W(p_0)$, $CP(\varphi^t(p;\cdot); f, u_0)$ admits one and only one weak solution u. Now we denote by $\mathcal{W}(p_0, u_0)$ the set of weak solutions u of $CP(\varphi^t(p;\cdot); f, u_0)$ for all $p \in \mathcal{X}_W(p_0)$, namely

$$\mathcal{W}(p_0, u_0) := \{u \mid u \text{ is the weak solution of } CP(\varphi^t(p;\cdot); f, u_0),\ p \in \mathcal{X}_W(p_0)\}.$$

We first observe from (4.3) that $\sup_{u \in \mathcal{W}(p_0,u_0)} \{|u|_{C([0,T];H)} + \Phi_0(u)\} < \infty$. Let u be the weak solution of $CP(\varphi^t(p;\cdot); f, u_0)$ with $p \in \mathcal{X}_W(p_0)$. Let $\eta \in \mathcal{K}_0(p)$ be a test function satisfying (4.3). Then it follows from (4.3) that $|\eta|^2_{C([0,T];H)} \le M_1^*$, $|\eta'|^2_{L^2(0,T;H)} \le M_1^*$ and $\int_0^T \varphi^t(p;\eta(t))dt \le M_1^*$ for some positive constant M_1^* depending only on M^*. Moreover we have by (2.7)

$$\int_0^t (\eta'(\tau), u(\tau) - \eta(\tau))_H d\tau + \int_0^t \varphi^\tau(p;u(\tau))d\tau + \frac{1}{2}|u(t) - \eta(t)|_H^2$$

$$\le \int_0^t \varphi^\tau(p;\eta(\tau))d\tau + \int_0^t (f(\tau), u(\tau) - \eta(\tau))_H d\tau + \frac{1}{2}\left|u_0 - \eta(0)\right|_H^2$$

for all $t \in (0, T]$. From this inequality we get that

$$\begin{aligned} &\int_0^t \varphi^\tau(p; u(\tau))d\tau + \frac{1}{2}|u(t) - \eta(t)|_H^2 - \nu \int_0^t |u(\tau) - \eta(\tau)|_H^2 d\tau \\ &\le \int_0^T \varphi^\tau(p; \eta(\tau))d\tau + \frac{1}{4\nu}\int_0^T (|\eta'(\tau)|_H^2 + |f(\tau)|_H^2)d\tau + \frac{1}{2}|u_0 - \eta(0)|_H^2 \end{aligned} \tag{4.4}$$

for all $t \in [0, T]$ and any positive number ν. For instance, choosing $\nu = \frac{1}{4T}$, we conclude from (4.4) that

$$\int_0^t \varphi^\tau(p; u(\tau))d\tau + \frac{1}{4}|u(t) - \eta(t)|_H^2 \le M_1^* + \frac{1}{4\nu}\left(M_1^* + |f|_{L^2(0,T;H)}^2\right) + |u_0|_H^2 + M_1^* =: M_2^*$$

for all $t \in [0, T]$, which implies that $|u(t)|_H \le |\eta(t)|_H + 2\sqrt{M_2^*} \le \sqrt{M_1^*} + 2\sqrt{M_2^*} =: M_3^*$ for all $t \in [0, T]$. As a consequence, $|u|_{C([0,T];H)} + \int_0^T \varphi^\tau(p; u(\tau))d\tau \le M_2^* + M_3^* =: M_4^*$. Thus, by (2.3) in (Φ), we have

$$\sup_{u \in \mathcal{W}(p_0, u_0)} \{|u|_{C([0,T];H)} + \Phi_0(u)\} \le M_4^*. \tag{4.5}$$

Next, let us consider the operator $S_{u_0} : \mathcal{X}_W(p_0) \to \mathcal{W}(p_0, u_0)$ which assigns to each $p \in \mathcal{X}_W(p_0)$ the weak solution u of $CP(\varphi^t(p; \cdot); f, u_0)$. Then it follows that S_{u_0} is continuous from $\mathcal{X}_W(p_0)$ with respect to the topology of $\mathcal{X}$ into $\mathcal{W}(p_0, u_0)$ with respect to that of $C([0, T]; H)$ (cf. Theorem 2.1). Moreover we see from condition $(\Lambda1) - (\Lambda3)$ that the composition $\mathcal{A} := \Lambda_{p_0} S_{u_0}$ is well-defined as a mapping in the closed convex subset $\mathcal{X}_W(p_0)$ of $\mathcal{X}_W$ and it is continuous with respect to the topology of $\mathcal{X}$ and its range $\mathcal{A}(\mathcal{X}_W(p_0))$ is relatively compact in $\mathcal{X}$ by (4.5). Therefore, by virtue of the Schauder's fixed point theorem, $\mathcal{A}$ has at least one fixed point p in $\mathcal{X}_W(p_0)$, namely $p = \mathcal{A}p$. It is easy to see that the pair $\{u, p\}$ with $u := S_{u_0}p$ is a weak solution of $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$. $\diamondsuit$

The next examples illustrate very well conditions $(\Lambda1) - (\Lambda3)$, although they are quite artificial.

Example 4.1. Let H, X, $\mathcal{X}$, $\mathcal{X}_0$, $K^t(p)$ and $\varphi^t(p; \cdot)$ be the same as in Example 2.1. Also, for functions $w_1, w_2, \cdots, w_N$ prescribed in $\mathcal{X}_0$, we denote their closed convex hull by $\overline{co}[w_1, w_2, \cdots, w_N]$ and put

$$\mathcal{K}(v_0) := \{w \in \overline{co}[w_1, w_2, \cdots, w_N] \mid w(0) = v_0\}.$$

We note that (4.3) holds, since we can take the constant 0 as η satisfying (4.3).

Next, let $\gamma(\cdot) \in C(\mathbf{R})$ whose range is $[d_*, d^*]$. Then, as a feedback system we define an operator Λ_{p_0}, with $p_0 = [v_0, d_0] \in \mathbf{R}^4$, from $C([0, T]; \mathbf{R}^3)$ into $\mathcal{X}_0$ by

$$p := [v, d] = \Lambda_{p_0} u \Longleftrightarrow \begin{cases} |u - v|_{L^2(0,T;\mathbf{R}^3)} = \min_{w \in \mathcal{K}(v_0)} |u - w|_{L^2(0,T;\mathbf{R}^3)}, \quad v(0) = v_0, \\ d(t) = \gamma(|v(t)|), \ \forall t \in [0, T], d_0 = \gamma(|v_0|), \end{cases}$$

for each $u \in C([0, T]; \mathbf{R}^3)$; Λ_{p_0} is well defined as a single-valued mapping, since $\mathcal{K}(v_0)$ is compact convex in $L^2(0, T; \mathbf{R}^3)$. It is easy to see that Λ_{p_0} satisfies conditions $(\Lambda1) -$

($\Lambda 3$). Therefore, by applying Theorem 4.1 to problem $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$ for each $f \in L^2(0,T;\mathbf{R}^3)$ and $u_0 \in \mathbf{R}^3$ with $|u_0 - v_0| \le \gamma(|v_0|)$, we conclude that there are functions u and v such that

$$u \in C([0,T;\mathbf{R}^3),\ u(0) = u_0,\ v \in \mathcal{K}(v_0)$$

$$|u - v|_{L^2(0,T;\mathbf{R}^3)} = \min_{w \in \mathcal{K}(v_0)} |u - w|_{L^2(0,T;\mathbf{R}^3)},$$

$$|u(t) - v(t)| \le \gamma(|v(t)|),\ \forall t \in [0,T],$$

$$\int_0^t (\eta'(\tau), u(\tau) - \eta(\tau))d\tau + \frac{1}{2}\int_0^t |u(\tau)|^2 d\tau + \frac{1}{2}|u(t) - \eta(t)|^2 \tag{4.6}$$

$$\le \int_0^t (f(\tau), u(\tau) - \eta(\tau))d\tau + \frac{1}{2}\int_0^t |\eta(\tau)|^2 d\tau + \frac{1}{2}|u_0 - \eta(0)|^2,$$

$$\forall \eta \in W^{1,2}(0,T;\mathbf{R}^3) \text{ with } |\eta(t) - v(t)| \le \gamma(|v(t)|),\ \forall t \in [0,T].$$

Example 4.2. Let $H, X, V_{R_0,R_1,R_2}, \mathcal{X},\ \mathcal{X}_0 = \mathcal{X}_W,\ K^t(p)$ and $\varphi^t(p;\cdot)$ be the same sets and convex functions as in Example 2.2. Also, let $\gamma(\cdot)$ be the same continuous function as in Example 4.1.

Condition (4.3) is easily checked by taking as $\eta \in \mathcal{K}_0(p)$ in (4.3) the function v for each $p = [v,d] \in \mathcal{X}_W$. Now, in order to set up the feedback system Λ_{p_0} with $p_0 := [v_0, d_0],\ v_0 \in V_{R_0,R_1,R_2},\ d_0 = \gamma(|v_0|)$, we consider the ordinary differential inclusion in $\mathbf{R}^3$ for each function $u \in C([0,T];\mathbf{R}^3)$

$$v'(t) + \partial I_{V_{R_0,R_1,R_2}}(v(t)) \ni u(t), \quad v(0) = v_0. \tag{4.7}$$

By the classical result on the nonlinear evolution inclusions (cf. [5]), (4.7) has one and only one strong solution $v \in W^{1,2}(0,T;\mathbf{R}^3)$. Now we define $\Lambda_{p_0} u = [v, \gamma(|v|)] =: p$. As is easily checked, conditions $(\Lambda 1) - (\Lambda 3)$ are satisfied. Therefore, by applying Theorem 4.1 we see that for each $f \in L^2(0,T;\mathbf{R}^3)$, $u_0 \in \mathbf{R}^3$ and $v_0 \in V_{R_0,R_1,R_2}$ with $(u_0 - v_0, v_0) = 0$ and $|u_0 - v_0| \le \gamma(|v_0|)$, the problem $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$ possesses a weak solution $\{u, p\}$, $p = [v.\gamma(|v|)]$; namely

$$u \in C([0,T];\mathbf{R}^3),\ u(0 = u_0,\ v \in W^{1,2}(0,T;\mathbf{R}^3),\ v(0) = v_0,$$

$$v'(t) + \partial I_{V_{R_0,R_1,R_2}}(v(t)) \ni u(t) \text{ for a.e. } t \in [0,T],$$

$$(u(t) - v(t), v(t)) = 0,\ |u(t) - v(t)| \le \gamma(|v(t)|),\ \forall t \in [0,T],$$

$$\int_0^t (\eta'(\tau), u(\tau) - \eta(\tau))d\tau + \frac{1}{2}\int_0^t |u(\tau)|^2 d\tau + \frac{1}{2}|u(t) - \eta(t)|^2 \tag{4.8}$$

$$\le \int_0^t (f(\tau), u(\tau) - \eta(\tau))d\tau + \frac{1}{2}\int_0^t |\eta(\tau)|^2 d\tau + \frac{1}{2}|u_0 - \eta(0)|^2,$$

$$\forall \eta \in W^{1,2}(0,T;\mathbf{R}^3) \text{ with } (\eta(t) - v(t), v(t)) = 0,\ |\eta(t) - v(t)| \le \gamma(|v(t)|),\ \forall t \in [0,T].$$

5 Perturbation problems

In this section perturbation problems for $CP(\varphi^t(p;\cdot); f, u_0)$ and $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$ are discussed with their weak solvability. Our perturbation problems are treated under a little bit more restricted setup than the above.

Let $H,\ X,\ \mathcal{X}$ and $\mathcal{X}_0$ be the same as in section 2 and V be a (real) uniformly convex Banach space with uniformly convex dual space V^* such that $V \subset H$ with continuous and dense embedding. Hence, we have the usual triplet

$$V \subset H \subset V^* \text{ with continuous and dense embeddings,} \tag{5.1}$$

and for simplicity denote the duality $\langle\cdot,\cdot\rangle_{V^*,V}$ by $\langle\cdot,\cdot\rangle$. For a proper, l.s.c. and convex function $\varphi(\cdot)$ on H or on V, we recall on the notation for subdifferentials that $\partial\varphi$ is the subdifferential of φ from H into itself and $\partial_*\varphi$ is that from V into V^*.

We consider as $\varphi_0(\cdot)$ in section 2 the function

$$\varphi_0(z) := \begin{cases} \dfrac{\nu}{2}|z|_V^2, & \forall z \in V, \\ \infty, & \text{otherwise,} \end{cases} \tag{5.2}$$

where ν is a positive constant. We denote by F the (single-valued) duality mapping from V onto V^* and suppose that F is uniformly monotone on V, namely

$$\langle Fz_1 - Fz_2, z_1, z_2\rangle \geq C_F|z_1 - z_2|_V^2,\ \forall z_1, z_2 \in V, \tag{5.3}$$

for a positive constant C_F. By (5.2) we have $\partial_*\varphi_0(z) = \nu Fz$ for all $z \in V$ and $\partial\varphi_0(z) = \nu Fz$, if $Fz \in H$.

In this section we suppose that for each $p \in \mathcal{X}_0$ and $t \in [0,T]$ the function $\varphi^t(p;\cdot)$ is of the following form:

$$\varphi^t(p;z) := \varphi_0(z) + \psi^t(p;z),\quad \forall z \in H, \tag{5.4}$$

which is the sum of $\varphi_0(\cdot)$ and a p-dependent, non-negative, proper, l.s.c. and convex function $\psi^t(p;\cdot)$ on H. For this function $\varphi^t(p;\cdot)$ given by (5.4) the classes $\mathcal{X}_S \neq \emptyset$ and $\mathcal{X}_W$ are similarly defined.

Furthermore we suppose that

$$L_{p\bar{p}}(t) \in B(V),\quad \forall p,\ \bar{p} \in \mathcal{X}_W,\ \forall t \in [0,T],$$

where $B(V)$ stands for the space of all linear continuous operators in V, and conditions (2.9) and (2.11) are strengthened as follows:

$$|L_{p\bar{p}} - I|_{C([0,T];B(V))} + |L'_{p\bar{p}}|_{L^2(0,T;B(V))} \leq L_0|p-\bar{p}|_{\mathcal{X}}, \forall p,\ \bar{p} \in \mathcal{X}_W \tag{5.5}$$

and

$$\begin{gathered}\sup_{t\in[0,T]} \varphi_0(\sigma_{p\bar{p},\varepsilon}(t)) \leq \sigma_0,\quad |\sigma_{p\bar{p},\varepsilon}|_{W^{1,2}(0,T;V)} \leq \sigma_0(|p-\bar{p}|_{\mathcal{X}} + \varepsilon), \\ \forall p,\ \bar{p} \in \mathcal{X}_W,\ \forall \varepsilon \in (0,1],\end{gathered} \tag{5.6}$$

where L_0 and σ_0 are positive constants.

Our perturbation problem for $CP(\varphi^t(p;\cdot); f, u_0)$ is of the form

$$u'(t) + \partial\varphi^t(p; u(t)) + G(p; t, u(t)) \ni f(t), \quad 0 < t < T, \quad u(0) = u_0, \tag{5.7}$$

where $G = G(p; t, z)$ is supposed to be a continuous mapping from $\mathcal{X}_W \times [0, T] \times V$ into H and to satisfy that

$$|G(p; t, z) - G(\bar{p}; t, \bar{z})|_H \le C_G(|p - \bar{p}|_{\mathcal{X}} + |z - \bar{z}|_V), \quad \forall t \in [0, T], \ \forall z, \bar{z} \in V, \tag{5.8}$$

for a positive constant C_G.

Definition 5.1. Given $p \in \mathcal{X}_W$, $f \in L^2(9, T; H)$ and $u_0 \in H$, we say that u is a weak solution of $CP(\varphi^t(p;\cdot), G(p;\cdot); f, u_0)$, if $u \in C([0.T]; H) \cap L^2(0, T; V)$, $u(0) = u_0$ and the following variational inequality holds:

$$\int_0^t (\eta'(\tau), u(\tau) - \eta(\tau))_H d\tau + \int_0^t \varphi^\tau(p; u(\tau)) d\tau + \int_0^t (G(p; \tau, u(\tau)), u(\tau) - \eta(\tau))_H d\tau$$

$$+\frac{1}{2}|u(t) - \eta(t)|_H^2 \le \int_0^t \varphi^\tau(p; \eta(\tau)) d\tau + \int_0^t (f(\tau), u(\tau) - \eta(\tau))_H d\tau + \frac{1}{2}\left|u_0 - \eta(0)\right|_H^2, \tag{5.9}$$

$$\forall \eta \in \mathcal{K}_0(p), \quad \forall t \in (0, T].$$

Lemma 5.1. *Suppose that the family $\{\varphi^t(p;\cdot) \mid p \in \mathcal{X}_W, \ t \in [0, T]\}$ given by (5.4) satisfies conditions (Φ), (Φ_S) and $(A1) - (A3)$ and $\mathcal{X}_W$ is convex in $\mathcal{X}$. Then, we have:*

(1) For any $p \in \mathcal{X}_W$ and $t \in [0, T]$, $z^* \in \partial\varphi^t(p; z)$ if and only if $z \in D(\psi^t(p;\cdot)) \cap V$, $z^* \in H$ and

$$(z^*, w - z)_H \le \nu\langle Fz, w - z\rangle + \psi^t(p; w) - \psi^t(p; z), \quad \forall w \in V. \tag{5.10}$$

(2) There is a non-negative continuous function $c_2(\cdot)$ on $[0, 1]$ with $c_2(0) = 0$ having the property that for any $\varepsilon \in (0, 1]$ there is a positive number δ_ε $(< \varepsilon)$ such that for any $p, \bar{p} \in \mathcal{X}_W$ with $|\bar{p} - p|_{\mathcal{X}} \le \delta_\varepsilon$, any $t \in [0, T]$ and any $z \in D(\varphi^t(p;\cdot))$ the element $\bar{z} := (1 + c_0(\varepsilon))L_{p\bar{p}}(t)z + \sigma_{p\bar{p},\varepsilon}(t)$ satisfies

$$\psi^t(\bar{p}; \bar{z}) \le \psi^t(p; z) + c_2(\varepsilon)(1 + |z|_V^2 + \psi^t(p; z)). \tag{5.11}$$

(3) For any $p \in \mathcal{X}_W$, $f \in L^2(0, T; H)$, $u_0 \in \overline{D(\varphi^0(p;\cdot))}$ and $\bar{u} \in L^2(0, T; V)$, $CP(\varphi^t(p;\cdot); f - G(p;\cdot, \bar{u}), u_0)$ has one and only one weak solution in $C([0, T]; H) \cap L^2(0, T; V)$.

(4) Let $p \in \mathcal{X}_W$, $f_i \in L^2(0, T; H)$, $u_{i0} \in \overline{D(\varphi^0(p;\cdot))}$ and $\bar{u}_i \in L^2(0, T; V)$ for $i = 1, 2$. Let u_i, $i = 1, 2$, be the weak solutions of $CP(\varphi^t(p;\cdot); f_i - G(p;\cdot, \bar{u}_i), u_{i0})$. Then,

$$e^{-C_\delta t}|u_1(t) - u_2(t)|_H^2 + 2\nu C_F \int_0^t e^{-C_\delta \tau}|u_1(\tau) - u_2(\tau)|_V^2 d\tau$$

$$\leq |u_{10}-u_{20}|_H^2 + \int_0^t e^{-C_\delta \tau}(f_1(\tau)-f_2(\tau), u_1(\tau)-u_2(\tau))_H d\tau \tag{5.12}$$
$$+\delta C_G \int_0^t e^{-C_\delta \tau}|\bar{u}_1(\tau)-\bar{u}_2(\tau)|_V^2 d\tau$$

for every $t \in [0,T]$, where δ is any positive number and $C_\delta := \frac{C_G}{\delta}$.

Proof. In order to prove (1), assume that $z^* \in \partial\varphi^t(p;z)$. Then for any $\tilde{w} \in D(\psi^t(p;\cdot)) \cap V$ we have by the definition of subdifferential

$$(z^*, \tilde{w}-z)_H \leq \frac{\nu}{2}|\tilde{w}|_V^2 - \frac{\nu}{2}|z|_V^2 + \psi^t(p;\tilde{w}) - \psi^t(p;z).$$

Now, take as $\tilde{w}$ the element $z + r(w-z)$ with any $w \in D(\psi^t(p;\cdot)) \cap V$ and $0 < r < 1$ to get

$$(z^*, w-z)_H \leq \frac{1}{r}\left\{\frac{\nu}{2}|z+r(w-z)|_V^2 - \frac{\nu}{2}|z|_V^2\right\} + \psi^t(p;w) - \psi^t(p;z).$$

Letting $r \downarrow 0$ yields (5.10). The converse is clear.

Next we show (2). For any $\varepsilon \in (0,1]$, let δ_ε be the same number in condition $(A3)$. Then we have for any $p,\ \bar{p} \in \mathcal{X}_W$ with $|p-\bar{p}|_{\mathcal{X}} \leq \delta_\varepsilon\ (< \varepsilon)$

$$\frac{1}{2}|\bar{z}|_V^2 + \psi^t(\bar{p};\bar{z}) \leq \frac{1}{2}|z|_V^2 + \psi^t(p;z) + c_1(\varepsilon)\left(1 + \frac{1}{2}|z|_V^2 + \psi^t(p;z)\right), \tag{5.13}$$

where $\bar{z} = (1+c_0(\varepsilon))L_{p\bar{p}}(t)z + \sigma_{p\bar{p},\varepsilon}(t)$. Since $\bar{z}-z = (L_{p\bar{p}}(t)-I)z + c_0(\varepsilon)L_{p\bar{p}}(t)z + \sigma_{p\bar{p},\varepsilon}(t)$, it follows from (5.5) that

$$\begin{aligned}
|\bar{z}-z|_V &\leq |L_{p\bar{p}}(t)-I|_{B(V)}|z|_V + |c_0(\varepsilon)||L_{p\bar{p}}(t)|_{B(V)}|z|_V + |\sigma_{p\bar{p},\varepsilon}(t)|_V \\
&\leq L_0|\bar{p}-p|_{\mathcal{X}}|z|_V + c_0(\varepsilon)(L_0|\bar{p}-p|_{\mathcal{X}}+1)|z|_V + |\sigma_{p\bar{p},\varepsilon}(t)|_V \\
&\leq L_0\{\varepsilon + c_0(\varepsilon) + \varepsilon c_0(\varepsilon)\}|z|_V + |\sigma_{p\bar{p},\varepsilon}(t)|_V.
\end{aligned}$$

Putting $c_1'(\varepsilon) := L_0(\varepsilon + c_0(\varepsilon) + \varepsilon c_0(\varepsilon))$, we have by the above inequalities

$$|\bar{z}-z|_V \leq c_1'(\varepsilon)|z|_V + |\sigma_{p\bar{p},\varepsilon}(t)|_V$$

and similarly

$$|\bar{z}|_V + |z|_V \leq 2|z|_V + c_1'(\varepsilon)|z|_V + |\sigma_{p\bar{p},\varepsilon}(t)|_V.$$

Also, it follows from (5.6) that there is a positive constant σ_1 such that

$$|\sigma_{p\bar{p},\varepsilon}|_{C([0,T];V)} \leq \sigma_1(|p-\bar{p}|_{\mathcal{X}} + \varepsilon) \leq 2\sigma_1\varepsilon$$

for all $p,\ \bar{p} \in \mathcal{X}_W$ with $|p-\bar{p}|_{\mathcal{X}} \leq \delta_\varepsilon$, $\varepsilon \in (0,1]$ and $t \in [0,T]$. Therefore

$$\begin{aligned}
\psi^t(\bar{p};\bar{z}) &\leq \psi^t(p;z) + \frac{1}{2}\left(|z|_V^2 - |\bar{z}|_V^2\right) + c_1(\varepsilon)\left(1 + \frac{1}{2}|z|_V^2 + \psi^t(p;z)\right) \\
&\leq \psi^t(p;z) + \frac{1}{2}|\bar{z}-z|_V\left(|\bar{z}|_V + |z|_V\right) + c_1(\varepsilon)\left(1 + \frac{1}{2}|z|_V^2 + \psi^t(p;z)\right) \\
&\leq \psi^t(p;z) + \frac{1}{2}(c_1'(\varepsilon)|z|_V + 2\sigma_1\varepsilon)(2|z|_V + c_1'(\varepsilon)|z|_V + 2\sigma_1\varepsilon) \\
&\quad + c_1(\varepsilon)\left(1 + \frac{1}{2}|z|_V^2 + \psi^t(p;z)\right).
\end{aligned}$$

As a consequence we obtain an inequality of the form (5.11) for a certain positive continuous function $c_2(\cdot)$ on $[0,1]$ with $c_2(0) = 0$; for instance, $c_2(\varepsilon) := 2c_1(\varepsilon) + c_1'(\varepsilon) + c_1'(\varepsilon)^2 + 3\sigma_1\varepsilon + 2\sigma_1^2\varepsilon^2$.

The assertion (3) is an immediate consequence of Theorem 2.1. Next we prove (4). Choose a sequence $\{p_n\}$ in $\mathcal{X}_S$ such that $p_n \to p$ in $\mathcal{X}$ (as $n \to \infty$) and a sequence $\{u_{i0,n}\}$, $i = 1,2$, such that $u_{i0,n} \in D(\varphi^0(p_n;\cdot))$ and $u_{i0,n} \to u_{i0}$ in H. Also, note by (5.8) that $G(p_n;\cdot,\bar{u}_i) \to G(p;\cdot,\bar{u}_i) \in L^2(0,T;H)$, since $\bar{u}_i \in L^2(0,T;V)$. Then, on account of Theorem 2.1, for each n problem $CP(\varphi^t(p_n;\cdot); f - G(p_n;\cdot,\bar{u}_i), u_{i0,n})$ has one and only one weak solution u_{in} and converges u_i in $C([0,T];H)$ with

$$0 \le \int_0^t \varphi^\tau(p; u_i(\tau))d\tau \le \liminf_{n\to\infty} \int_0^t \varphi^\tau(p_n; u_{in}(\tau))d\tau < \infty.$$

Hence $u_{in} \to u_i$ weakly in $L^2(0,T;V)$.

Now, recalling (3.26) in the proof of Theorem 2.2, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{d\tau}|u_{1n}(\tau) - u_{2n}(\tau)|_H^2 + (\xi_{1n}(\tau) - \xi_{2n}(\tau), u_{1n}(\tau) - u_{2n}(\tau))_H \\ = \;& (f_1(\tau) - f_2(\tau) - G(p_n;\tau,\bar{u}_1(\tau)) + G(p_n;\tau,\bar{u}_2(\tau)), u_{1n}(\tau) - u_{2n}(\tau))_H, \end{aligned}$$

where $\xi_{in} \in \partial\varphi^t(p_n; u_{in})$, $i = 1,2, n = 1,2,\cdots$. Moreover, we get from this inequality with (5.3) and (5.10) in (1) that for any n and a.e. $\tau \in [0,T]$

$$\begin{aligned} & \frac{1}{2}\frac{d}{d\tau}|u_{1n}(\tau) - u_{2n}(\tau)|_H^2 + \nu C_F|u_{1n}(\tau) - u_{2n}(\tau)|_V^2 \\ \le \;& (f_1(\tau) - f_2(\tau), u_{1n}(\tau) - u_{2n}(\tau))_H \\ & + \frac{1}{2}\delta C_G|\bar{u}_1(\tau) - \bar{u}_2(\tau)|_V^2 + \frac{C_G}{2\delta}|u_{1n}(\tau) - u_{2n}(\tau)|_H^2. \end{aligned}$$

Hence, multiplying the above inequality by $e^{-C_\delta\tau}$, with $C_\delta = \frac{C_G}{\delta}$, integrating the resultant in τ on $[0,t]$ yields that

$$\begin{aligned} & e^{-C_\delta t}|u_{1n}(t) - u_{2n}(t)|_H^2 + 2\nu C_F \int_0^t e^{-C_\delta\tau}|u_{1n}(\tau) - u_{2n}(\tau)|_V^2 d\tau \\ \le \;& |u_{10,n} - u_{20,n}|_H^2 + \int_0^t e^{-C_\delta\tau}(f_1(\tau) - f_2(\tau), u_{1n}(\tau) - u_{2n}(\tau))_H d\tau \\ & + \delta C_G \int_0^t e^{-C_\delta\tau}|\bar{u}_1(\tau) - \bar{u}_2(\tau)|_V^2 d\tau \end{aligned}$$

Now, passing to the limit as $n \to \infty$, we obtain (5.12). $\diamondsuit$

The weak solvability of a perturbation problem for $CP(\varphi^t(p;\cdot); f, u_0)$ is stated as follows.

Theorem 5.1. *Suppose that conditions* (Φ), (Φ_S), $(A1)-(A3)$ *and (51)-(5.6) are satisfied, and that* $\mathcal{X}_W$ *is convex in* $\mathcal{X}$. *Let* $p \in \mathcal{X}_W$, $f \in L^2(0,T;H)$ *and* $u_0 \in \overline{D(\varphi^0(p;\cdot))}$. *Then* $CP(\varphi^t(p;\cdot), G(p;\cdot); f, u_0)$ *has one and only one weak solution* u, *namely the variational inequality (5.9) holds:*

Proof. Thanks to Lemma 5.1, for every $\bar{u} \in L^2(0,T;V)$ there is one and only one weak solution u of $CP(\varphi^t(p;\cdot); f - G(p;\cdot,\bar{u}), u_0)$ and we define the mapping $\mathcal{S}$ which assigns to each $\bar{u} \in L^2(0,T;V)$ the weak solution u of $CP(\varphi^t(p;\cdot); f - G(p;\cdot,\bar{u}), u_0)$. On account of (5.12) of Lemma 5.1, $\mathcal{S}$ satisfies that for any $\bar{u}_i \in L^2(0,T;V)$, $i = 1,2$,

$$2\nu C_F \int_0^T e^{-C_\delta t} |(\mathcal{S}\bar{u}_1)(t) - (\mathcal{S}\bar{u}_2)(t)|_V^2 dt \le \delta C_G \int_0^T e^{-C_\delta t} |\bar{u}_1(t) - \bar{u}_2(t)|_V^2 dt. \tag{5.13}$$

We note here that the functional $u \to \rho(u) := |\int_0^T e^{-C_\delta t}|u(t)|_V^2 dt|^{\frac{1}{2}}$ is an equivalent norm to the usual one of $L^2(0,T;V)$ and (5.13) implies that the mapping $\mathcal{S}$ is a strictly contractive mapping from $L^2(0,T;V)$ into itself with respect to $\rho(\cdot)$, if $\delta > 0$ is chosen so that $\delta C_G < 2\nu C_F$. Therefore $\mathcal{S}$ has one and only one fixed point u, $u = \mathcal{S}u$, which is the required weak solution of $CP(\varphi^t(p;\cdot); f - G(p;\cdot,u), u_0)$. $\diamondsuit$

One of big advantages brought by the form (5.4) is stated in the following lemma.

Lemma 5.2. *Assume that (4.3) holds, namely there is a positive constant* M^* *such that for any* $p \in \mathcal{X}_W$ *there is a function* $\eta \in \mathcal{K}_0(p)$ *such that*

$$|\eta|_{W^{1,2}(0,T;H)} + \int_0^T \left\{ \frac{1}{2}|\eta(t)|_V^2 + \psi^t(p;\eta(t)) dt \right\} \le M^*. \tag{5.14}$$

Let $\{p_n\} \subset \mathcal{X}_W$ *and* $p \in \mathcal{X}_W$ *such that* $p_n \to p$ *in* $\mathcal{X}$ *(as* $n \to \infty$*). Also, let* $\{f_n\}$ *and* $\{u_{n0}\}$ *be sequences in* $L^2(0,T;H)$ *and* H, *respectively, such that* $f_n \to f$ *in* $L^2(0,T;H)$, $u_{n0} \in \overline{D(\varphi^0(p_n;\cdot))}$ *and* $u_{n0} \to u_0$ *in* H. *Then the weak solution* u_n *of* $CP(\varphi^t(p_n;\cdot); f_n, u_{n0})$ *converges to the weak solution* u *of* $CP(\varphi^t(p;\cdot); f, u_0)$ *in* $C([0,T];H) \cap L^2(0,T;V)$.

Proof. We may assume that $p_n \in \mathcal{X}_S$ and $u_{n0} \in D(\varphi^0(p_n;\cdot))$, since $\mathcal{X}_W$ is the closure of $\mathcal{X}_S$ in $\mathcal{X}$. Let u_n be the strong solution of $CP(\varphi^t(p_n;\cdot); f_n, u_{n0})$. By making use of Lemma 5.1, (1), we have

$$\begin{aligned}(u_n'(t), u_n(t) - z)_H + \nu\langle Fu_n(t), u_n(t) - z\rangle_{V^*,V} + \psi^t(p_n; u_n(t)) \\ \le \psi^t(p_n; z) + (f_n(t), u_n(t) - z)_H, \quad \forall z \in D(\psi^t(p_n;\cdot)) \cap V.\end{aligned} \tag{5.15}$$

As z in (5.15), take a function $\eta_n(t)$ with $\eta_n \in \mathcal{K}_0(p_n)$ satisfying (5.14). Then we see (cf. (4.5)) that there is a positive constant M_4^* such that

$$|u_n|_{C([0,T];H)} + \frac{1}{2}\int_0^T |u_n(t)|_V^2 dt + \int_0^T \psi^t(p_n; u_n(t)) dt \le M_4^*. \tag{5.16}$$

Next for any n, m and any positive number $\varepsilon \in (0,1]$ we use the same notations for $L_{mn}(t)$ and $\sigma_{mn,\varepsilon}$ as in the proof of Theorem 2.1 of section 3. Here take as z of (5.15) the element $\tilde{u}_{mn}(t) := (1 + c_0(\varepsilon))L_{mn}(t)u_m(t) + \sigma_{mn,\varepsilon}(t)$. Then, similarly to (3.9) we have that

$$(u_n'(t), u_n(t) - u_m(t))_H + \nu\langle Fu_n(t), u_n(t) - u_m(t)\rangle + \psi^t(p_n; u_n(t)) \tag{5.17}$$

$$\le \psi^t(p_m; u_m(t)) + (f(t), u_n(t) - u_m(t))_H + \Gamma_{nm,\varepsilon}(t) + \tilde{\Gamma}_{nm,\varepsilon}(t) + H_{nm,\varepsilon}(t),$$

where $\Gamma_{nm,\varepsilon}(t)$ and $\tilde{\Gamma}_{nm,\varepsilon}(t)$ are the same expressions as (3.10) and (3.11) and

$$\begin{aligned} H_{nm,\varepsilon}(t) : \ = \ & (f_n(t) - f(t), u_n(t) - u_m(t))_H \\ & -(f_n - f(t), (L_{mn}(t) - I)u_m + c_0(\varepsilon)L_{mn}(t)u_m(t) + \sigma_{mn,\varepsilon}(t))_H \\ & +\nu\langle Fu_n, (L_{mn}(t) - I)u_m(t) + c_0(\varepsilon)L_{mn}(t)u_m(t) + \sigma_{mn,\varepsilon}(t)\rangle; \end{aligned}$$

note from (5.5), (5.6) and uniform estimate (5.16) that $H_{nm,\varepsilon} \to 0$ in $L^2(0,t)$ uniformly in $t \in [0,T]$ as $n, m \to \infty$ and $\varepsilon \to 0$.

Exchanging n and m in (5.17), we obtain

$$(u_m'(t), u_m(t) - u_n(t))_H + \nu\langle Fu_m(t), u_m(t) - u_n(t)\rangle + \psi^t(p_m; u_m(t)) \tag{5.18}$$

$$\le \psi^t(p_n; u_n(t)) + (f(t), u_m(t) - u_n(t))_H + \Gamma_{mn,\varepsilon}(t) + \tilde{\Gamma}_{mn,\varepsilon}(t) + H_{mn,\varepsilon}(t).$$

Adding (5.17) and (5.18) yields that

$$\frac{1}{2}\frac{d}{dt}|u_n(t) - u_m(t)|_H^2 + \nu|u_n(t) - u_m(t)|_V^2 \tag{5.19}$$

$$\le \Gamma_{nm,\varepsilon}(t) + \Gamma_{mn,\varepsilon}(t) + \tilde{\Gamma}_{nm,\varepsilon}(t) + \tilde{\Gamma}_{mn,\varepsilon}(t) + H_{nm,\varepsilon}(t) + H_{mn,\varepsilon}(t)$$

Finally integrate the above inequality in time t on the time interval $[0,s]$, $s \in (0,T]$, to get

$$\frac{1}{2}|u_n(s) - u_m(s)|_H^2 + \nu\int_0^s |u_n(t) - u_m(t)|_V^2 dt \le \frac{1}{2}|u_{n0} - u_{m0}|_H^2 \tag{5.19}$$

$$+\int_0^s \left\{\Gamma_{nm,\varepsilon}(t) + \Gamma_{mn,\varepsilon}(t) + \tilde{\Gamma}_{nm,\varepsilon}(t) + \tilde{\Gamma}_{mn,\varepsilon}(t) + H_{nm,\varepsilon}(t) + H_{mn,\varepsilon}(t)\right\} dt.$$

Moreover, noting that the integral of the right hand side of (5.19) converges to 0 in $L^1(0,s)$ uniformly in $s \in [0,T]$ as $n, m \to \infty$ and $\varepsilon \to 0$ (cf. Lemma 3.2), we conclude that $\{u_n\}$ is a Cauchy sequence in $C([0,T];H) \cap L^2(0,T;V)$ and the limit u is the weak solution of $CP(\varphi^t(p;\cdot); f, u_0)$. $\diamondsuit$

Remark 5.1. Since 1980s, so many perturbation problems have been discussed in the theory of nonlinear evolution equations associated with time-dependent subdifferential operators. We refer to papers [23, 26] among them for the related works. Furthermore, recently, it was pointed out that variational inequalities of the Navier-Stokes type are handled in our framework of perturbations, and the heat convection problem has been treated under various temperature-dependent constraints (cf. [6, 8, 9]).

Now we formulate a perturbation problem for $QVP(\varphi^t, \Lambda_{p_0}; f, u_0)$.

Definition 5.2. Let $f \in L^2(0,T;H)$, $p_0 \in X$ and $u_0 \in H$. Then $\{u, p\}$ is called a weak solution of $QVP(\varphi^t, \Lambda_{p_0}, G; f, u_0)$, if $u \in C([0,T];H) \cap L^2(0,T;V)$ with $u(0) = u_0$, $p = \Lambda_{p_0}u$ and the variational inequality (5.9) holds.

The next theorem is concerned with the weak solvability for $QVP(\varphi^t, \Lambda_{p_0}, G; f, u_0)$.

Theorem 5.2. *Suppose that conditions* (Φ), (Φ_S) *and* $(A1)-(A3)$ *are satisfied and that* $\mathcal{X}_W$ *is convex in* $\mathcal{X}$. *Let* p_0 *be an element in* X *such that* $\mathcal{X}_W(p_0) := \{p \in \mathcal{X}_W \mid p(0) = p_0\}$ *is non-empty in* $\mathcal{X}_W$ *and* Λ_{p_0} *be a feedback system such that* $(\Lambda1)-(\Lambda3)$ *hold. Suppose that there is a positive constant* M^* *such that for any* $p \in \mathcal{X}_W$ *there is a function* η *in* $\mathcal{K}_0(p)$ *satisfying (4.3). Moreover, let* $f \in L^2(0,T;H)$ *and* $u_0 \in H$ *such that* $u_0 \in D(\varphi^0(\tilde{p};\cdot))$ *for some* $\tilde{p} \in \mathcal{X}_W(p_0)$. *Then, there exists at least one weak solution of* $QVP(\varphi^t, \Lambda_{p_0}, G; f, u_0)$.

Proof. Let us consider the mapping $\mathcal{S}_1$ which assigns to each $p \in \mathcal{X}_W(p_0)$ the weak solution $u \in C([0,T];H) \cap L^2(0,T;V)$ of $CP(\varphi^t(p;\cdot), G(p;\cdot); f, u_0)$. This mapping $\mathcal{S}_1$ is well-defined on account of Theorem 5.1 and for some positive constant M_4^* we have

$$|\mathcal{S}_1 p|_{C([0,T];H)} + \int_0^T \varphi^t(p; \mathcal{S}_1 p(t))dt \le M_4^*, \quad \forall p \in \mathcal{X}_W(p_0), \tag{5.20}$$

which is obtained just as (4.5). Now we put $\mathcal{A}_1 := \Lambda_{p_0}\mathcal{S}_1$ which is the composition of $\mathcal{S}_1$ and Λ_{p_0}. Clearly $\mathcal{A}_1$ is a mapping from $\mathcal{X}_W(p_0)$ into itself. Next we show that $\mathcal{A}_1$ is continuous in $\mathcal{X}$. To do so, let $\{p_n\}$ be a sequence in $\mathcal{X}_W(p_0)$ and $\{u_{n0}\}$ be a sequence in V such that $p_n \to p$ in $\mathcal{X}$, $u_{n0} \in D(\varphi^0(p_n;\cdot))$ and $u_{n0} \to u_0$ in H, and let u_n be the weak solution of $CP(\varphi^t(p_n;\cdot), G(p_n;\cdot); f, u_{n0})$ as well as u be the weak solution of $CP(\varphi^t(p;\cdot), G(p;\cdot); f, u_0)$. Moreover, let $\bar{u}_n$ be the weak solution of $CP(\varphi^t(p_n;\cdot); f - G(p_n;\cdot,u), u_n 0)$. Now, by Lemma 5.2, $\bar{u}_n$ converges to the weak solution u of $CP(\varphi^t(p;\cdot); f - G(p;\cdot,u), u_0)$ in $C([0,T];H) \cap L^2(0,T;V)$. Next, comparing u_n with $\bar{u}_n$, we observe from Lemma 5.1, (4) that

$$\begin{aligned}
& e^{-C_\delta t}|u_n(t) - \bar{u}_n(t)|_H^2 + 2\nu C_F \int_0^t e^{-C_\delta \tau}|u_n(\tau) - \bar{u}_n(\tau)|_V^2 d\tau \\
\le\ & \int_0^t e^{-C_\delta \tau}(G(p_n;\tau,u_n(\tau)) - G(p_n;\tau,u(\tau)), u_n(\tau) - \bar{u}_n(\tau))_H d\tau \\
\le\ & \int_0^t e^{-C_\delta \tau}(G(p_n;\tau,u_n(\tau)) - G(p_n;\tau,\bar{u}_n(\tau)), u_n(\tau) - \bar{u}_n(\tau))_H d\tau \\
& + \int_0^t e^{-C_\delta \tau}(G(p_n;\tau,\bar{u}_n(\tau)) - G(p;\tau,u(\tau)), u_n(\tau) - \bar{u}_n(\tau))_H d\tau \\
\le\ & \delta C_G \int_0^t e^{-C_\delta \tau}|u_n(\tau) - \bar{u}_n(\tau)|_V^2 d\tau + \frac{C_G}{4\delta}\int_0^t e^{-C_\delta \tau}|u_n(\tau) - \bar{u}_n(\tau)|_H^2 d\tau \\
& + \int_0^t e^{-C_\delta \tau}(G(p_n;\tau,\bar{u}_n(\tau)) - G(p;\tau,u(\tau)), u_n(\tau) - \bar{u}_n(\tau))_H d\tau
\end{aligned}$$

Now choose a positive constant δ so as to satisfy $2\nu C_F - \delta C_G \ge \delta$ to have

$$\begin{aligned}
& e^{-C_\delta t}|u_n(t) - \bar{u}_n(t)|_H^2 + \delta \int_0^t e^{-C_\delta \tau}|u_n(\tau) - \bar{u}_n(\tau)|_V^2 d\tau \\
\le\ & \frac{C_G}{4\delta}\int_0^t e^{-C_\delta \tau}|u_n(\tau) - \bar{u}_n(\tau)|_H^2 d\tau
\end{aligned} \tag{5.21}$$

$$+\int_0^t e^{-C_\delta \tau}(G(p_n;\tau,\bar{u}_n(\tau))-G(p;\tau,u(\tau)),u_n(\tau)-\bar{u}_n(\tau))_H d\tau.$$

Neglecting the second term of the left hand side of (5.21), we apply the Gronwall's lemma to (5.21) and pass to the limit as $n\to\infty$ to obtain that $u_n-\bar{u}_n\to 0$ in $C([0,T];H)$, since $G(p_n;\cdot,\bar{u}_n)\to G(p;\cdot,u)$ in $L^2(0,T;H)$. Moreover, going back to (5.21), we see that $u_n-\bar{u}_n\to 0$ in $L^2(0,T;V)$. Consequently it results that $u_n\to u$ in $C([0,T];H)\cap L^2(0,T;V)$. This shows that $\mathcal{S}_1 p_n\to\mathcal{S}_1 p$ in $C([0,T];H)\cap L^2(0,T;V)$ and $\mathcal{A}_1 p_n=\Lambda_{p_0}(\mathcal{S}_1 p_n)\to\Lambda_{p_0}(\mathcal{S}_1 p)=\mathcal{A}_1 p$ by condition ($\Lambda 2$). Thus $\mathcal{A}_1$ is continuous in $\mathcal{X}_W(p_0)$ with respect to the topology of $\mathcal{X}$. Also, by ($\Lambda 3$) and (5.20), the range of $\mathcal{A}_1$ is relatively compact in $\mathcal{X}_W(p_0)$. Accordingly, it results from the Schauder's fixed-point theorem that $\mathcal{A}_1$ has at least one fixed point p, $\mathcal{A}_1 p=p$. The pair $\{u:=\mathcal{S}_1 p,p\}$ gives a weak solution of $QVP(\varphi^t,\Lambda_{p_0},G;f,u_0)$. $\diamondsuit$

Remark 5.2. The elliptic theory of quasi-variational inequalities initiated in 1970s in connection with optimal control problems (cf. [4]), and until now many interesting results have been established in various free boundary problems (see the book [3] and its references, and [10, 14, 22, 24]). The abstract theory of quasi-variational inequalities was established in [28], where an existence result for a class of elliptic quasi-variational inequalities was proved. Subsequently the parabolic type of problems was treated [13, 20, 27]. Moreover, some other different approaches were proposed in [19, 24] and these ideas have been applied to various problems with complex structure (cf. [9, 11, 12, 15]).

6 Application

We consider a 3-component system, $\mathbf{w}:=(w^{(1)},w^{(2)},w^{(3)})$; each component $w^{(k)}:=w^{(k)}(x,t), k=1,2,3,$ is a function on $Q=\Omega\times(0,T)$, where Ω is a bounded domain in $\mathbf{R}^3$ with smooth boundary Γ. Our problem is formally described by the following system:

$$\mathbf{w}_t-\nu\cdot\Delta\mathbf{w}+\partial I_{\mathbf{E}(\theta)}(\mathbf{w})\ni\mathbf{f}(\theta,\mathbf{w})\ \text{ in } Q, \tag{6.1}$$

$$\frac{\partial\mathbf{w}}{\partial n}=0\ \text{ on } \Sigma:=\Gamma\times(0,T),\quad \mathbf{w}(\cdot,0)=\mathbf{w}_0\ \text{ in }\Omega, \tag{6.2}$$

$$\theta_t-\kappa\Delta\theta=h(t,x,\theta,\mathbf{w})\ \text{ in } Q, \tag{6.3}$$

$$\frac{\partial\theta}{\partial n}+n_0\theta=0\ \text{ on }\Sigma,\quad \theta(\cdot,0)=\theta_0\ \text{ in }\Omega, \tag{6.4}$$

where

$$\mathbf{w}_t:=\begin{bmatrix} w_t^{(1)}\\ w_t^{(2)}\\ w_t^{(3)}\end{bmatrix},\ \nu:=\begin{bmatrix}\nu^{(1)}\\ \nu^{(2)}\\ \nu^{(3)}\end{bmatrix},\ \nu\cdot\Delta\mathbf{w}:=\begin{bmatrix}\nu^{(1)}\Delta w^{(1)}\\ \nu^{(2)}\Delta w^{(2)}\\ \nu^{(3)}\Delta w^{(3)}\end{bmatrix},\ \mathbf{f}(\theta,\mathbf{w}):=\begin{bmatrix} f^{(1)}(\theta,\mathbf{w})\\ f^{(2)}(\theta,\mathbf{w})\\ f^{(3)}(\theta,\mathbf{w})\end{bmatrix},$$

and $\nu^{(k)}$ is a positive constant, $w_t^{(k)}=\frac{\partial}{\partial t}w^{(k)}$ for $k=1,2,3$, κ and n_0 are positive constants, $\mathbf{w}_0$, θ_0 are initial data, $f(\theta,\mathbf{w})$ is a vector field from $\mathbf{R}^4$ into $\mathbf{R}^3$ and $h(x,t,\theta,\mathbf{w})$ is a function on $\overline{Q}\times\mathbf{R}^4$.

For simplicity we use the following notations:

$$|\cdot| := |\cdot|_{\mathbf{R}^N},\ 1 \le N < \infty,\quad (\cdot,\cdot) := (\cdot,\cdot)_{\mathbf{R}^N},$$

$$H = L^2(\Omega)^3,\quad (\cdot,\cdot)_H = (\cdot,\cdot)_{L^2(\Omega)^3},\quad V = H^1(\Omega)^3,\quad \langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{V^*,V}.$$

Now we consider the family of compact convex constraints $\mathbf{E}(\theta)$ as follows. Let $\mathbf{a}(\cdot) := (a^{(1)}(\cdot), a^{(2)}(\cdot), a^{(3)}(\cdot))$ be a vector field of C^2-class from $\mathbf{R}$ into $\mathbf{R}^3$ such that

$$|a^{(k)}(\theta)| \le R_0,\ k = 1,2,\quad R_1 \le a^{(3)}(\theta) \le R_2,\quad \forall\theta \in \mathbf{R}$$

for positive constants $R_0,\ R_1,\ R_2\ (> R_1)$. Then we define

$$\mathbf{E}(\theta) := \{\mathbf{z} = (z^{(1)}, z^{(2)}, z^{(3)}) \in \mathbf{R}^3 \mid (\mathbf{a}(\theta), \mathbf{z} - \mathbf{a}(\theta)) = 0, |\mathbf{z} - \mathbf{a}(\theta)| \le \gamma(\theta)\},\quad \forall\theta \in \mathbf{R},$$

where $\gamma(\cdot)$ is a continuous function on $\mathbf{R}$ such that $d_* \le \gamma(\theta) \le d^*$ for all $\theta \in \mathbf{R}$ with constants d_*, d^* satisfying $0 < d_* < d^*$.

For prescribed data $\mathbf{w}_0,\ \theta_0,\ \mathbf{f}(\cdot,\cdot)$ and $h(x,t,\cdot,\cdot)$ we suppose that

(i) $\mathbf{w}_0 \in H$ and $\theta_0 \in H^2(\Omega)$ such that $\frac{\partial\theta_0}{\partial n} + n_0\theta_0 = 0$ a.e. on Γ and $\mathbf{w}_0(x) \in \mathbf{E}(\theta_0(x))$ for a.e. $x \in \Omega$.

(ii) $\mathbf{f}$ is Lipschitz continuous from $\mathbf{R}^4$ into $\mathbf{R}^3$, namely

$$|\mathbf{f}(\theta,\mathbf{w}) - \mathbf{f}(\bar\theta,\bar{\mathbf{w}})| \le L_{\mathbf{f}}(|\theta - \bar\theta| + |\mathbf{w} - \bar{\mathbf{w}}|),\quad \forall\theta,\bar\theta \in \mathbf{R},\ \forall\mathbf{w},\bar{\mathbf{w}} \in \mathbf{R}^{\mathbf{3}},$$

where $L_{\mathbf{f}}$ is a positive constant.

(iii) h is Lipschitz continuous from $\overline{Q} \times \mathbf{R}^4$ into $\mathbf{R}$, namely

$$|h(x,t,\theta,\mathbf{w}) - h(\bar x,\bar t,\bar\theta,\bar{\mathbf{w}})| \le L_h(|x - \bar x| + |t - \bar t| + |\theta - \bar\theta| + |\mathbf{w} - \bar{\mathbf{w}}|),$$

$$\forall(x,t),(\bar x,\bar t) \in \overline{Q}, \forall\theta,\bar\theta \in \mathbf{R}, \forall\mathbf{w},\bar{\mathbf{w}} \in \mathbf{R}^3,$$

where L_h is a positive constant.

Furthermore, in order to treat system (6.1)-(6.4) in our framework of quasi-variational problems we setup function spaces and convex functions as follows. We put

$$\mathcal{X} := \left\{ p := [\mathbf{v},\theta,d] \;\middle|\; \begin{array}{l} \mathbf{v} \in W^{1,2}(0,T;V) \cap C([0,T];W^{1,4}(\Omega)^3), \\ \theta \in W^{1,2}(0,T;H_0^1(\Omega)) \cap C(\overline{Q}), \\ d \in C(\overline{Q}) \end{array} \right\}, \tag{6.5}$$

$$\mathcal{X}_0 := \left\{ p := [\mathbf{v},\theta,d] \in \mathcal{X} \;\middle|\; \begin{array}{l} \mathbf{v} := (v^{(1)}, v^{(2)}, v^{(3)}),\ |\mathbf{v}|_{W^{1,2}(0,T;H^1(\Omega)^3)} \le m^*, \\ |\mathbf{v}|_{L^\infty(0,T;H^2(\Omega)^3)} \le m^*, \\ |v^{(k)}| \le R_0,\ k = 1,2,\ R_1 \le v^{(3)} \le R_2 \text{ in } Q, \\ \theta \in W^{1,2}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega)), \\ d \in C(\overline{Q}),\ d_* \le d \le d^* \text{ in } Q \end{array} \right\}, \tag{6.6}$$

where m^* is a positive constant, and

$$\varphi^t(p;\mathbf{z}) := \frac{1}{2}\sum_{k=1}^{3}\nu_k|\nabla z^{(k)}|^2_{L^2(\Omega)} + I_{K^t(p)}(\mathbf{z}), \quad \forall \mathbf{z} \in L^2(\Omega)^3,\ \forall p := [\mathbf{v},\theta,d] \in \mathcal{X}_0,$$

where $\mathbf{z} = (z^{(1)}, z^{(2)}, z^{(3)})$ and

$$K^t(p) := \left\{ \mathbf{z} \in H^1_0(\Omega)^3 \ \middle|\ \begin{array}{l} |z^{(k)}| \le R_0\ (k=1,2),\ R_1 \le z^{(3)} \le R_2 \text{ a.e. in } \Omega, \\ (\mathbf{v}(\cdot,t), \mathbf{z} - \mathbf{v}(\cdot,t)) = 0,\ |\mathbf{z} - \mathbf{v}(\cdot,t)| \le d(\cdot,t) \text{ a.e. in } \Omega \end{array} \right\}$$

Next, for each $p := [\mathbf{v},\theta,d]$ and $\bar{p} := [\bar{\mathbf{v}},\bar{\theta},\bar{d}] \in \mathcal{X}_0$, we consider the rotation $R_{\mathbf{v}\bar{\mathbf{v}}}(x,t)$ in $\mathbf{R}^3$ with the angle $\alpha_{\mathbf{v}\bar{\mathbf{v}}}(x,t)$, given by $\cos\alpha_{\mathbf{v}\bar{\mathbf{v}}}(x,t) = \frac{(\mathbf{v}(x,t),\bar{\mathbf{v}}(x,t))}{|\mathbf{v}(x,t)||\bar{\mathbf{v}}(x,t)|}$, around the axis $\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(x,t) := \frac{\mathbf{v}(x,t)\times\bar{\mathbf{v}}(x,t)}{|\mathbf{v}(x,t)\times\bar{\mathbf{v}}(x,t)|}$, provided that $\hat{\mathbf{v}}(x,t) := \frac{\mathbf{v}(x,t)}{|\mathbf{v}(x,t)|} \ne \hat{\bar{\mathbf{v}}}(x,t) := \frac{\bar{\mathbf{v}}(x,t)}{|\bar{\mathbf{v}}(x,t)|}$; by definition $R_{\mathbf{v}\bar{\mathbf{v}}}(x,t) = I$, when $\hat{\mathbf{v}}(x,t) = \hat{\bar{\mathbf{v}}}(x,t)$.

Now, we define $L_{p\bar{p}}(t)$ and $\sigma_{p\bar{p},\varepsilon}$ by

$$\begin{cases} [L_{p\bar{p}}(t)\mathbf{z}](x) := R_{\mathbf{v}\bar{\mathbf{v}}}(x,t)\mathbf{z}(x), \quad \text{a.e. } x \in \Omega,\ \forall \mathbf{z} \in H, \\ \sigma_{p\bar{p},\varepsilon}(x,t) := \left(1 - \dfrac{|\mathbf{v}(x,t)|}{|\bar{\mathbf{v}}(x,t)|}\right)\bar{\mathbf{v}}(x,t) + \varepsilon\left(\dfrac{|\mathbf{v}(x,t)|}{|\bar{\mathbf{v}}(x,t)|}\right)\bar{\mathbf{v}}(x,t), \quad (x,t) \in Q, \end{cases} \tag{6.7}$$

(cf. (2.20), (2.21)). As to $\sigma_{p\bar{p},\varepsilon}$, we immediately see that

$$|\sigma_{p\bar{p},\varepsilon}(t)|_V \le C_{\mathcal{X}_0}(|\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_V + \varepsilon)$$

for some positive constant $C_{\mathcal{X}_0}$; hereafter we use the same notation $C_{\mathcal{X}_0}$ to denote a generic positive constant depending only on the class $\mathcal{X}_0$.

(Verification of (A1)-(A3))

Conditions (A1)-(A3) are verified in a quite similar way to that in Example 2.2 as follows. Let $p = [\mathbf{v},\theta,d]$, $\bar{p} = [\bar{\mathbf{v}},\bar{\theta},\bar{d}] \in \mathcal{X}_0$ and let $R_{\mathbf{v}\bar{\mathbf{v}}}(x,t)$ be the rotation with angle $\alpha_{\mathbf{v}\bar{\mathbf{v}}}(x,t)$ around the axis $\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(x,t)$ as defined above. We first show $R_{\mathbf{v}\bar{\mathbf{v}}}(\cdot,t) \in B(V)$. Also, we remember the relation (cf. (2.24))

$$(\sin\alpha_{\mathbf{v}\bar{\mathbf{v}}}(x,t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(x,t) = \hat{\mathbf{v}}(x,t) \times \hat{\bar{\mathbf{v}}}(x,t). \tag{6.8}$$

Since the right side of (6.8) is written as $(\hat{\mathbf{v}}(x,t) - \hat{\bar{\mathbf{v}}}(\mathbf{x},\mathbf{t})) \times \hat{\bar{\mathbf{v}}}(\mathbf{x},\mathbf{t})$, we obtain an estimate of the form

$$|(\sin\alpha_{\mathbf{v}\bar{\mathbf{v}}}(x,t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(x,t)| \le C_{\mathcal{X}_0}|\mathbf{v}(x,t) - \bar{\mathbf{v}}(x,t)|, \quad (x,t) \in Q,$$

whence

$$|(\sin\alpha_{\mathbf{v}\bar{\mathbf{v}}}(t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(t)|_H \le C_{\mathcal{X}_0}|\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_H.$$

We note here that the right side of (6.8) is differentiable in $x \in \Omega$ and $t \in (0,T)$, so that

$$\frac{\partial}{\partial x_j}(\sin\alpha_{\mathbf{v}\bar{\mathbf{v}}}(x,t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(x,t) = \hat{\mathbf{v}}_{x_j}(x,t) \times \hat{\bar{\mathbf{v}}}(x,t) + \hat{\mathbf{v}}(x,t) \times \hat{\bar{\mathbf{v}}}_{x_j}(x,t),\ j = 1,2,3,$$

and

$$\frac{\partial}{\partial t}(\sin \alpha_{\mathbf{v}\bar{\mathbf{v}}}(x,t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(x,t) = \hat{\mathbf{v}}'(x,t) \times \hat{\bar{\mathbf{v}}}(x,t) + \hat{\mathbf{v}}(x,t) \times \hat{\bar{\mathbf{v}}}'(x,t),$$

where $(\cdot)_{x_j} = \frac{\partial}{\partial x_j}$ and $(\cdot)' = \frac{\partial}{\partial t}$ or $\frac{d}{dt}$. Furthermore,

$$\frac{\partial^2}{\partial x_j \partial t}(\sin \alpha_{\mathbf{v}\bar{\mathbf{v}}}(x,t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(x,t) = \hat{\mathbf{v}}'_{x_j}(x,t) \times \hat{\bar{\mathbf{v}}}(x,t) + \hat{\mathbf{v}}_{x_j}(x,t) \times \hat{\bar{\mathbf{v}}}'(x,t)$$
$$+ \hat{\mathbf{v}}(x,t) \times \hat{\bar{\mathbf{v}}}'_{x_j}(x,t) + \hat{\mathbf{v}}'(x,t) \times \hat{\bar{\mathbf{v}}}_{x_j}(x,t).$$

By virtue of the fundamental properties of 3-dimensional exterior product, these inequalities give us estimates of the form:

$$\left|\frac{\partial}{\partial x_j}(\sin \alpha_{\mathbf{v}\bar{\mathbf{v}}}(t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(t)\right|_H \le C_{\mathcal{X}_0}(|\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_V + |\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_{C(\overline{\Omega})^3}), \tag{6.9}$$

$$\left|\frac{d}{dt}(\sin \alpha_{\mathbf{v}\bar{\mathbf{v}}}(t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(t)\right|_H \tag{6.10}$$
$$\le C_{\mathcal{X}_0}(|\mathbf{v}'(t)|_H + |\bar{\mathbf{v}}'(t)|_H)(|\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_{C(\overline{\Omega})^3} + |\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_H),$$

$$\left|\frac{\partial^2}{\partial x_j \partial t}(\sin \alpha_{\mathbf{v}\bar{\mathbf{v}}}(t))\hat{\ell}_{\mathbf{v}\bar{\mathbf{v}}}(t)\right|_H \tag{6.11}$$
$$\le C_{\mathcal{X}_0}\{(|\mathbf{v}'(t)|_V + |\bar{\mathbf{v}}'(t)|_V)|\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_{W^{1,4}(\Omega)^3} + |\mathbf{v}'(t) - \bar{\mathbf{v}}'(t)|_V\},$$

for $j = 1, 2, 3$ and for a.e. $t \in [0, T]$. By the expression of $R_{\mathbf{v}\bar{\mathbf{v}}}(x,t)$ in the form (2.22), the estimates (6.9)-(6.11) yield that

$$|R_{\mathbf{v}\bar{\mathbf{v}}}(t) - I|_{B(V)} \le C_{\mathcal{X}_0}|\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_V, \tag{6.12}$$

and

$$|R'_{\mathbf{v}\bar{\mathbf{v}}}(t)|_{B(V)} \le C_{\mathcal{X}_0}\{(|\mathbf{v}'(t)|_V + |\bar{\mathbf{v}}'(t)|_V)|\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_{W^{1,4}(\Omega)^3} + |\mathbf{v}'(t) - \bar{\mathbf{v}}'(t)|_V\}, \tag{6.13}$$

which implies that $L_{p\bar{p}}$ and $\sigma_{p\bar{p},\varepsilon}$ given by (6.7) satisfy the strong type of conditions (5.5) and (5.6). Thus condition (A1) is satisfied as well as (A2).

Moreover, we see that for any small positive number $\varepsilon \in (0, 1]$ the mapping $\mathbf{z} \to \bar{\mathbf{z}}$ given by

$$\bar{\mathbf{z}} = (1 - \varepsilon)L_{p\bar{p}}(t)\mathbf{z} + \sigma_{p\bar{p},\varepsilon}(t) \ (= (1 - \varepsilon)R_{\mathbf{v}\bar{\mathbf{v}}}(\cdot, t)\mathbf{z} + \sigma_{p\bar{p},\varepsilon}(t))$$

maps $K^t(p)$ onto $K^t(\bar{p})$ and

$$|\bar{\mathbf{z}} - \mathbf{z}|_V \le C_{\mathcal{X}_0}(1 + |\mathbf{z}|_V)(|\mathbf{v}(t) - \bar{\mathbf{v}}(t)|_V + \varepsilon). \tag{6.14}$$

Now, from (6.14) it is easy to verify condition (A3) for an appropriate function of the form $c_1(\varepsilon) = c_2\varepsilon$ with a positive constant c_2.

(Verification of (Φ_S))

Let $\mathcal{X}_1 := \{p = [\mathbf{v}, \theta, d] \in \mathcal{X}_0 \mid \mathbf{v} \in C^1(\overline{Q})^3, \theta \in C^1(\overline{Q}), d \in C^1(\overline{Q})\}$ and $p := [\mathbf{v}, \theta, d]$ be in $\mathcal{X}_1$. Also, denote by $\alpha_{\mathbf{v}}(x, s, t)$ the angle made by $\mathbf{v}(x, s)$ and $\mathbf{v}(x, t)$ for each

$x \in \Omega$ and $s, t \in [0, T]$. For simplicity we write $\hat{\mathbf{v}}(x, s)$ and $\hat{\mathbf{v}}(x, t)$ for $\frac{\mathbf{v}(x,s)}{|\mathbf{v}(x,s)|}$ and $\frac{\mathbf{v}(x,t)}{|\mathbf{v}(x,t)|}$. Now, consider the rotation mapping $R_{\mathbf{v}}(x, s, t)$ with angle $\alpha_{\mathbf{v}}(x, s, t)$ around the axis $\hat{\ell}_{\mathbf{v}}(x, s, t) := \frac{\mathbf{v}(x,s)\times\mathbf{v}(x,t)}{|\mathbf{v}(x,s)\times\mathbf{v}(x,t)|}$, when $\hat{\mathbf{v}}(x, s) \neq \hat{\mathbf{v}}(x, t)$; by definition put $R_{\mathbf{v}}(x, s, t) = I$, when $\hat{\mathbf{v}}(x, s) = \hat{\mathbf{v}}(x, t)$.

Replacing $\{\mathbf{v}(x, t),\ \bar{\mathbf{v}}(x, t)\}$ by $\{\mathbf{v}(x, s),\ \mathbf{v}(x, t)\}$, and $\alpha_{\mathbf{v}\bar{\mathbf{v}}}(x, t)$ by $\alpha_{\mathbf{v}}(x, s, t)$ as well as $R_{\mathbf{v}\bar{\mathbf{v}}}(x, t)$ by $R_{\mathbf{v}}(x, s, t)$, we repeat a similar argument to that in (Verification of (A1)-(A3)) to get

$$|R_{\mathbf{v}}(\cdot, s, t) - I|_{B(V)} \le C_{\mathcal{X}_0}|\mathbf{v}(s) - \mathbf{v}(t)|_V. \tag{6.15}$$

Now, just as in Example 2.2, consider the mapping $\mathbf{z} \to \bar{\mathbf{z}}$ in H given by

$$\bar{\mathbf{z}}(x) = \frac{d(x, s)}{d(x, t)}R_{\mathbf{v}}(x; s, t)\mathbf{z}(x) + \left(1 - \frac{d(x, t)|\mathbf{v}(x, s)|}{d(x, s)|\mathbf{v}(x, t)|}\right)\mathbf{v}(x, t), \quad x \in \Omega, \tag{6.16}$$

or equivalently

$$\bar{\mathbf{z}}(x) - \mathbf{v}(x, t) = \frac{d(x, s)}{d(x, t)}R_{\mathbf{v}}(x; s, t)(\mathbf{z}(x) - \mathbf{v}(x, s)), \quad x \in \Omega.$$

Then we see from (6.13) that the mapping (6.16) maps $K^s(p)$ onto $K^t(p)$ and for any $\mathbf{z} \in K^s(p)$ its image $\bar{\mathbf{z}}$ satisfies

$$|\bar{\mathbf{z}} - \mathbf{z}|_V \le C_{\mathcal{X}_0}(1 + |\mathbf{z}|_V)\int_s^t (|\mathbf{v}'(\tau)|_V + |d'(\tau)|_{C([0,T])} + 1)d\tau. \tag{6.17}$$

It is easy to verify condition (Φ_S) by (6.17) and $\mathcal{X}_1 \subset \mathcal{X}_S$. As a result, we have $\mathcal{X}_W = \mathcal{X}_0$, since $\mathcal{X}_1$ is dense in $\mathcal{X}_0$ with respect to the topology of $\mathcal{X}$.

(Setup of the feedback system Λ_{p_0})

Let $\mathbf{w}$ be any function in $L^2(0, T; V)$ and let θ be a unique solution of (6.3)-(6.4). Then, on account of a regularity result (cf. [7]), we have $\theta \in W^{1,2}(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$ and put $p = [\mathbf{a}(\theta), \theta, \gamma(\theta)]$. Since $W^{1,2}(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$ is compactly embedded in $C(\overline{Q})$, it follows that $p \in \mathcal{X}_0$. Now for the initial datum $p_0 := [\mathbf{a}(\theta_0), \theta_0, \gamma(\theta_0)]$ we define $\Lambda_{p_0} : C([0, T]; H) \cap L^2(0, T; H^1(\Omega)) \to (W^{1,2}(0, T; V) \cap C([0, T]; W^{1,4}(\Omega)^3)) \times (W^{1,2}(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)) \times C(\overline{Q})$ by

$$p := [\mathbf{a}(\theta), \theta, \gamma(\theta)] = \Lambda_{p_0}\mathbf{w}.$$

It is easy to see that the operator Λ_{p_0} satisfies conditions $(\Lambda 1) - (\Lambda 3)$.

Now, we can apply Theorem 5.2 in the present case of $\varphi^t(p; \mathbf{z})$ and $G(p; t, \mathbf{z}) := \mathbf{f}(\theta, \mathbf{z})$ to obtain that the quasi-variational inequality

$$\begin{cases} \mathbf{w}'(t) + \partial\varphi^t(p; \mathbf{w}(t)) \ni G(p; t, \mathbf{w}(t)), \quad 0 < t < T, \quad \mathbf{w}(0) = \mathbf{w}_0, \\ p = \Lambda_{p_0}\mathbf{w} \end{cases}$$

possess at least one weak solution $\{\mathbf{w}, p\}$. Namely there is at least one pair of functions $\{\mathbf{w}, \theta\}$ such that

$$\mathbf{w} \in C([0, T]; H) \cap L^2(0, T; V),\ \theta \in W^{1,2}(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)),$$

$$\mathbf{w} \in \mathbf{E}(\theta) \text{ a.e. in } Q, \quad \mathbf{w}(\cdot,0) = \mathbf{w}_0 \text{ a.e. in } \Omega, \quad \theta(\cdot,0) = \theta_0 \text{ in } \Omega,$$

$$\theta_t - \kappa\Delta\theta = h(t,x,\theta,\mathbf{w}) \ \text{ a.e. in } Q, \quad \frac{\partial\theta}{\partial n} + n_0\theta = 0 \text{ a.e. on } \Sigma,$$

$$\sum_{k=1}^{3}\int_0^t\int_\Omega \eta_\tau^{(k)}(w^{(k)} - \eta^{(k)})dxd\tau + \sum_{k=1}^{3}\int_0^t\int_\Omega \nu^{(k)}\nabla w^{(k)}\cdot\nabla(w^{(k)} - \eta^{(k)})dxd\tau$$

$$+\frac{1}{2}|\eta(t) - \mathbf{w}(t)|_H^2 \le \sum_{k=1}^{3}\int_0^t\int_\Omega f^{(k)}(\theta,\mathbf{w})(w^{(k)} - \eta^{(k)})dxd\tau + \frac{1}{2}|\eta(0) - \mathbf{w}_0|_H^2,$$

$$\forall t \in [0,T], \ \forall\eta \in W^{1,2}(0,T;H) \cap L^2(0,T;V) \text{ with } \eta \in \mathbf{E}(\theta) \text{ a.e. in } Q,$$

where $\eta = (\eta^{(1)}, \eta^{(2)}, \eta^{(3)})$ and $\eta_\tau^{(k)} = \frac{\partial}{\partial\tau}\eta^{(k)}$.

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