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TRANSIENT THERMOELASTIC PROBLEM IN A CONFOCAL ELLIPTICAL DISC WITH INTERNAL HEAT SOURCES

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Abstract. An exact solution is found for the thermoelastic responses in an elliptical disc due to interior heat generation within the solid, under thermal boundary conditions that are subjected to arbitrary initial temperature on the upper and lower face at zero temperature, with radiation boundary conditions on both surfaces. The method of integral transformation technique is used to generate an exact solution of heat conduction equation in which sources are generated according to the linear function of the temperature. The determination of displacement and stresses was performed by means of Airy's stress function approach. The numerical results obtained using these computational tools are accurate enough for practical purposes.

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1 Introduction

The theoretical study of the heat flow within a hollow elliptical structures are of considerable practical importance in a wide range of sectors such as mechanical, aerospace and food engineering fields for the past few decades. Unfortunately, there are only few studies concerned with steady and transient state heat conduction problems in elliptical objects. McLachlan [11],[12] obtained mathematical solution of the heat conduction problem for elliptical cylinder in the form of an infinite Mathieu function series considering special case with neglecting surface resistance. Gupta [5] introduced a finite transform involving Mathieu functions and used for obtaining the solutions of boundary value problem involving elliptic cylinders. Choubey [1] also introduced a finite Mathieu transform whose kernel is given by Mathieu function to solve heat conduction in a hollow elliptic cylinder with radiation. Kirkpatic et al. [8] extended the McLachlan's solution with the involvement of numerical calculation. Erdogdu et al. [2],[3] investigated the heat conduction within an elliptical cylinder by means of a finite difference method. Sugano et al. [13] dealt with transient thermal stress in a confocal hollow elliptical structures with both face surfaces insulated perfectly and obtained the analytical solution with couple-stresses. Sato [14] subsequently obtained heat conduction problem of an infinite elliptical cylinder during heating and cooling considering the effect of the surface resistance. Recently El Dhaba [4] used boundary integral method to solve the problem of plane, uncoupled linear thermoelasticity with heat sources for an infinite cylinder with elliptical cross section, subjected to a uniform pressure and to a thermal radiation condition on its boundary. However, there aren't many investigations done or studied to successfully eliminate thermoelastic problems.

Researchers haven't considered any thermoelastic problem expressed in elliptical coordinates with boundary conditions of radiation type, in which sources are generated according to the linear function of the temperatures, which satisfies the time-dependent heat conduction equation. It has been proved that ample cases of heat production in solids have led to various technical problems in mechanical applications in which heat produced is rapidly sought to be transferred or dissipated. For instance, gas turbines blades, walls of I.C. engine, outer surface of a space vehicle and other factors all depend for their durability on rapid heat transfer from their surfaces. Reviewing the previous studies, it was observed by the author that no analytical procedure has been established, considering internal heat sources generation within the body. Actually, by considering a circle as a special kind of ellipse, it is shown that the temperature distribution and history in a circular solution can be derived as a special case from the present mathematical solution for the elliptical disc.

The object of this paper is to study the problem of heat transfer in the region where heat is generated in the system and is transferred which may be taken as the zero on the temperature scale. The success of this research mainly lies with the mathematical procedures which present much simpler approach for optimization for the design in terms of material usage and performance in engineering problem, particularly in the determination of thermoelastic behavior in elliptical disc engaged as the foundation of pressure vessels, furnaces, etc. In this paper we have extended the integral transformation defined Choubey [1] involving ordinary and modified Mathieu functions of first and second kind

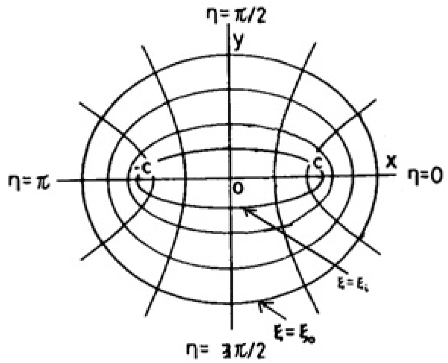

Figure 1: Shows the geometry of the problem

of order n which is analogous to the finite Hankel transform. Integral transformation and it Inversion formula is established and some properties are mentioned at the Appendix section. We do not claim to have obtained new integral transformation but certainly we have modified integral transform suiting to our boundary conditions and applied the transformation to determine the temperature distribution in a nonhomogeneous finite elliptical disc occupying the space $D = \{(\xi, \eta, z) \in R^3 : \xi_i \le \xi \le \xi_o, 0 \le \eta \le 2\pi, z = \ell\}$. For illustrating the practical usage of the research, a particular case with realistic example is explained for further clarification.

2 Formulation of the problem

The thermoelastic problem of an elliptical disc subjected to radiation type boundary conditions on the outside and inside surfaces can be rigorously analyzed by introducing the elliptical coordinates (ξ, η, z), which are related to the rectangular coordinates (x, y, z) by the relation

$$x = c\cosh\xi\cos\eta,\ y = c\sinh\xi\sin\eta,\ z = \ell, \tag{1}$$

where c is the semi-focal length as shown in Fig. 1. From the above equations, one obtains a group of confocal ellipses and hyperbolas with the common foci for various values of ξ and η, respectively.

2.1 Transient Heat Conduction Analysis

The governing equation of heat conduction with internal heat source, the initial condition and boundary conditions in elliptical cylindrical coordinates are given, respectively as

$$\kappa h^2 (\partial_{,\xi\xi} + \partial_{,\eta\eta})\, \theta\ (\xi,\, \eta,\, t) + \Theta\,(\xi,\, \eta,\, t,\, \theta) = \theta(\xi,\, \eta,\, t)_{,t}\,, \tag{2}$$

$$\theta(\xi,\, \eta,\, t)|_{t=0} = \ 0, \ \text{ for } t = 0, \ \text{ at } \xi_i \le \xi \le \xi_o,\ 0 < \eta < 2\pi, \tag{3}$$

$$\theta(\xi,\, \eta,\, t) + k_1\, \theta(\xi,\, \eta,\, t)_{,\xi} = \ 0, \ \text{ for } \xi = \xi_i, \ \text{ at } 0 < \eta < 2\pi, \tag{4}$$

$$\theta(\xi,\, \eta,\, t) + k_2\, \theta(\xi,\, \eta,\, t)_{,\xi} = \ 0, \ \text{ for } \xi = \xi_o, \ \text{ at } 0 < \eta < 2\pi, \tag{5}$$

where $\theta(\xi, \eta, t)$ is the temperature function, $\Theta(\xi, \eta, t, \theta)$ is the source function for the problem, $k_i\,(i = 1, 2)$ are radiation coefficients, $\kappa = \lambda/\rho C$ represents thermal diffusivity in which λ being the thermal conductivity of the material, ρ is the density and C is the calorific capacity, assumed to be constant.
We assume source functions as the superposition of the simpler function [9]

$$\Theta(\xi, \eta, t, \theta) = \Phi(\xi, \eta, t) + \psi(t)\,\theta(\xi, \eta, t), \tag{6}$$

and

$$\left.\begin{aligned} T(\xi, \eta, t) &= \theta(\xi, \eta, t)\,\exp\left[-\int_0^t \psi(\zeta)\,d\zeta\right], \\ \chi(\xi, \eta, t) &= \Phi(\xi, \eta, t)\,\exp\left[-\int_0^t \psi(\zeta)\,d\zeta\right]. \end{aligned}\right\} \tag{7}$$

Substituting equations (6) and (7) in the heat conduction equation (2), we obtain

$$\kappa\,h^2(\partial_{,\xi\xi} + \partial_{,\eta\eta})\,T(\xi, \eta, t) + \chi(\xi, \eta, t) = T(\xi, \eta, t)_{,t}\,. \tag{8}$$

For the sake of brevity, we consider

$$\chi(\xi, \eta, t) = \exp(-\omega t)\delta(\xi - \xi_0)\delta(\eta - \eta_0)/\xi_0\,\eta_0, \xi_i \le \xi_0 \le \xi_o, 0 \le \eta_0 \le 2\pi, \omega > 0. \tag{9}$$

The equations (2) to (9) constitute the mathematical formulation for temperature change within elliptical disc with internal heat source under consideration.

2.2 Displacement and Thermal Stress Analysis

Following Gosh [6] and Jeffery [7], the displacements are given by

$$\left.\begin{aligned} (2\mu)\,u/h &= -\phi(\xi, \eta, t)_{,\xi} + P(\xi, \eta, t)_{,\eta}/h^2, \\ (2\mu)\,v/h &= -\phi(\xi, \eta, t)_{,\eta} + P(\xi, \eta, t)_{,\xi}/h^2, \end{aligned}\right\} \tag{10}$$

where (u, v) are displacements in the directions normal to the curves (ξ, η) constant, P satisfies the equations

$$\left.\begin{aligned} &\nabla^2 P = 0, \\ &(\lambda + \mu)\,[(h^{-2}P_{,\eta})_{,\xi} + (h^{-2}P_{,\xi})_{,\eta}] = (\lambda + 2\mu)[(\partial_{,\xi\xi} + \partial_{,\eta\eta})\,\phi], \end{aligned}\right\} \tag{11}$$

and stress function in equation (10) satisfies below equation (12) of the fourth order

$$h^2\nabla^2 h^2\nabla^2\phi = -\,h^2\nabla^2\theta. \tag{12}$$

The components of the stresses [13] are represented as

$$\left.\begin{aligned} \sigma_{\xi\xi} &= h^2\phi_{,\eta\eta} + (e^2h^4/2)\,\sinh 2\xi\;\phi_{,\xi} - (e^2h^4/2)\,\sin 2\eta\;\phi_{,\eta}\,, \\ \sigma_{\eta\eta} &= h^2\phi_{,\xi\xi} - (e^2h^4/2)\,\sinh 2\xi\;\phi_{,\xi} + (e^2h^4/2)\,\sin 2\eta\;\phi_{,\eta}\,, \\ \sigma_{\xi\eta} &= -\,h^2\phi_{,\xi\eta} + (e^2h^4/2)\,\sin 2\eta\;\phi_{,\xi} + (e^2h^4/2)\,\sinh 2\xi\;\phi_{,\eta}\,. \end{aligned}\right\} \tag{13}$$

For traction free surface the stress functions

$$\sigma_{\xi\xi} = \sigma_{\xi\eta} = 0 \;\text{ at }\; \xi = \xi_i,\, \xi_o. \tag{14}$$

The set of equations (12) to (14) constitute mathematical formulation for displacement and thermal stresses developed within solid due to temperature change.

3 Solution for the Problem

3.1 Transient Heat Conduction Analysis

In order to solve fundamental differential equation (8), we firstly introduce the extended integral transformation of order n and m over the variable ξ and η as

$$\bar{f}(q_{n,m}) = \int_{\xi_i}^{\xi_o} \int_0^{2\pi} f(\xi,\eta)(\cosh 2\xi - \cos 2\eta) S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m}) d\xi d\eta, \tag{15}$$

Inversion theorem of (15) as

$$f(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{f}(q_{n,m})\, S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})/\, C_{n,m}, \tag{16}$$

where, the kernel $S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})$is given in equation (37) and

$$C_{n,m} = \int_{\xi_i}^{\xi_o} \int_0^{2\pi} (\cosh 2\xi - \cos 2\eta)\; S^2_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})\, d\xi\, d\eta. \tag{17}$$

Performing above integral transformation under the conditions (4) and (5), we obtain

$$\overline{T}_{,t}\,(n,m,t) + \alpha_{n,m}\overline{T}(n,m,t) = \exp(-\omega t), \tag{18}$$

where $\bar{T}(n,m,t)$ is the transformed function of $T(\xi,\eta,t)$ and $\alpha_{n,m} = 2\,\kappa\, q_{n,m}\, h^2$. On solving (18) under initial boundary condition given in equation (3), one obtain

$$\overline{T}(n,m,t) = \wp_{n,m}\; \exp(-\alpha_{n,m}\, t), \tag{19}$$

where

$$\wp_{n,m} = 1 + [\exp\,(\,\alpha_{n,m} - \omega)\, t/(\alpha_{n,m} - \omega)].$$

On applying inversion theorems defined in (16), one obtain the expression for temperature as

$$T(\xi,\eta,\;t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \wp_{n,m}\, S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})\; \exp(-\alpha_{n,m}\, t)/C_{n,m}. \tag{20}$$

Taking into account the first equation of equation (7), the temperature distribution as

$$\begin{aligned} \theta(\xi,\eta,\;t) = {} & \textstyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \wp_{n,m}\, S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m}) \\ & \times\; \exp(-\alpha_{n,m}\, t)/C_{n,m}. \end{aligned} \tag{21}$$

The function given in equation (21) represents the temperature at every instant and at all points of elliptical annulus of finite height under the influence of radiation's conditions.

3.2 Thermoelastic Solution

Assuming Airy's stress function $\phi(\xi,\,\eta,t)$ which satisfies condition (12) as,

$$\begin{aligned}\phi(\xi,\,\eta,t) = \textstyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\xi+X_{n,m}\,\xi^2+Y_{n,m}\,\eta^2}{C_{n,m}\,(\omega-\alpha_{n,m})}\right)\;S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})\\ \times\,\wp_{n,m}\;\exp(-\alpha_{n,m}\,t)\;\exp[\textstyle\int_0^t\psi\,(\zeta)\;d\,\zeta].\end{aligned} \tag{22}$$

Arbitrary functions $X_{n,m}$ and $Y_{n,m}$ are determined using equation (22) and (13) in equation (14). Thus equation (22) becomes

$$\begin{aligned}\phi =\; \textstyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{\begin{array}{c}(-\omega+\alpha_{n,m})\exp[t\,(-\omega+\alpha_{n,m})]\\ \times C_1\,S_{n,m}(\xi,\eta)+C_2\,S_{n,m}(\xi_0,\eta)\end{array}}{\begin{array}{c}2C_{n,m}(-\omega+\alpha_{n,m})^2\xi_0[2A_1+C_3S_{n,m}[\xi_0,\eta]+C_4]\\ +4\,\eta\,C_5\,S_{n,m}[\xi_0,\eta]\end{array}}\right\}\\ \times\;\exp(-\alpha_{n,m}\,t)\;\;\exp\,[\textstyle\int_0^t\psi(\xi)\,d\xi].\end{aligned} \tag{23}$$

Now using equation (23) in equation (13), one obtains the stresses expressions as

$$\begin{aligned}\sigma_{\xi\xi} =\; \textstyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{\begin{array}{c}(\omega-\alpha_{n,m})\exp[t\,(-\omega+\alpha_{n,m})]\\ [D_1+S_{n,m}[\xi_0,\eta](\overline{D})]-2S_{n,m}[\xi,\eta](\overline{\overline{D}})\end{array}}{\begin{array}{c}4C_{n,m}(-\omega+\alpha_{n,m})^2\xi_0[2A_1+C_3\,S_{n,m}[\xi_0,\eta]+C_4]\\ +4\,\eta\,C_5\,S_{n,m}[\xi_0,\eta]\end{array}}\right\}\\ \times\;\exp(-\alpha_{n,m}\,t)\;\exp\,[\textstyle\int_0^t\psi(\xi)\,d\xi].\end{aligned} \tag{24}$$

where

$$\begin{aligned}A_1 &= e^2h^2[4\sin 2\eta +\; e^2h^2\eta\,(\cos 4\eta-\cosh 4\xi_0)]\;S^2_{n.m}[\xi_0,\eta],\\ C_1 &= e^2h^2\xi\,(e^2h^2\eta\,(\cos 4\eta-\cosh 4\xi_0)+4\sin 2\eta)(\xi-2\xi_0)S^2_{n,m}[\xi_0,\eta]\\ &\quad +4\,[\,(e^2h^2\eta(\xi-\xi_0)^2\sin 2\eta+4\xi(-2\xi+\xi_0(4e^2h^2(\xi-\xi_0)\sinh 2\xi_0))\\ &\quad \times\partial_\eta S^2_{n.m}[\xi_0,\eta]+2\,\xi\,\xi_0(\xi-\xi_0)\,\partial_\xi S_{n,m}[\xi_0,\eta](2\partial_{\eta\eta}S_{n,m}[\xi_0,\eta]\\ &\quad +e^2h^2\sinh 2\xi_0\partial_\xi S_{n,m}[\xi_0,\eta])+(\xi-\xi_0)\,\partial_\eta S_{n,m}[\xi_0,\eta](2\eta(-\xi+\xi_0)\\ &\quad \times\partial_{\eta\eta}S_{n,m}[\xi_0,\eta]+e^2h^2(2\xi\xi_0\;\sin 2\eta+\eta\sinh 2\xi_0(-\xi+\xi_0))\partial_\xi S_{n,m}[\xi_0,\eta]\\ &\quad -8\xi\xi_0\partial_{\xi\eta}S_{n,m}[\xi_0,\eta]],\end{aligned}$$

$$
\begin{aligned}
C_2 = {} & [-16\xi^2 - e^4h^4\eta^2\xi^2\cosh 4\xi_0 + 24e^2h^2\eta\xi\sin 2\eta(\xi - 2\xi_0) \\
& + e^4h^4\eta^2(\xi-\xi_0)^2\cos 4\eta + 32\xi\xi_0 + 2e^4h^4\eta^2\xi\xi_0\cosh 4\xi_0 \\
& + 8e^2h^2\eta^2\xi^2\xi_0\sinh 2\xi_0 - e^4h^4\eta^2\xi_0^2\cosh 4\xi_0 \\
& - 8e^2h^2\xi\xi_0^2\sinh 2\xi_0]\,\partial_\eta S_{n,m}[\xi_0,\eta] \\
& + 2\,[2e^2h^2\eta(\xi-\xi_0)(\eta(\xi-\xi_0)\sin 2\eta - 2\xi\xi_0\sinh 2\xi_0)\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& + e^2h^2\xi(4\xi\eta\sinh 2\xi_0 + \xi_0(-8\eta\sinh 2\xi_0 + e^2h^2\eta(\xi-\xi_0)\cos 4\eta \\
& + 4(\xi-\xi_0)\sin 2\eta + e^2h^2\eta(-\xi+\xi_0)\cosh 4\xi_0)\partial_\xi S_{n,m}[\xi_0,\eta] \\
& + 2(\xi-\xi_0)(e^2h^2\eta^2\xi\sinh 2\xi_0 + \xi_0(-4\xi + e^2h^2\eta\xi\sin 2\eta - e^2h^2\eta^2\sinh 2\xi_0)) \\
& \times\ \partial_{\xi\eta}S_{n,m}[\xi_0,\eta]],
\end{aligned}
$$

$$
\begin{aligned}
C_3 = {} & [-16 + e^4h^4\eta^2\cos 4\eta - e^4h^4\eta^2\cos 4\xi_0 + 24e^2h^2\eta\sin 2\eta + 4e^2h^2\xi_0\sinh 2\xi_0)] \\
& \times\partial_\eta S_{n,m}[\xi_0,\eta] + 4e^2h^2\eta\,(\eta\sin 2\eta - \xi_0\sinh 2\xi_0)\,\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& + 8e^2h^2\eta\sinh 2\xi_0\partial_\xi S_{n,m}[\xi_0,\eta] \\
& + e^2h^2[e^2h^2\eta\,(\cos 4\eta - \cosh 4\xi_0) + 4\sin 2\eta]\,\xi_0\,\partial_\xi S_{n,m}[\xi_0,\eta],
\end{aligned}
$$

$$
C_4 = 4e^2h^2\eta^2\sinh 2\xi_0\partial_{\xi\eta}S_{n,m}[\xi_0,\eta] + 4\,(-2 + e^2h^2\eta\sin 2\eta)\xi_0\partial_{\xi\eta}S_{n,m}[\xi_0,\eta],
$$

$$
\begin{aligned}
C_5 = {} & -8 + e^2h^2(\eta\sin 2\eta + 2\xi_0\sinh 2\xi_0)\partial_\eta S^2_{n.m}[\xi_0,\eta] \\
& + \xi_0\partial_\xi S_{n,m}[\xi_0,\eta]\,(2\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] + e^2h^2\sinh 2\xi_0\partial_\xi S_{n,m}[\xi_0,\eta]) \\
& -\partial_\eta S_{n,m}[\xi_0,\eta]\ [2\eta\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] + e^2h^2(\eta\sinh 2\xi_0 \\
& - \xi_0\sin 2\eta]\,\partial_\xi S_{n,m}[\xi_0,\eta] + 4\xi_0\partial_{\xi\eta}S_{n,m}[\xi_0,\eta],
\end{aligned}
$$

$$
\overline{D} = -2D_2 + D_3 + 4\eta D_4,\ \overline{\overline{D}} = D_5 + D_6 + 4D_7,
$$

$$
\begin{aligned}
D_1 = {} & 2e^2h^2\xi[e^2h^2\eta(\cos 4\eta - \cosh 4\xi_0) + 4\sin 2\eta] \\
& \times\ (\xi - 2\xi_0)S^2_{n.m}[\xi_0,\eta](-2\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& + e^2h^2(\sin 2\eta - \sinh 2\xi)\partial_\xi S_{n,m}[\xi_0,\eta]),
\end{aligned}
$$

$$
\begin{aligned}
D_2 = {} & \partial_{\eta\eta}S_{n,m}[\xi,\eta] + e^2h^2\cosh\xi\sinh h\xi\partial_\xi S_{n,m}[\xi,\eta] \\
& \times\ (2(e^2h^2\xi[e^2h^2\eta(\cos 4\eta - \cosh 4\xi_0) \\
& + 4\sin 2\eta](\xi - 2\xi_0)S^2_{n.m}[\xi_0,\eta](-2\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& + e^2h^2(\sin 2\eta - \sinh 2\xi)\partial_\xi S_{n,m}[\xi_0,\eta]) \\
& + (-16\xi^2 - e^4h^4\eta^2\xi^2\cosh 4\xi_0 + 24e^2h^2\eta\xi\sin 2\eta(\xi - 2\xi_0) \\
& + e^4h^4\eta^2\cos 4\eta(\xi-\xi_0)^2 + 32\xi\xi_0 \\
& + 2e^4h^4\eta^2\xi\xi_0\cosh 4\xi_0 + 8e^2h^2\xi_0\xi^2\sinh 2\xi_0 - e^4h^4\eta^2\xi_0^2\cosh 4\xi_0 \\
& - 8e^2h^2\xi\xi_0^2\sinh 2\xi_0)\partial_\eta S_{n,m}[\xi_0,\eta] \\
& + e^2h^2\eta(\xi-\xi_0)(\eta\sin 2\eta(\xi-\xi_0) - 2\xi\sinh 2\xi_0)\partial_{\eta\eta}S_{n,m}[\xi_0,\eta]) \\
& + e^2h^2\xi(4\eta\xi\sinh 2\xi_0 + \xi_0(-8\eta\sinh 2\xi_0 + e^2h^2\eta\cos 4\eta(\xi-\xi_0) \\
& + 4(\xi-\xi_0)\sin 2\eta + e^2h^2\eta(-\xi+\xi_0)\cosh 4\xi_0))\partial_\xi S_{n,m}[\xi_0,\eta] \\
& + 2(\xi-\xi_0)(e^2h^2\eta\xi\sinh 2\xi_0 \\
& + (-4\xi + 2e^2h^2\eta\sin 2\eta - e^2h^2\eta^2\sinh 2\xi_0)\xi_0)\partial_{\xi\eta}S_{n,m}[\xi_0,\eta])),
\end{aligned}
$$

$$
\begin{aligned}
D_3 = {} & e^2h^2\partial_\eta S_{n,m}[\xi,\eta](\partial_\eta S_{n,m}[\xi_0,\eta](\xi^2\sin 2\eta(-16+e^4h^4\eta^2\cos 4\eta \\
& -e^4h^4\eta^2\cosh 4\xi_0+24e^2h^2\eta\sin 2\eta) \\
& +2\xi\sin 2\eta\,(16-e^4h^4\eta^2\cos 4\eta+e^4h^4\eta^2\cosh 4\xi_0-24e^2h^2\eta\sin 2\eta \\
& +4e^2h^2\xi\sinh 2\xi_0)\,\xi_0+e^2h^2(\eta(\cos 4\eta-\cosh 4\xi_0)(-8+e^2h^2\eta\sin 2\eta) \\
& -8\xi\sin 2\eta\sinh 2\xi_0)\xi_0^2)+2(2\eta\sin 2\eta(e^2h^2\eta\sin 2\eta(\xi-\xi_0)^2 \\
& +2\xi_0(-4\xi_0+e^2h^2\xi(-\xi+\xi_0)\sinh 2\xi_0))\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& +e^2h^2\xi\sin 2\eta(4\eta\xi\sinh 2\xi_0 \\
& +\xi_0(8\eta\sinh 2\xi_0+e^2h^2\eta(\xi-\xi_0)\cos 4\eta+4(\xi-\xi_0)\sin 2\eta \\
& +e^2h^2\eta(-\xi+\xi_0)\cosh 4\xi_0))\partial_\xi S_{n,m}[\xi_0,\eta] \\
& +2(e^2h^2\xi\xi_0(-\xi+\xi_0)\cos 4\eta+\eta\xi_0(e^2h^2\xi(\xi-\xi_0)-8\xi_0\sinh 2\xi_0) \\
& +(-\xi+\xi_0)\sin 2\eta(4\xi\xi_0+e^2h^2\eta^2(-\xi+\xi_0)\sinh 2\xi_0))\partial_{\xi\eta}S_{n,m}[\xi_0,\eta])),
\end{aligned}
$$

$$
\begin{aligned}
D_4 = {} & e^2h^2\eta\sin 2\eta(\xi-\xi_0)^2+4\xi(-2\xi+(4+e^2h^2\xi_0(\xi-\xi_0)\sinh 2\xi_0))\partial \\
& \times_\eta S^2_{n,m}[\xi_0,\eta]+2\xi\xi_0(\xi-\xi_0)\partial_\xi S_{n,m}[\xi_0,\eta](2\partial_{\eta\eta}S_{n,m}[\xi_0,\eta]+e^2h^2\sinh 2\xi_0 \\
& \times\partial_\xi S_{n,m}[\xi_0,\eta]+(\xi-\xi_0)\partial_\eta S_{n,m}[\xi_0,\eta](2\eta((-\xi+\xi_0)\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& +e^2h^2(2\xi\xi_0\sin 2\eta+\eta\sinh 2\xi_0(-\xi+\xi_0))\partial_\xi S_{n,m}[\xi_0,\eta] \\
& -8\xi\xi_0\partial_{\xi\eta}S_{n,m}[\xi_0,\eta]-2\partial_{\eta\eta}S_{n,m}[\xi_0,\eta]+e^2h^2\cosh\xi\sinh\xi\partial_\xi S_{n,m}[\xi_0,\eta] \\
& +\partial_\eta S_{n,m}[\xi,\eta](e^2h^2(\xi-\xi_0)\sin 2\eta(-8\xi+e^2h^2\eta\sin 2\eta(\xi-\xi_0) \\
& +4\xi_0(2+e^2h^2\xi\sinh 2\xi_0)\partial_\eta S^2_{n,m}[\xi_0,\eta] \\
& +2e^2h^2\sin 2\eta(\xi-\xi_0)\xi_0\partial_\xi S_{n,m}[\xi_0,\eta](2\partial_{\eta\eta}S_{n,m}[\xi_0,\eta]+e^2h^2\xi_0\sinh 2\xi_0) \\
& \times\partial_\eta S^2_{n,m}[\xi_0,\eta]-\partial_\eta S_{n,m}[\xi_0,\eta](\,2(e^2h^2\sin 2\eta(\xi-\xi_0)^2-8\xi_0^2)\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& +e^2h^2(e^2h^2\eta\sin 2\eta\sinh 2\xi_0(\xi-\xi_0)^2+\xi\xi_0e^2h^2\cos 4\eta(\xi-\xi_0) \\
& +\xi_0(-8\xi_0\sinh 2\xi_0+e^2h^2\xi(-\xi+\xi_0))\partial_\xi S_{n,m}[\xi_0,\eta] \\
& +8\xi\xi_0(\xi-\xi_0)\sin 2\eta\partial_{\xi\eta}S_{n,m}[\xi_0,\eta])),
\end{aligned}
$$

$$
D_5 = 2e^4h^4(e^2h^2\eta(\cos 4\eta-\cosh 4\xi_0)+4\sin 2\eta)\sinh 2\xi(\xi-\xi_0)S^2_{n,m}[\xi_0,\eta],
$$

$$
\begin{aligned}
D_6 = {} & e^2h^2S_{n,m}[\xi_0,\eta](\partial_\eta S_{n,m}[\xi_0,\eta](\xi(-16+e^4h^4\eta^2\cos 4\eta \\
& -e^4h^4\eta^2\cosh 4\xi_0+24e^2h^2\eta\sin 2\eta)\sinh 2\xi \\
& +\sinh 2\xi(16-e^4h^4\eta^2\cos 4\eta+e^4h^4\eta^2\cosh 4\xi_0-24e^2h^2\eta\sin 2\eta \\
& +8\xi e^2h^2\sinh 2\xi_0))\xi_0 \\
& -e^2h^2((\cos 4\eta-\cosh 4\xi_0)(-2+e^2h^2\eta\sin 2\eta)+4\xi_0^2\sinh 2\xi\sinh 2\xi_0)) \\
& +4(e^2h^2\eta^2\xi\sin 2\eta\sinh 2\xi+\xi_0(-e^2h^2\eta\sinh 2\xi(\eta\sin 2\eta+2\xi\sinh 2\xi_0) \\
& +(\sin 2\eta(2-e^2h^2\eta\sin 2\eta) \\
& +e^2h^2\eta\sinh 2\xi\sinh 2\xi_0)\xi_0))\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& +e^2h^2\sinh 2\xi(8\eta\xi\sinh 2\xi_0+\xi_0(-8\eta\sinh 2\xi_0+e^2h^2\eta\cos 4\eta(2\xi-\xi_0) \\
& +e^2h^2\eta\cosh 4\xi_0(-2\xi+\xi_0)-4\sin 2\eta(-2\xi+\xi_0))\partial_\xi S_{n,m}[\xi_0,\eta] \\
& +(e^2h^2\eta^2\xi\sinh 2\xi\sinh 2\xi_0+\sinh 2\xi(-4\xi+2e^2h^2\eta\xi\sin 2\eta \\
& -e^2h^2\eta^2\sinh 2\xi_0)\xi_0-(-2+e^2h^2\eta\sin 2\eta)(\sinh 2\xi+\sinh 2\xi_0)\xi_0^2) \\
& \times\partial_{\xi\eta}S_{n,m}[\xi_0,\eta]),
\end{aligned}
$$

$$
\begin{aligned}
D_7 = & \, e^2h^2\partial_\eta S^2_{n,m}[\xi_0,\eta](\eta\xi(-8+e^2h^2\eta\sin 2\eta)\sinh 2\xi_0 \\
& +\xi_0\eta\sinh 2\xi(8-e^2h^2\eta\sin 2\eta+4e^2h^2\xi\sinh 2\xi_0 \\
& -(\sin 2\eta(-2+e^2h^2\eta\sin 2\xi\sinh 2\xi_0)\xi_0)) \\
& +e^2h^2\eta\sin 2\xi(2\xi-\xi_0)\xi_0\partial_\xi S_{n,m}[\xi_0,\eta](2\partial_{\eta\eta}S_{n,m}[\xi_0,\eta] \\
& +e^2h^2\sinh 2\xi_0\partial_\xi S_{n,m}[\xi_0,\eta] \\
& +\partial_\eta S_{n,m}[\xi_0,\eta](e^2h^2((e^2h^2\eta\xi_0\sin 2\eta(\sinh 2\xi(2\xi-\xi_0) \\
& +\sinh 2\xi_0(-2\xi_0^2+e^2h^2\eta\sinh 2\xi(-\xi+\xi_0))\partial_\xi S_{n,m}[\xi_0,\eta] \\
& +4\eta\xi_0\sinh 2\xi((-2\xi+\xi_0)\partial_{\xi\eta}S_{n,m}[\xi_0,\eta]-4(e^2h^2\eta^2\xi\cosh\xi\sinh\xi \\
& +\xi_0(e^2h^2\eta^2\cosh\xi\sinh\xi+\xi_0-e^2h^2\eta\xi_0\cos\eta\sinh\eta))\partial_{\eta\eta}S_{n,m}[\xi_0,\eta]).
\end{aligned}
$$

The other resulting equations of stresses (i.e. $\sigma_{\eta\eta}$ and $\sigma_{\xi\eta}$) which are also rather lengthy, and consequently are omitted here for the sake of brevity, but considered during graphical discussion described in below section.

4 Numerical Results, Discussion and Remarks

For the sake of simplicity of calculation, we introduce the following dimensionless values

$$
\left.
\begin{array}{l}
\bar{b}_o = b_o/a_o,\ \overline{b_i} = b_i/a_i,\ e = c/a_o,\ \bar{h}^2 = h^2a_o^2,\ \tau = \kappa\, t/a_o^2, \\
\overline{\theta}(\xi,\eta,\ t) = \theta(\xi,\eta,\ t)/\theta_k,\ (\overline{\theta}_i,\overline{\theta}_o) = (\theta_i,\theta_o)/\theta_k\ \ (k=i,o), \\
\bar{\phi}(\xi,\ \eta,t) = \phi(\xi,\ \eta,t)/E\alpha_t\theta_k a_o^2,\ \bar{\sigma}_{ij} = \sigma_{ij}/E\alpha_t\theta_k\ \ (i,j=\xi,\eta).
\end{array}
\right\}
\tag{25}
$$

Here E stands for Young's modulus, α_t for Thermal expansion coefficient, respectively. Then, setting

$$
\psi(\zeta) = -\,\zeta, T_0 = 0, \tag{26}
$$

$$
\Rightarrow \quad \int_0^t \psi(\zeta)\,d\zeta = -\,t^2/2, \bar{T}_0^* = 0. \tag{27}
$$

Substituting the value of equation (27) in equations (21), (23) and (24), we obtained the expressions for the temperature, displacement and stresses respectively for our numerical discussion. The numerical computations have been carried out for Aluminum metal with parameter a = 2.65 cm, b = 3.22 cm, h = 2.00 cm, Modulus of Elasticity $E = 6.9 \times 10^6$ N/cm^2, Shear modulus $G = 2.7 \times 10^6$ N/cm^2, Poisson ratio $\upsilon = 0.281$, Thermal expansion coefficient, α_t= 25.5$\times$ 10^6 cm/cm-^{0}C, Thermal diffusivity $\kappa = 0.86$ cm^2/sec, Thermal conductivity $\lambda = 0.48$ cal sec^{-1}/cm ^{0}C with $q_{n,m}$ =0.0986, 0.3947, 0.8882, 1.5791, 2.4674, 3.5530, 4.8361, 6.3165, 7.9943, 9.8696, 11.9422, 14.2122, 16.6796, 19.3444, 22.2066, 25.2661, 28.5231, 31.9775, 35.6292, 39.4784 are the positive & real roots of the transcendental equation (35). The foregoing analysis are performed by setting the radiation coefficients constants, $k_i = 0.86$ $(i = 1,\ 2)$ so as to obtain considerable mathematical simplicities. In order to examine the influence of uniform heating on the disc, we performed the numerical calculation for time $\tau = 0.001,\ 0.05,\ 0.12,\ 0.30,\ 0.70..\infty$ and numerical calculations are depicted in the following figures with the help of MATHEMATICA software. The theoretical analysis on the heat conduction & its thermal stress in a confocal hollow elliptical plate without internal heat source subjected to non-axisymmetric heating on internal and outer elliptical boundaries was investigated by integral transform by Sugano et al. [13], where kernel was expressed in the form of Mathieu and modified Mathieu functions. The thermoelastic effects on the temperature, displacements, and thermal stresses without internal heat source was fully discussed in research paper [13]. For the sake of brevity, discussion of these effects was omitted here and graphical illustration for the thermoelastic response for an elliptical disc considering interior heat generation was investigated during our research. Figs. 2-6 illustrates the numerical results of dimensionless temperature and stresses of elliptical disc due to interior heat generation within the solid, under thermal boundary condition that are subjected to arbitrary initial temperature on the upper and lower face at zero temperature and boundary conditions of radiation type on the outside and inside surfaces, with independent radiation constants in radial direction at $\eta = 90^0$ for different values of time. As shown in Fig. 2, the temperature drops as the time proceeds along radial direction and is greatest in a steady and initial state. From Fig. 2, it can be seen that the temperature change on the heated surface decreases when the radius of plate increases. The variation of normal stresses $\bar{\sigma}_{\xi\xi}$, $\bar{\sigma}_{\eta\eta}$, and $\bar{\sigma}_{\xi\eta}$ is shown in Figs. 3, 4 and 5, respectively.

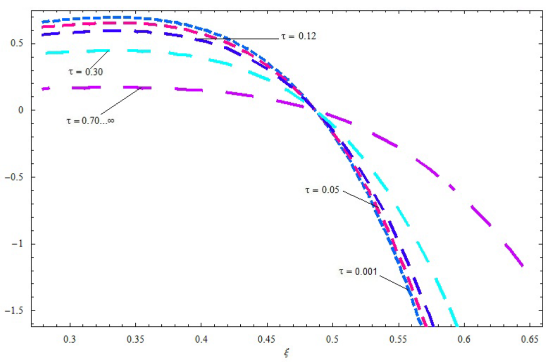

Figure 2: $\overline{\theta}(\xi, \eta, t)$ versus ξ at $\eta = 90$ deg. for different values of time

From Fig. 3, the large compressive stress occurs on the inner heated surface and the tensile stress occurs on the inner surface which drops along the radial direction. It is also

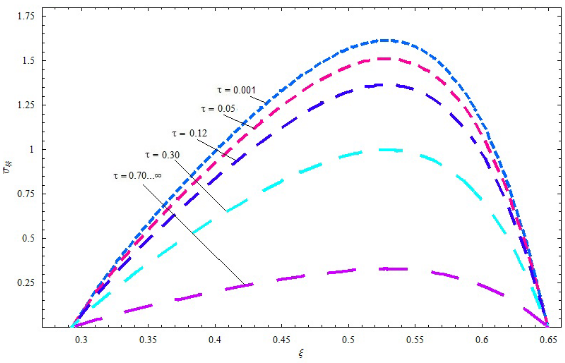

Figure 3: $\bar{\sigma}_{\xi\xi}$ versus ξ at $\eta = 90$ deg. for different values of time

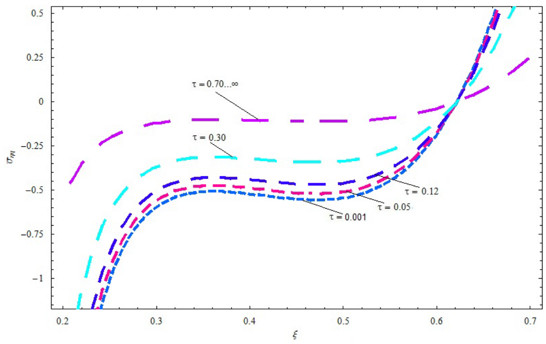

Figure 4: $\bar{\sigma}_{\eta\eta}$ versus ξ at η= 90 deg. for different values of time

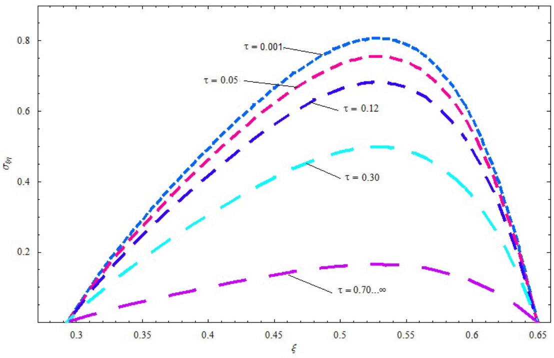

Figure 5: $\bar{\sigma}_{\xi\eta}$ versus ξ at η= 90 deg. for different values of time

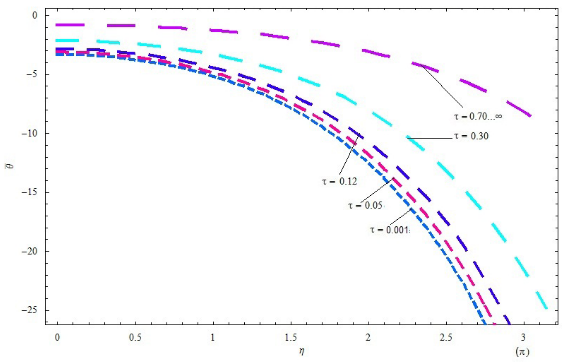

Figure 6: $\overline{\theta}(\xi, \eta,\, t)$ versus η at ξ= 0.45 for different values of time

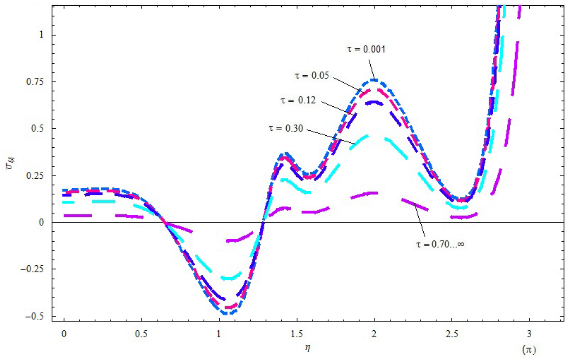

Figure 7: $\bar{\sigma}_{\xi\xi}$ versus η at ξ= 0.45 for different values of time

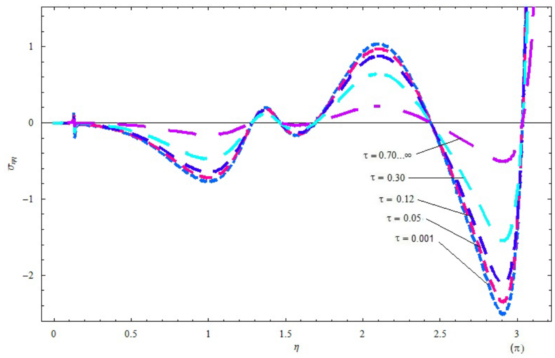

Figure 8: $\bar{\sigma}_{\eta\eta}$ versus η at ξ= 0.45 for different values of time

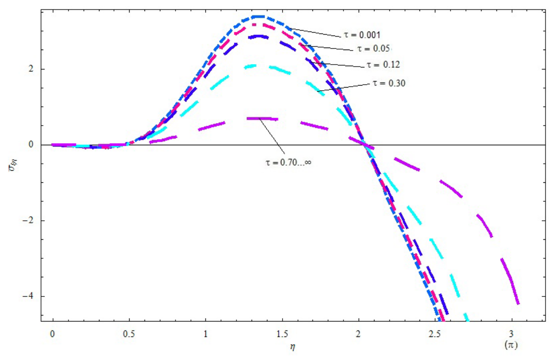

Figure 9: $\bar{\sigma}_{\xi\eta}$ versus η at ξ= 0.45 for different values of time

noted that Fig. 3 agrees with the traction free surface conditions as quoted in equation (14). From Fig. 4, the compressive stress occurs on the outer edge of the ellipse and the absolute value rises as the time proceeds towards outer surface. From Fig. 5, the maximum tensile stress occurs during uniform heating inside the core of the disc which follows assumed traction free property as in equation (14). Figs. 6 to 9 shows dimensionless temperature and thermal stresses along angular direction.

Fig. 6 shows the time variation of temperature distribution along angular direction of the disc. The temperature decreases with time, and the maximum value of temperature magnitude occurs at higher steady state with available internal heat energy. The aforementioned results agrees with the results [13] as soon as internal heat is not considered while doing thermoelastic analysis. The distribution of the dimensionless temperature gradient at each time decreases in the unheated area of the outer ellipse boundary tending below zero in one direction. The stress distributions are shown from Figs. 7 to 9. It is observed that the stress patterns from elliptical inner hole to mid core part which follows the similar pattern of the applied mechanical boundary conditions. The radial stress $\bar{\sigma}_{\xi\xi}$, circumferential stress $\bar{\sigma}_{\eta\eta}$ and shear stress $\bar{\sigma}_{\xi\eta}$ at inner surface are nearly zero due to the assumed traction free boundary conditions. It is noted that maximum tensile stress occurs near the outer surface and the compressive stress occurs inside the disc and its absolute value increases with time.

5 Transition to annular-circular disc

When the elliptical disc degenerates into an annular circular disc with the thickness$h \to 0$, internal radiusξ_i, and external radius $\xi_o \to \infty$, occupying the space $D' = \{(x, y, z) \in R^3 : a \le (x^2 + y^2)^{1/2} \le b,\ z = \ell\}$, where $r = (x^2 + y^2)^{1/2}$ in such a way that $h\ \exp(\xi)/2 \to r$, $h\ \exp(\xi_i)/2 \to a$, and $h\ \exp(\xi_0)/2 \to b$ ([11] pp. 367-368) and taking θ independent of η. For that we take,

$$\begin{array}{l} n = 0,\ q \to 0,\ e \to 0,\ \cosh 2\xi\, d\xi \to 2rh^2 dr,\ A_2^{(0)} \to 0,\ A_0^{(0)} \to 1/\sqrt{2}, \\ \lambda_{0,m}^2 \to \alpha_m^2,\ ce_0(\eta,\ q_{0,m}) \to 1/\sqrt{2},\ ce_0(\xi,\ q_{0,m}) \to J_0(\alpha_m r), \\ Fey_0(\xi_0,\ q_{0,m}) \to Y_0(\alpha_m r),\ \alpha_m (= \alpha_{0,m}) \text{are the roots of} \\ J_0(k_1, \alpha_a)\, Y_0(k_2, \alpha_b) - J_0(k_2, \alpha_b)\, Y_0(k_2, \alpha_a) = 0, \\ Ce_0(k_1, \xi, \eta, q_{0,m}) \to Ce_0(k_1, r\alpha_m), \\ Fey_0(k_1, \xi, \eta, q_{0,m}) \to Fey_0(k_1, r\alpha_m), \\ S_{0,m}(k_1, k_2, \xi, \eta, q_{0,m}) \to S_{0,m}(k_1, k_2, r\alpha_m)\ \ (= S_m(k_1, k_2, r\alpha_m)), \end{array}$$

where

$$\left.\begin{array}{l} J_0(k_j, \alpha_i\, r)\ = J_0(\alpha_i r) + k_j J_0'(\alpha_i r), \\ Y_0(k_j, \alpha_i\, r)\ = Y_0(\alpha_i r) + k_j\, Y_0'(\alpha_i r), \end{array}\right\} i =\ 1,\ 2.$$

Equation (21) degenerates into temperature distribution in hollow circular disc

$$\begin{array}{rl} \theta\,(r,\ z,\ t) = & \sum_{m=1}^{\infty} \frac{1}{C_m} \left(1 + \frac{\exp\,(\alpha_m - \omega)\, t}{\alpha_m - \omega}\right) S_m(k_1, k_2, r\alpha_m) \\ & \times\ \exp(-\alpha_m\, t)\ \exp[\int_0^t \psi\,(\zeta)\ d\,\zeta], \end{array} \tag{28}$$

where

$$C_m = \int_a^b r\, S_m^2(k_1, k_2, r\alpha_m)\, dr,$$

and kernel as

$$\begin{aligned} S_m(k_1, k_2, r\alpha_m) = {} & J_0(r\alpha_m)\, [\, Y_0(k_1, a\alpha_m) + Y_0(k_2, b\alpha_m)\,] \\ & - Y_0(r\alpha_m)\, [\, J_0(k_1, a\alpha_m) + J_0(k_2, b\alpha_m)\,]. \end{aligned}$$

The aforementioned results agrees with the results [10].

6 Conclusion

The proposed analytical solution of transient plane thermal stress problem of the confocal elliptical region was handled in elliptical coordinate system with the presence of a source of internal heat. To author's knowledge there have been no reports of solution so far in which sources are generated according to the linear function of the temperature in mediums in the form of elliptical disc of finite height with boundaries conditions of the radiation type. The analysis of non-stationary two-dimensional equation of heat conduction is investigated with the integral transformation method as when there are conditions of radiation type contour acting on the object under consideration. With proposed integral transformation method, it is possible to apply widely to analysis stationary as well as non-stationary temperatures. Also by using the Airy's stress function induced by Sugano [13], we have proposed an exact solution theoretically and illustrated graphically for better understanding. The following results were obtained to carry away during our research are:

1. The advantage of this method is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions.

2. The maximum tensile stress is shifting from central core to outer region may be due to heat, stress, concentration or available internal heat sources under considered temperature field.

3. Finally, the maximum tensile stress occurs in the circular hole on the major axis compared to elliptical hole indicates that the distribution of weak heating. It may be due to insufficient penetration of heat through elliptical inner surface. The aforementioned integral transform will also be extended to other elliptical objects having finite height with conditions of radiation type contour during further research work.

7 Nomenclature

$\xi, \eta =$	Elliptical Coordinates.
$q =$	Parameter of Mathieu equation.

$ce_n(\eta,\, q) =$ Ordinary Mathieu function of first kind of order n.
$Ce_n(\xi,\, q) =$ Modified Mathieu function of second kind of order n.
$h =$ Interfocal distance.
$1/h^2 =$ $c^2(\cosh\xi - \cos 2\eta)/2$.
$k =$ Thermal conductivity.
$H =$ Surface conductance.
$Ce_n(k_j, \xi_i, q) =$ Mathieu function defined in equation (33).
$Fey_n(k_j, \xi_o, q) =$ Mathieu function defined in equation (33).
$S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m}) =$ Mathieu function defined in equation (37).
$q_{n,m} =$ Parametric roots of equation (35).
$\bar{f}(q_{n,m}) =$ Mathieu transform of $f(\xi, \eta)$.
$\theta\ (\xi,\ \eta,\ t) =$ The temperature distribution at any time t.
$\Theta\ (\xi,\ \eta,\ t,\ \theta) =$ The internal source function
$\phi =$ The Airy's stress functions.
$T_0 =$ The reference temperature.
$f(\eta, t) =$ The heat supply available on curved surface.
$2c =$ Focal length, $= 2\sqrt{a_i^2 - b_i^2} = 2\sqrt{a_o^2 - b_o^2}$,[13].
$\xi_i,\, \xi_o =$ $\tanh^{-1}(b_i/a_i)$, $= \tanh^{-1}(b_o/a_o)$,[13].
$C_{n,m} =$ $\int_{\xi_i}^{\xi_o} \int_0^{2\pi} (\cosh 2\xi - \cos 2\eta)\ S_{n,m}^2(k_1, k_2, \xi, \eta, q_{n,m})\, d\xi\, d\eta$.

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8 Appendix

8.1 The extended finite Mathieu transformation

Let us seek the solution of the Mathieu differential equation

$$\partial_{,\xi\xi}\, y(\xi,\eta) + \partial_{,\eta\eta}\, y(\xi,\eta) + 2q\,[\cosh(2\xi) - \cos(2\eta)]\, y(\xi,\eta) = 0, \tag{29}$$

for the modified boundary conditions of the radiation type as compared with

$$\left.\begin{array}{l} y(\xi,\eta) + k_1\,\partial_{,\xi}\, y(\xi,\eta)\,|_{\xi=\xi_i} = 0, \\ y(\xi,\eta) + k_2\,\partial_{,\xi}\, y(\xi,\eta)\,|_{\xi=\xi_o} = 0, \end{array}\right\} \text{ as } \xi_i < \xi_o. \tag{30}$$

If the general solution of the equation (29) is

$$y(\xi,\eta) = [C_1\, Ce_n(\xi,\, q) + C_2\, Fey_n(\xi,\, q)]\, ce_n(\eta,\, q), \tag{31}$$

where the comma denotes the differentiation with the following variable, C_1 and C_2 are arbitrary constants, $ce_n(\xi,\, q)$ ([11], p.21) is a Mathieu function of the first kind of order n, $Ce_n(\xi,\, q)$ ([11], p.159) is a modified Mathieu function of the first kind of order n, $Fey_n(\xi,\, q)$ ([11], p.159) is a Y-type modified Mathieu function of the second order n. To obtain a solution of equation (29) that satisfies conditions (30)), we will substitute equation (31) into (30), one obtains

$$\begin{aligned} C_1[Ce_n(\xi_i,q) + k_1\, C'e_n(\xi_i,q)] &= -C_2[Fey_n(\xi_i,q) + k_1\, F'ey_n(\xi_i,q)], \\ C_1[Ce_n(\xi_o,q) + k_1\, C'e_n(\xi_o,q)] &= -C_2[Fey_n(\xi_o,q) + k_1\, F'ey_n(\xi_o,q)]\,, \end{aligned} \tag{32}$$

Taking

$$\left.\begin{array}{ll} Ce_n(k_j,\xi_i,q) & = Ce_n(\xi_i,q) + k_j\, C'e_n(\xi_i,q), \\ Fey_n(k_j,\xi_o,q) & = Fey_n(\xi_o,q) + k_j\, F'ey_n(\xi_o,q). \end{array}\right\} \tag{33}$$

We get equation (32) as

$$\left.\begin{array}{l} C_1[Ce_n(k_1,\xi_i,\eta,q)] = -C_2[Fey_n(k_i,\xi_i,\eta,q)], \\ C_1[Ce_n(k_2,\xi_o,\eta,q)] = -C_2[Fey_n(k_2,\xi_o,\eta,q)]\,. \end{array}\right\} \tag{34}$$

Eliminating $C_1\& \, C_2$ from (34), we see that the frequency equation exist only if

$$Ce_n(k_1,\xi_i,\eta,q)\, Fey_n(k_2,\xi_o,\eta,q) - Ce_n(k_2,\xi_0,\eta,q)\, Fey_n(k_1,\xi_i,\eta,q) = 0. \tag{35}$$

Denote by $q_{n,m}$ the roots of equation (35). From the first equation of (34) and (31), and on the other hand, by using the second equation of (34) and (31), we obtain

$$\left.\begin{array}{rl} y(\xi,\eta) = & \frac{C_1 ce_n(\eta,q_{n,m})}{Fey_n(k_1,\xi_i,\eta,q_{n,m})}\, [Ce_n(\xi,q_{n,m})\, Fey_n(k_1,\xi_i,\eta,q_{n,m}) \\ & \quad - Fey_n(\xi,q_{n,m})\, Ce_n(k_1,\xi_i,\eta,q_{n,m})], \\ y(\xi,\eta) = & \frac{C_2 ce_n(\eta,q_{n,m})}{Fey_n(k_2,\xi_0,\eta,q_{n,m})}\, [Ce_n(\xi,q_{n,m})\, Fey_n(k_1,\xi_o,\eta,q_{n,m}) - \\ & \quad Fey_n(\xi,q_{n,m})\, Ce_n(k_1,\xi_o,\eta,q_{n,m})]\,. \end{array}\right\} \tag{36}$$

The linear combination of the two preceding equations of (36) leads to the functions

$$\begin{array}{rl} S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m}) = & Ce_n(\xi,q_{n,m})\ ce_n(\eta,q_{n,m}) \\ & \times[\, Fey_n(k_1,\xi_i,q_{n,m}) +\ Fey_n(k_2,\xi_o,q_{n,m})\,] \\ & + Fey_n(\xi,q_{n,m})\ ce_n(\eta,q_{n,m}) \\ & \times[Ce_n(k_1,\xi_i,q_{n,m})\ + Ce_n(k_2,\xi_o,q_{n,m})]. \end{array} \tag{37}$$

8.2 Essential properties

Differential property of $S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})$

Differentiate equation (37) with respect to ξ,

$$\begin{array}{rl} S'_{n,m}(k_1,k_2,\xi,\eta,q_{n,m}) = & C'_{e_n}(\xi,q_{n,m})\ ce_n(\eta,q_{n,m}) \\ & \times[\, Fey_n(k_1,\xi_i,q_{n,m}) +\ Fey_n(k_2,\xi_o,q_{n,m})\,] \\ & - F'_{ey_n}(\xi,q_{n,m})\ ce_n(\eta,q_{n,m}) \\ & \times[Ce_n(k_1,\xi_i,q_{n,m})\ + Ce_n(k_2,\xi_o,\eta,q_{n,m})]. \end{array} \tag{38}$$

From equations (37) and (38), we obtain

$$\frac{S'_{n,m}}{S_{n,m}} = \frac{\left\{\begin{array}{l} C'_{e_n}(\xi,q_{n,m})\, ce_n(\eta,q_{n,m}) \\ \times[Fey_n(k_1,\xi_i,q_{n,m}) + Fey_n(k_2,\xi_o,q_{n,m})] \\ -F'_{ey_n}(\xi,q_{n,m})\, ce_n(\eta,q_{n,m}) \\ \times[Ce_n(k_1,\xi_i,q_{n,m}) + Ce_n(k_2,\xi_o,\eta,q_{n,m})] \end{array}\right\}}{\left\{\begin{array}{l} Ce_n(\xi,q_{n,m})\ ce_n(\eta,q_{n,m}) \\ \times[Fey_n(k_1,\xi_i,q_{n,m}) + Fey_n(k_2,\xi_o,q_{n,m})\,] \\ -Fey_n(\xi,q_{n,m})\ ce_n(\eta,q_{n,m}) \\ \times[Ce_n(k_1,\xi_i,q_{n,m}) + Ce_n(k_2,\xi_o,\eta,q_{n,m})] \end{array}\right\}}. \tag{39}$$

Since

$$\left.\begin{array}{ll} C'_{e_n}(\xi, q_{n,m}) & = (1/k_1)\ [Ce_n(k_1, \xi, \eta, q_{n,m}) - Ce_n(\xi, q_{n,m})], \\ F'_{ey_n}(\xi, q_{n,m}) & = (1/k2)[Fey_n(k_1, \xi, \eta, q_{n,m}) - Fye_n(\xi, q_{n,m})]. \end{array}\right\} \tag{40}$$

Substituting equation (40) in (39) leads to

$$S'_{n,m}(k_1, k_2, \xi_i, \eta, q_{n,m})/S_{n,m}(k_1, k_2, \xi_i, \eta, q_{n,m}) = -1/k_1 \mathrm{at} \xi = \xi_i, \tag{41}$$

$$S'_{n,m}(k_1, k_2, \xi_o, \eta, q_{n,m})/S_{n,m}(k_1, k_2, \xi_o, \eta, q_{n,m}) = -1/k_2 \mathrm{at} \xi = \xi_o. \tag{42}$$

Orthogonal property of eigenfunction $S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})$

If $q_{n,m}$ and $q_{p,r}$ be the roots of equation (35) then equation $S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})$ and $S_{p,r}(k_1, k_2, \xi, \eta, q_{p,r})$ satisfy the differential equation (29) so that we have

$$\partial_{\xi\xi}\ S_{n,m} + \partial_{\eta\eta} S_{n,m} + 2q_{n,m}\ [\cosh(2\xi) - \cos(2\eta)]\ S_{n,m} = 0, \tag{43}$$

$$\partial_{\xi\xi}\ S_{p,r} + \partial_{\eta\eta} S_{p,r} + 2q_{p,r}\ [\cosh(2\xi) - \cos(2\eta)]\ S_{p,r} = 0. \tag{44}$$

Multiply (43) by $S_{p,r}(k_1, k_2, \xi, \eta, q_{p,r})$ and (44) by $S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})$ and subtracting (43) from (44) we obtain

$$\begin{array}{l} \partial_\xi[S_{p,r}\ \partial_\xi S_{n,m} - S_{n,m} \partial_\xi S_{p,r}] + \partial_\eta[S_{p,r}\ \partial_\eta S_{n,m} - S_{n,m}\ \partial_\eta S_{p,r}] \\ +2(q_{n,m} - q_{p,r})[\cosh(2\xi) - \cos(2\eta)] s_{n,m} s_{p,r} = 0. \end{array} \tag{45}$$

Integrating equation (45) with respect to ξ and η

$$\begin{array}{l} \int_0^{2\pi} [S_{p,r} \partial_\xi S_{n,m} - S_{n,m} \partial_\xi S_{p,r}]_{\xi_i}^{\xi_o} d\eta \\ + \int_{\xi_i}^{\xi_o} [S_{p,r} \partial_\xi S_{n,m} - S_{n,m} \partial_\xi S_{p,r}]_0^{2\pi} d\xi \\ +2(q_{n,m} - q_{p,r})\ \int_{\xi_i}^{\xi_o}\ \int_0^{2\pi}\ [\cosh(2\xi) - \cos(2\eta)] s_{n,m} s_{p,r} d\xi d\eta = 0. \end{array} \tag{46}$$

Thus we can easily obtain

$$\int_{\xi_i}^{\xi_o} \int_0^{2\pi} [\cosh(2\xi) - \cos(2\eta)] s_{n,m}\, s_{p,r}\, d\xi\, d\eta = 0,\ p \neq n,\ \ r \neq m. \tag{47}$$

Properties of finite Mathieu transform

$$\begin{array}{l} \int_{\xi_i}^{\xi_o} \int_0^{2\pi} [\partial_{\xi\xi} y(\xi, \eta) + \partial_{\eta\eta} y(\xi, \eta)]\ S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})\, d\xi\, d\eta \\ = \ \int_0^{2\pi} S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})\ \left[\partial_\xi y - y \frac{S'_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})}{S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})}\right]_{\xi_i}^{\xi o} d\eta \\ + \int_{\xi_i}^{\xi_o} \int_0^{2\pi} y(\xi, \eta)\ [\partial_{\xi\xi} + \partial_{\eta\eta}]\ S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})\, d\xi\, d\eta \\ + \int_{\xi_i}^{\xi_o} [S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})\ [\partial_\xi y] - y\ [\partial_\eta y]\ S_{n,m}(k_1, k_2, \xi, \eta, q_{n,m})]_0^{2\pi} d\xi, \end{array} \tag{48}$$

since $S_{n,m}$ is the solution of $\partial_{\xi\xi}S_{n,m} + \partial_{\eta\eta}S_{n,m} + 2q\ [\cosh(2\xi) - \cos(2\eta)]\ S_{n,m} = 0$, so we obtain

$$\begin{aligned}&\int_{\xi_i}^{\xi_o}\int_0^{2\pi}[\partial_{\xi\xi}y(\xi,\eta) + \partial_{\eta\eta}y(\xi,\eta)]\, S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})\, d\xi\, d\eta \\ &= -2q_{n,m}\overline{v} + \int_0^{2\pi}(h/\sqrt{2})\ \sqrt{\cosh 2\xi - \cos 2\eta}\ \{S_{n,m}(k_1,k_2,\xi_i,\eta,q_{n,m}) \\ &\times[k_2\partial_\xi y + y]_{\xi=\xi_i} d\eta\ + S_{n,m}(k_1,k_2,\xi_i,\eta,q_{n,m})\ [k_1\partial_\xi y + y]_{\xi=\xi_o}\}.\end{aligned} \tag{49}$$

Convergence
In order to portray the solution to be meaningful, the series expressed in equation (21) should converge for all $\xi_i \le \xi \le \xi_o$ and $0 \le \eta \le 2\pi$. The temperature equation (21) can be expressed as

$$\begin{aligned}\theta\,(\xi,\,\eta,\,t)\ =\ &\sum_{n=0}^{M}\sum_{m=1}^{M'}\frac{\bar{\chi}\,(n,m,t)}{\alpha_{n,m}C_{n,m}}\,S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m})\ \exp(-\alpha_{n,m}\,t) \\ &\times\exp[\int_0^t \psi\,(\zeta)\,d\zeta].\end{aligned} \tag{50}$$

But it should be noted that $Ce_n(\xi_i, q_{n,m}) = 0$ (for $n = 0, 1, 2, 3..$) and $Ce_n(\eta, q_{n,m}) = 0$ (for $0 \le \eta \le 2\pi$). The above roots define a series of a confocal nodal hyperbola. Now, $Ce_n(\eta, q_{n,m})$, each have n zeros in $0 \le \eta \le 2\pi$ so for a given n, each function gives rise to a n nodal hyperbolas, Thus from above it is clear that $S_{n,m}(k_1,k_2,\xi,\eta,q_{n,m}) \to 0$ as $n \to \infty$, $m \to \infty$. Also, from orthogonally property, $C_{n,m} \ne 0$, which shows that $\theta\,(\xi,\,\eta,\,t)$ converges with the finite value as $n \to \infty, m \to \infty$. Thus, from physical consideration, it is clear that $\theta\,(\xi,\,\eta,\,t)$ must be continuous function of $(\xi,\,\eta)$ within the ellipse and it vanishes on the boundary. Thus, any function of $(\xi,\,\eta)$ continuous and single valued within the ellipse and vanishes on the boundary may be expanded at any point of the interior in the form of the double series given in inversion theorem, which proves the convergence of modified Mathieu function.