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ANALYSIS OF FOREST KINEMATIC MODEL WITH NONLINEAR DEGENERATE DIFFUSION

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Abstract. This paper is concerned with the forest kinematic model with nonlinear degenerate diffusion. The model is described as a system of two ordinary differential equations and one reaction diffusion equation. For this system, we will give complete information on the structure of nonnegative stationary solutions when the spatial dimension is one. In particular, we will show that the stationary problem has a continuum of nonnegative solutions with compact support in a domain under suitable conditions. The existence of such a type of solutions is completely different from the linear diffusion case.

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1 Introduction

This paper is concerned with the following forest kinematic model with nonlinear degenerate diffusion:

$$\begin{cases} u_t = \beta\delta w - \gamma(v)u - fu & \text{in } \Omega \times (0,\infty), \\ v_t = fu - hv & \text{in } \Omega \times (0,\infty), \\ w_t = (w^m)_{xx} - \beta w + \alpha v & \text{in } \Omega \times (0,\infty), \\ w = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 \geq 0 & \text{in } \Omega, \\ v(\cdot,0) = v_0 \geq 0 & \text{in } \Omega, \\ w(\cdot,0) = w_0 \geq 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $m > 1$ and Ω is identical with one dimensional interval $(-L, L)$ for a positive number L. The unknown functions $u(x,t)$ and $v(x,t)$ denote the tree densities of young age class and of old age class at position $x \in \Omega$ and time $t \in [0,\infty)$, respectively. The third unknown function $w(x,t)$ denotes the density of seeds at $x \in \Omega$ and $t \in [0,\infty)$. In (1.1), $\alpha > 0$ and $\beta > 0$ denote seed production and seed deposition rates, respectively. A constant $0 < \delta < 1$ is an establishment rate of seeds, $f > 0$ is an aging rate, $h > 0$ is a mortality of old trees and $\gamma(v) > 0$ is a mortality of young trees which is given by

$$\gamma(v) = a(v-b)^2 + c \tag{1.2}$$

where a, b, c are all positive constants.

When w has a linear diffusion term, i.e., $m = 1$, this problem was first proposed by Kuznetsov, Antonovsky, Biktashev and Aponina [9] in 1994. For such a model, Shirai, Chuan and A. Yagi ([13, 14]) studied the existence of a unique global solution and its asymptotic behavior as $t \to \infty$ for two dimensional domain Ω. They also investigated the structure of solutions for the corresponding stationary problem under Dirichlet boundary condition in [15]. See also the works [2, 3], [4] and [5] in case of Neumann boundary condition for w.

In the present paper we take nonlinear diffusion of the form $(w^m)_{xx}$ with $m > 1$ for the dispersion of seeds. If we consider such a degenerate diffusion of porous media type in population dynamics, it is well known that an initial population which is confined in a bounded domain spreads out at a finite speed and remains confined in a bounded domain for any time (see the work of Gurtin and MacCamy [7]). This effect is due to the degeneracy of nonlinear diffusion with $m > 1$. Therefore, it is sometimes plausible to take account of nonlinear degenerate diffusion in population dynamics and such nonlinear degenerate diffusion models have been studied by lots of authors (see, e.g., [6], [7], [8], [10],[11, 12] and [16]).

The purpose of the present paper is to study the structure of nonnegative solutions for the following stationary problem related to (1.1):

$$\text{(SP)} \quad \begin{cases} \beta\delta w - \gamma(v)u - fu = 0 & \text{in } (-L,L), \\ fu - hv = 0 & \text{in } (-L,L), \\ (w^m)_{xx} - \beta w + \alpha v = 0 & \text{in } (-L,L), \\ w(-L) = w(L) = 0, \end{cases}$$

where $m > 1$, α, β, δ, f and h are positive constants and $\gamma(v)$ is given by (1.2) with positive constants a, b, c. We intend to show that the presence of nonlinear diffusion brings about different results from the linear diffusion case. Take α or δ as a parameter. If $\alpha\delta$ is large ($\alpha\delta > h(ab^2 + c + f)/f$), then it will be proved that (SP) has at least one positive solution for every $L > 0$ in case $m > 1$. This fact is different from the linear diffusion case $m = 1$ where (SP) has no positive solution for small $L > 0$. Moreover, when $\alpha\delta$ lies in a certain range ($\alpha\delta \in (h(c + f)/f, h(ab^2 + c + f)/f)$) and $m > 1$, we will show that, if L is sufficiently large, then (SP) has an infinite number of nonnegative solutions with compact supports as well as usual positive solutions. This is a big difference from the linear diffusion case. Ecologically, such a solution with compact support is very interesting because it corresponds to a patchy pattern of forests in a real world. We will prove that (SP) provides rich structure of nonnegative solutions of (SP).

The contents of this paper are as follow. In Section.2, we will consider (SP) and reduce it to a boundary value problem for a single equation of the form:

$$\text{(SPW)} \quad \begin{cases} (w^m)_{xx} + g(w) = 0 & \text{in } (-L, L), \\ w(-L) = w(L) = 0, \end{cases}$$

where $g(w)$ is a suitable continuous function satisfying $g(0) = 0$. Phase plane analysis is effective for the analysis of (SPW). When we look for nonnegative solutions for (SPW), nonlinearity g can be classified into two types; monostable type and bistable type according as the number of zero points of g. In Section.3, we will construct a set of positive solutions of stationary problem (SPW) (and, therefore, (SP)) in the case where g is of monostable type. In Section.4, we will discuss the case for bistable type of g and look for all nontrivial and nonnegative solutions of (SPW). In particular, we will prove that (SPW) has a continuum nonnegative solutions with compact supports in $(-L, L)$ provided that L is larger than a critical value.

2 Phase plane analysis

In this section we will look for nontrivial and nonnegative solutions of (SP). From the first and second equations of (SP)

$$u = \frac{h}{f}v \quad \text{and} \quad w = \frac{h}{\beta\delta f}\{\gamma(v) + f\}v =: G(v) \geq 0. \tag{2.1}$$

Here $\gamma(v)$ is given by (1.2); so that $G(v)$ is a cubic function. Furthermore, it is possible to show that $G(v)$ is strictly increasing if

$$ab^2 \leq 3(c + f), \tag{2.2}$$

which assures that G has its inverse function $v = G^{-1}(w)$ for $w \geq 0$.

In what follows, we always assume (2.2). By virtue of (2.1) and $v = G^{-1}(w)$, (SP) is reduced to (SPW) with

$$g(w) = \alpha G^{-1}(w) - \beta w. \tag{2.3}$$

Noting $g(0) = 0$ we have the following result.

Lemma 2.1. *Define $g(w)$ by (2.3). Then it has the following properties.*

(i) *If $\alpha\delta < h(c+f)/f$, then $g(w) < 0$ for $w > 0$.*

(ii) *If $\alpha\delta = h(c+f)/f$, then $g(w) < 0$ for $w \in (0, G(b)) \cup (G(b), \infty)$ and $g(G(b)) = 0$.*

(iii) *If $h(c+f)/f < \alpha\delta < h(ab^2+c+f)/f$, then there exist two positive numbers w^- and w^+ ($w^- \leq w^+$) such that $g(w) > 0$ for $w \in (w^-, w^+)$, $g(w^-) = g(w^+) = 0$ and $g(w) < 0$ for $w \in (0, w^-) \cup (w^+, \infty)$.*

(iv) *If $\alpha\delta \geq h(ab^2+c+f)/f$, then there exists a positive number w^+ such that $g(w) > 0$ for $w \in (0, w^+)$, $g(w^+) = 0$ and $g(w) < 0$ for $w \in (w^+, \infty)$.*

Remark 2.1. By the implicit function theorem, it is possible to show

$$g'(0) = \alpha \frac{d}{dw} G^{-1}(w)\bigg|_{w=0} - \beta = \frac{\alpha}{G'(0)} - \beta = \beta \left\{ \frac{\alpha\delta f}{h(ab^2+c+f)} - 1 \right\}.$$

Therefore, we see $g'(0) < 0$ in case (iii) and $g'(0) \geq 0$ in case (iv). We say that g is of *bistabe type* (resp. *monostable type*) if it satisfies assumption (iii) (resp. (iv)) of Lemma 2.1.

Proof. We first study zero points of $g(w)$ for $w > 0$. If $g(w^*) = 0$ with $w^* > 0$, then it follows from (2.3) that $v^* := G^{-1}(w^*)$ satisfies

$$\alpha v^* = \beta w^* = \beta G(v^*) = \frac{h}{\delta f}\{\gamma(v^*) + f\} v^*.$$

By $v^* > 0$, we obtain from (1.2)

$$a(v^* - b)^2 + c + f = \frac{\alpha\delta f}{h} \quad \text{with} \quad v^* > 0. \tag{2.4}$$

One can prove that (2.4) has no solution in case of (i), (2.4) has a unique solution $v^* = b$ in case of (ii), (2.4) has two solutions

$$v^{\pm} := b \pm \sqrt{\frac{\alpha\delta f - h(c+f)}{ah}} > 0$$

in case of (iii) and (2.4) has a unique solution $v^+ > 0$ in case of (iv). Hence setting $w^{\pm} := G(v^{\pm}) > 0$ we see $g(w^{\pm}) = 0$ in case of (iii) and $g(w^+) = 0$ with $w^+ > 0$ in case of (iv).

In order to complete the proof, we note that $g(w) > 0$ (resp. $g(w) < 0$) for $w > 0$ is equivalent to $\alpha v - \beta G(v) > 0$ (resp. $\alpha v - \beta G(v) < 0$) for $v > 0$. Here it should be noted that $\alpha v - \beta G(v)$ vanishes at $v = 0$ and that

$$\alpha - \beta G'(0) = \alpha - \frac{h(ab^2+c+f)}{\delta f}. \tag{2.5}$$

Thus $g(w) > 0$ for small $w > 0$ in case of (iv) and $g(w) < 0$ for small $w > 0$ in cases of (i),(ii) and (iii). Since $\lim_{w\to\infty} g(w) = -\infty$, the conclusion follows by taking account of (2.5) and the number of zero points of $g(w)$. □

We will construct nonnegative solutions of (SPW). For this purpose, it is convenient to study the following initial value problem:

$$\begin{cases} (w^m)_{xx} + g(w) = 0, \\ w(0) = \mu, \quad (w^m)_x(0) = 0 \qquad \text{for} \quad \mu > 0. \end{cases} \tag{2.6}$$

Let $w(x;\mu)$ be the solution of (2.6), which is sometimes denoted by $w(x)$. Define $X(\mu)$ by

$$X(\mu) = \inf\{x > 0 \mid w(x;\mu) = 0\}. \tag{2.7}$$

If one can find $\mu^* > 0$ satisfying $X(\mu^*) = L$, then $w(x;\mu^*)$ becomes a positive solution of (SPW) (note $w(-x;\mu) = w(x;\mu)$).

We will use the phase plane method to construct all nontrivial and nonnegative solutions of (SP). This method is standard and we will follow the arguments of Aronson, Crandall and Peletier [1] who studied a degenerate nonlinear diffusion equation similar to ours. Define a new function p by $p(x) = (w^m(x))_x$. Then it is seen that (w,p) satisfies

$$\begin{cases} mw^{m-1}w_x = p, \\ p_x = -g(w), \\ w(0) = \mu, p(0) = 0 \quad \text{for} \quad \mu \geq 0. \end{cases} \tag{2.8}$$

Multiply the first equation of (2.8) by $g(w)$ and multiply the second equation of (2.8) by p. Adding these resulting expressions we get

$$pp_x + mw^{m-1}w_x g(w) = \frac{d}{dx}\left\{\frac{1}{2}p^2 + mF(w)\right\} = 0,$$

where $F(w) = \displaystyle\int_0^w s^{m-1}g(s)ds$. Therefore,

$$\frac{1}{2}p(x;\mu)^2 + mF(w(x;\mu)) = mF(\mu). \tag{2.9}$$

In wp-plane, the solution $w(x;\mu)$ of (2.6) corresponds to an orbit $O(\mu)$ starting from $(w,p) = (\mu, 0)$. In order to look for a solution of (SPW), such an orbit $O(\mu)$ must hit the p-axis. Therefore, if $w(x^*,\mu) = 0$ for some $x^* > 0$, then it follows from (2.9) that

$$mF(\mu) = \frac{1}{2}p(x^*;\mu)^2 + mF(w(x^*,\mu)) = \frac{1}{2}p(x^*;\mu)^2 > 0. \tag{2.10}$$

Now recall Lemma 2.1. If (i) and (ii) hold, i.e., $\alpha\delta f \leq h(c+h)$, then $g(w) \leq 0$ for all $w \geq 0$. Since $F(w) < 0$ for $w \geq 0$ in this case, it is impossible to find μ satisfying (2.10). Thus one can derive the following result.

Proposition 2.1. *Assume $\alpha\delta \leq (c+f)/f$. Then* (SPW) *has no nontrivial and nonnegative solution and, therefore,* (SP) *also has no nontrivial and nonnegative solution.*

Hereafter we assume $\alpha\delta > (c+f)/f$ by virtue of Proposition 2.1. In particular, we will study (SPW) by dividing g into two types:

(A) Monostable type: $\alpha\delta > h(ab^2+c+f)/f$,
and
(B) **Bistable type**: $h(c+f)/f < \alpha\delta < h(ab^2+c+f)/f$ and $F(w^+) > 0$.

In case (B), take $\mu_0 \in (w^-, w^+)$ satisfying $F(\mu_0) = 0$. Then it can be seen from the phase plane analysis for (2.9) and the definition of $X(\mu)$ that, for each $\mu \in (\mu_0, w^+)$ in case (B) (resp. $\mu \in (0, w^+)$ in case (A)),

$$w(x;\mu) > 0 \quad \text{for} \quad x \in [0, X(\mu)) \ \text{ and } \ w(X(\mu);\mu) = 0.$$

Here note by (2.9) that

$$p(x;\mu) = (w^m(x;\mu))_x = -\sqrt{2m(F(\mu) - F(w(x;\mu)))} \quad \text{for} \ \ 0 \le x \le X(\mu),$$

which gives

$$\frac{w^{m-1}(x;\mu)}{\sqrt{F(\mu) - F(w(x;\mu))}}\frac{dw}{dx} = -\sqrt{\frac{2}{m}} \quad \text{for} \ \ 0 \le x \le X(\mu). \tag{2.11}$$

Then the integration of (2.11) over $(0, X(\mu))$ leads to

$$\sqrt{\frac{2}{m}}X(\mu) = \int_0^{\mu} \frac{w^{m-1}}{\sqrt{F(\mu) - F(w)}}dw = \mu^m \int_0^1 \frac{\tau^{m-1}}{\sqrt{F(\mu) - F(\mu\tau)}}d\tau. \tag{2.12}$$

Thus we get an integral representation of $X(\mu)$, which will enable us to derive its important properties in subsequent sections.

3 Monostable type

In this section we will show some basic properties of $X(\mu)$ defined by (2.7) (see also (2.12)) under the assumption that g satisfies monostable condition (A).

Lemma 3.1. *If $X(\mu)$ is defined by* (2.7) (*or, equivalently, by* (2.12)), *then it is a continuous function for $0 < \mu < w^+$ such that*

$$\lim_{\mu \to 0} X(\mu) = 0 \quad \textit{and} \quad \lim_{\mu \to w^+} X(\mu) = \infty.$$

Proof. Denote the right-hand side of (2.12) by $\Lambda(\mu)$. Clearly, $\Lambda(\mu)$ is continuous for $\mu \in (0, w^+)$.

We will first discuss asymptotic behavior of $\Lambda(\mu)$ as $\mu \to 0$. By virtue of $g'(0) > 0$, take a positive number c_0 such that $g'(0) > c_0$. Then $g(w) \ge c_0 w$ for sufficiently small $w > 0$ and, for any $0 < \tau < 1$ and any sufficiently small $\mu > 0$,

$$\begin{aligned} F(\mu) - F(\tau\mu) = \int_{\tau\mu}^{\mu} s^{m-1}g(s)ds &\ge \frac{c_0}{m+1}\mu^{m+1}(1-\tau^{m+1}) \\ &= c_1\mu^{m+1}(1-\tau^{m+1}) \end{aligned}$$

with $c_1 = c_0/(m+1)$. Thus we obtain

$$\Lambda(\mu) = \mu^m \int_0^1 \frac{\tau^{m-1}}{\sqrt{F(\mu) - F(\mu\tau)}} d\tau \le \frac{1}{\sqrt{c_1}} \mu^{\frac{m-1}{2}} \int_0^1 \frac{\tau^{m-1}}{\sqrt{1-\tau^{m+1}}} d\tau = c_2 \mu^{\frac{m-1}{2}}$$

with a positive constant c_2. Since $m > 1$, it follows that

$$\lim_{\mu \to 0} \Lambda(\mu) = 0.$$

We will next show the second assertion. Since $F(w^+) > 0, F'(w^+) = 0$ and $F''(w^+) = (w^+)^{m-1} g'(w^+) < 0$, $F(w)$ is approximated as

$$F(w) \ge -c_3 (w - w^+)^2 + F(w^+)$$

near $w = w^+$ with a positive constant $c_3 > 0$. Then for sufficiently small $\rho > 0$,

$$\begin{aligned} \lim_{\mu \to w^+} \int_{\mu-\rho}^{\mu} \frac{w^{m-1}}{\sqrt{F(\mu) - F(w)}} dw &= \int_{w^+-\rho}^{w^+} \frac{w^{m-1}}{\sqrt{F(w^+) - F(w)}} dw \\ &\ge \frac{1}{\sqrt{c_3}} \int_{w^+-\rho}^{w^+} \frac{w^{m-1}}{w^+ - w} dw \\ &= \infty. \end{aligned}$$

Therefore, one can conclude $\Lambda(\mu) \to \infty$ as $\mu \to w^+$. □

Remark 3.1. For the linear diffusion case $m = 1$, one can see

$$\lim_{\mu \to 0} X(\mu) = X_0 \quad \text{and} \quad \lim_{\mu \to w^+} X(\mu) = \infty,$$

where X_0 is a positive constant. This result is slightly different from Lemma 3.1 for $m > 1$.

Remark 3.2. In Lemma 3.1 we have assumed (A), that is, $\alpha\delta > h(ab^2 + c + f)/f$ which assures $g'(0) > 0$. In case $\alpha\delta = h(ab^2 + c + f)/f$, the asymptotic behavior of $X(\mu)$ near $\mu = 0$ is slightly different from the conclusion of Lemma 3.1. We can show

$$\lim_{\mu \to 0} X(\mu) = \begin{cases} 0 & \text{if } m > 2, \\ X_0 \quad \text{with a positive constant } X_0 & \text{if } m = 2, \\ \infty & \text{if } 1 < m < 2, \end{cases}$$

by some modification of the above proof.

Denote by $E_w(L)$ the set of positive solutions of (SPW). Clearly, the preceding consideration shows that

$$E_w(L) = \{ w(x;\mu) | X(\mu) = L \}, \tag{3.1}$$

where $w(x;\mu)$ is the solution of (2.6). Then it follows from Lemma 3.1 that, for each $L > 0$, there exists at least one positive number $\mu_L \in (0, w^+)$ such that $X(\mu_L) = L$. Moreover, it is possible to prove that $X(\mu)$ is analytic with respect to $\mu \in (0, w^+)$. This fact implies that the number of μ satisfying $X(\mu) = L$ is finite; so that $E_w(L)$ consists of a finite number of isolated elements. Hence we can obtain the following result.

Proposition 3.1. *Assume that g satisfies monostable condition* (A). *Then for each $L > 0$, $E_w(L)$ is given by* (3.1) *and consists of a finite number of isolated elements.*

Remark 3.3. If the equation for w has a linear diffusion term, i.e. $m = 1$, then we see from Remark 3.1 that $\inf_{\mu\in(0,w^+)} X(\mu) \geq X_0^*$ with some $X_0^* > 0$. This fact, in particular, implies that (SPW) has no nontrivial and nonnegative solution for $0 < L < X_0^*$. This is a big difference from the nonlinear diffusion case $m > 1$. For the stationary problem with linear diffusion under Dirichlet condition, see the works of Shirai, Chuan and Yagi [13, 14, 15].

By the phase plane method, any nontrivial and nonnegative solution of (SPW) must be a positive solution of (SPW). Proposition 3.1 implies that the number of nontrivial and nonnegative solutions of (SPW) is the same as the number of intersection points of $\nu = X(\mu)$ and $\nu = L$ for $\mu \in (0, w^+)$ in $\mu\nu$-plane.

We are ready to give complete information on the structure of solutions of (SP) in monostable case. Recall that any solution (u^*, v^*, w^*) of (SP) can be given by

$$u^* = \frac{h}{f} v^* \quad \text{and} \quad v^* = G^{-1}(w^*) \tag{3.2}$$

where $w^* \geq 0$ ($\not\equiv 0$) is a solution of (SPW) (see (2.1)). Define the following set:

$$E(L) := \{(u^*, v^*, w^*) |\ u^* \text{ and } v^* \text{ are given by (3.2) and } w^* \in E_w(L)\}. \tag{3.3}$$

Then we are ready to give information on nontrivial and nonnegative solutions of (SP) with use of Proposition 3.1.

Theorem 3.1. *Assume that g satisfies monostable condition* (A) *and let $\mathcal{E}(L)$ denote the set of nontrivial and nonnegative solutions of* (SP) *for $L > 0$. Then $\mathcal{E}(L)$ is identical with $E(L) \neq \emptyset$ and consists of a finite number of isolated elements, where $E(L)$ is a set defined by* (3.3).

Remark 3.4. Let $m = 1$ and let X_0^* be a positive constant in Remark 3.3. We see that, if $L \geq X_0^*$, then $\mathcal{E}(L) \neq \emptyset$; whereas, if $0 < L < X_0^*$, then $\mathcal{E}(L) = \emptyset$. So nonlinear degenerate diffusion yields a different result on the structure of positive solutions of (SP).

4 Bistable type

In this section we will study $X(\mu)$ under the assumption that g satisfies bistable condition (B). Recall that $X(\mu)$ is defined for $\mu \in (\mu_0, w^+)$ in this case.

Lemma 4.1. *If $X(\mu)$ is defined by* (2.7) (*or, equivalently, by* (2.12)), *then it satisfies the following properties.*

(i) $\lim_{\mu\to\mu_0} X(\mu) = L_1$ *with a positive number L_1 and* $\lim_{\mu\to w^+} X(\mu) = \infty$.

(ii) $\lim_{\mu\to\mu_0} X'(\mu) = -\infty$.

Proof. We first study $\lim_{\mu\to\mu_0} X(\mu)$. Observe that $F(0) = F(\mu_0) = 0$ and $F(w) < 0$ for $w \in (0, \mu_0)$. By (2.12)

$$\sqrt{\frac{2}{m}} \lim_{\mu\to\mu_0} X(\mu) = \lim_{\mu\to\mu_0} \Lambda(\mu) = \int_0^{\mu_0} \frac{w^{m-1}}{\sqrt{-F(w)}} dw.$$

Hence for sufficiently small $w > 0$

$$F(w) = \int_0^w s^{m-1} g(s) ds = \frac{g'(0)}{m+1} w^{m+1} + o(w^{m+1}) \tag{4.1}$$

with $g'(0) < 0$ and

$$F(w) = F'(\mu_0)(w - \mu_0) + o(w - \mu_0) \tag{4.2}$$

with $F'(\mu_0) > 0$ near $w = \mu_0$. Therefore, in view of $m > 1$, it is possible to see that $w^{m-1}/\sqrt{-F(w)}$ is integrable over $(0, \mu_0)$. Thus

$$\lim_{\mu\to\mu_0} \Lambda(\mu) = \int_0^{\mu_0} \frac{w^{m-1}}{\sqrt{-F(w)}} dw \in (0, \infty).$$

This fact assures the first assertion of (i).

The proof of $\lim_{\mu\to w^+} X(\mu) = \infty$ is the same as that of Lemma 3.1, so we omit it.

In order to study $\lim_{\mu\to\mu_0} X'(\mu)$, we will employ the arguments in the work of Aronson, Crandall and Peletier [1]. Note by (2.12)

$$\Lambda(\mu) = \mu^m \int_0^1 \frac{\tau^{m-1}}{\sqrt{F(\mu) - F(\mu\tau)}} d\tau.$$

Therefore,

$$\begin{aligned}\Lambda'(\mu) &= \mu^{m-1} \int_0^1 \frac{\tau^{m-1}}{\sqrt{F(\mu) - F(\tau\mu)}} \left\{ m - \frac{\mu(F'(\mu) - \tau F'(\tau\mu))}{2(F(\mu) - F(\tau\mu))} \right\} d\tau \\ &= \frac{1}{2\mu} \int_0^\mu \frac{1}{\sqrt{F(\mu) - F(w)}} \left\{ 2m - \frac{\mu^m g(\mu) - w^m g(w)}{F(\mu) - F(w)} \right\} w^{m-1} dw.\end{aligned} \tag{4.3}$$

If we set

$$\theta(w) = 2mF(w) - w^m g(w),$$

then it is seen from (4.3) that

$$\Lambda'(\mu) = \frac{1}{2\mu} \int_0^\mu \frac{\theta(\mu) - \theta(w)}{(F(\mu) - F(w))^{\frac{3}{2}}} w^{m-1} dw.$$

Then

$$\lim_{\mu\to\mu_0} \Lambda'(\mu) = \frac{1}{2\mu_0} \int_0^{\mu_0} \frac{\theta(\mu_0) - \theta(w)}{(-F(w))^{\frac{3}{2}}} w^{m-1} dw.$$

Recall that $F(w)$ vanishes only at $w = 0$ and $w = \mu_0$. It follows from (4.2) that, if w is close to μ_0 with $w < \mu_0$, then

$$\left|\frac{\theta(\mu_0) - \theta(w)}{(-F(w)^{\frac{3}{2}})}\right| \leq c^* \frac{\mu_0 - w}{(\mu_0 - w)^{\frac{3}{2}}} = \frac{c^*}{(\mu_0 - w)^{\frac{1}{2}}}$$

with a positive constant c^*. Therefore, for small $\rho > 0$,

$$\left|\int_{\mu_0-\rho}^{\mu_0} \frac{\theta(\mu_0) - \theta(w)}{(-F(w))^{\frac{3}{2}}} w^{m-1} dw\right| \leq 2c^* \rho^{\frac{1}{2}} \mu_0^{m-1}. \tag{4.4}$$

On the other hand, it follows from (4.1) that, if $w > 0$ is small, then

$$\theta(\mu_0) - \theta(w) = -\mu_0^m g(\mu_0) - \frac{m-1}{m+1} g'(0) w^{m+1} + o(w^{m+1}) \leq -\frac{1}{2}\mu_0^m g(\mu_0)$$

with $g(\mu_0) > 0$ and

$$-F(w) \geq c_0^* w^{m+1}$$

with a positive constant c_0^*. Therefore, for small $\rho > 0$,

$$\int_0^\rho \frac{\theta(\mu_0) - \theta(w)}{(-F(w))^{\frac{3}{2}}} w^{m-1} dw \leq \frac{-\mu_0^m g(\mu_0)}{2c_0^{*\frac{3}{2}}} \int_0^\rho w^{-\frac{m+5}{2}} dw = -\infty. \tag{4.5}$$

Thus we get the conclusion of (ii) by virtue of (4.4) and (4.5). □

Remark 4.1. If $m = 1$, then one can prove

$$\lim_{\mu \to w^+} X(\mu) = \infty, \quad \lim_{\mu \to \mu_0} X(\mu) = \infty, \lim_{\mu \to \mu_0} X'(\mu) = -\infty.$$

So $\min_{\mu \in (\mu_0, w^+)} X(\mu) = X_0^* > 0$.

By Lemma 4.1

$$0 < \min_{\mu \in (\mu_0, w^+)} X(\mu) =: L_0 < L_1 = X(\mu_0)$$

and $\lim_{\mu \to w^+} X(\mu) = \infty$. Recalling (3.1) we see

$$E_w(L) = \emptyset \quad \text{if } \ 0 < L < L_0 \qquad \text{and} \qquad E_w(L) \neq \emptyset \qquad \text{if } \ L_0 \leq L.$$

In particular, if $L_0 < L < L_1$, there exist at least two numbers $\mu_1, \mu_2 \in (\mu_0, w^+)$ with $\mu_1 < \mu_2$ such that $X(\mu_i) = L$ $(i = 1, 2)$. Here one can also prove the order relation between $w(x; \mu_1)$ and $w(x; \mu_2)$ by the phase plane analysis in the following manner:

Lemma 4.2. *Let μ_1 and μ_2 be two positive numbers satisfying $X(\mu_i) = L$ $(i = 1, 2)$ with $\mu_1 < \mu_2$. Let $w(x; \mu_i)$ $(i = 1, 2)$ be the solution of (2.6) such that $w(0; \mu_i) = \mu_i$ $(i = 1, 2)$. Then it holds that*

$$0 < w(x; \mu_1) < w(x; \mu_2) \quad \textit{for} \quad x \in (-L, L).$$

Making use of the same arguments as in Section 3 we can prove that both $\{\mu \in (\mu_0, w^+)|\ X(\mu) = L\}$ and $E_w(L)$ are finite sets. Define

$$\mu_{\max} = \max\{\mu \in (\mu_0, w^+)|\ X(\mu) = L\} \quad \text{and} \quad \mu_{\min} = \min\{\ \mu \in (\mu_0, w^+)|X(\mu) = L\}$$

and set

$$w_{\max}(x) = w(x; \mu_{\max}) \quad \text{and} \quad w_{\min}(x) = w(x; \mu_{\min}).$$

Then it follows from Lemma 4.2 that $w_{\max}(x) > w_{\min}(x)$ for $x \in (-L, L)$ if $L_0 < L < L_1$. On the other hand, if $L > L_1$, then $w_{\max}(x)$ and $w_{\min}(x)$ may coincide each other. Summarizing these results we have the following result on the set $E_w(L)$ of positive solutions for (SPW).

Proposition 4.1. *Define $E_w(L)$ by (3.1) and put $L_0 = \min_{\mu \in (\mu_0, w^+)} X(\mu)$. Then $E_w(L)$ consists of a finite number of isolated elements and it satisfies the following properties.*

(i) *If $0 < L < L_0$, then $E_w(L) = \emptyset$.*

(ii) *If $L \geq L_0$, then $E_w \neq \emptyset$ and, furthermore, if $L_0 < L < L_1$, then there exist a maximal $w_{\max} \in E_w(L)$ and a minimal $w_{\min} \in E_w(L)$ such that $w_{\max} > w_{\min}$ in $(-L, L)$. Here L_1 is a positive constant defined in Lemma 4.1.*

Here we should take account of some important properties of $w(\cdot; \mu_0)$, which yields nontrivial and nonnegative solutions of (SPW) differently from positive solutions in $E_w(L)$. The corresponding orbit $O(\mu_0)$ in $wp-$plane connects a point $(w, p) = (\mu_0, 0)$ with the origin $(w, p) = (0, 0)$. This fact allows us to see

$$w(\pm L_1; \mu_0) = 0 \quad \text{and} \quad (w^m)_x(\pm L_1, \mu_0) = 0$$

with $X(\mu_0) = L_1$. Define $\tilde{w}(x)$ by

$$\tilde{w}(x) = \begin{cases} w(x; \mu_0) & \text{for } x \in [-L_1, L_1], \\ 0 & \text{for } x \in \mathbb{R} \backslash [-L_1, L_1]. \end{cases}$$

Clearly $\tilde{w}$ satisfies

$$(\tilde{w}^m)_{xx} + g(\tilde{w}) = 0 \quad \text{in} \quad \mathbb{R}$$

and $\tilde{w}(\pm L) = 0$ if $L > L_1$. This fact implies that, if $L > L_1$, then $\tilde{w}$ becomes a nontrivial and nonnegative solution of (SPW). Furthermore, its translation, $\tilde{w}(x - \xi)$ with $|\xi| < L - L_1$, also satisfies (SPW). We say that $\tilde{w}$ has a *positive core* $(-L_1, L_1)$. By virtue of the degeneracy of the nonlinear diffusion, (SPW) has a continuum of nonnegative solutions, $\{\tilde{w}(x - \xi)|\ |\xi| < L - L_1\}$, where $\tilde{w}$ is a nonnegative solution with positive core $(-L_1, L_1)$. Such a class of solutions was discussed by Aronson, Crandall and Peletier [1] for reaction diffusion equations with nonlinear degenerate diffusion.

Analogous results also hold for positive solutions with multiple number of positive cores. For example, if we assume $L > 2L_1$, we can choose $\xi_1, \xi_2 \in (-L, L)$ with $\xi_1 < \xi_2$ satisfying

$$-L < \xi_1 - L_1 < \xi_1 + L_1 < \xi_2 - L_1 < \xi_2 + L_1 < L.$$

Define

$$\tilde{w}(x; \xi_1, \xi_2) = \begin{cases} w(x - \xi_1; \mu_0) & \text{for } x \in [\xi_1 - L_1, \xi_1 + L_1], \\ w(x - \xi_2; \mu_0) & \text{for } x \in [\xi_2 - L_1, \xi_2 + L_1], \\ 0 & \text{otherwise,} \end{cases}$$

then it becomes a nonnegative solution of (SPW) with two positive cores. See Figure 1 for a solution with two positive cores.

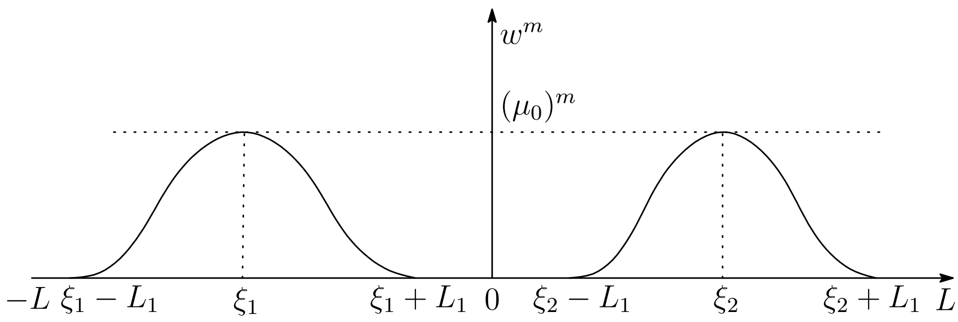

Figure 1: Graph of $w^m(x;\xi_1,\xi_2)$ with two positive cores

More generally, we can show the following result (see [1]).

Proposition 4.2. *Let n be a positive integer such that $L > nX(\mu_0) = nL_1$ and let $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in (-L, L)^n$ with $\xi_1 < \xi_2 < \cdots < \xi_n$ satisfy*

$$-L \leq \xi_1 - L_1 < \xi_1 + L_1 < \xi_2 - L_1 < \xi_2 + L_1 < \cdots < \xi_n - L_1 < \xi_n + L_n \leq L. \tag{4.6}$$

If $\tilde{w}(x;\xi)$ is defined by

$$\tilde{w}(x;\xi) = \begin{cases} w(x-\xi_i;\mu_0) & \text{for } x \in (\xi_i - L_1, \xi_i + L_1),\ i = 1,2,\cdots,n, \\ 0 & \text{for } x \in \mathbb{R} - \bigcup_{i=1}^{n} (\xi_i - L_1, \xi_i + L_1), \end{cases}$$

then it is a nonnegative solution of (SPW).

A function $\tilde{w}(x;\xi)$ defined in Proposition 4.2 is called a solution of (SPW) with n positive cores. Denote by $P_{w,n}(L)$ the collection of functions of the form $\tilde{w}(x;\xi)$ with $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ satisfying (4.6). In other words, this is a continuum of solutions of (SPW) with n positive cores.

Proposition 4.3. *Let $\mathcal{E}_w(L)$ be the set of nontrivial and nonnegative solutions of* (SPW)*. For $L > L_1$, let N be the largest integer satisfying $NL_1 = NX(\mu_0) < L$. Then it holds that*

$$\mathcal{E}_w(L) = E_w(L) \cup P_w(L) \qquad \text{with} \quad P_w(L) = \bigcup_{j=1}^{N} P_{w,j}(L),$$

where $E_w(L)$ is a finite set defined by (3.1).

Proof. Owing to the preceding arguments, nontrivial and nonnegative solutions of (SPW) are given by positive solutions in $E_w(L)$ and nonnegative solutions with positive cores (if they exist). When N is the largest integer satisfying $NL_1 < L$, it is easy to see that (SPW) has solutions with j positive cores for $j = 1, 2, \cdots, N$. Then the conclusion follows from Propositions 4.1 and 4.2. □

Proposition 4.3 implies that $\mathcal{E}_w(L)$ consists of a finite number of positive solutions and a continuum of nonnegative solutions with a finite number of positive cores if $L > L_1$.

We are ready to give precise information on the set $\mathcal{E}(L)$ of nontrivial and nonnegative solutions of (SP). Recall that any solution (u^*, v^*, w^*) of (SP) is given by (3.2) and that $E(L)$ is a set of positive solutions of (SP) defined by (3.3).

Theorem 4.1. *Assume that g satisfies bistable condition* (B). *Define* $L_0 = \min_{\mu \in (\mu_0, w^+)} X(\mu)$. *Then the following properties for $\mathcal{E}(L)$ hold true.*

(i) *If $0 < L < L_0$, then $\mathcal{E}(L) = \emptyset$.*

(ii) *If $L_0 \leq L \leq L_1$, then $\mathcal{E}(L) = E(L) \neq \emptyset$. In particular, if $L_0 < L < L_1$, then there exist a maximal element $(u_{\max}, v_{\max}, w_{\max}) \in E(L)$ and a minimal element $(u_{\min}, v_{\min}, w_{\min}) \in E(L)$ such taht*

$$u_{\max}(x) > u_{\min}(x) > 0, \quad v_{\max}(x) > v_{\min}(x) > 0, \quad w_{\max}(x) > w_{\min}(x) > 0$$

for $x \in (-L, L)$.

(iii) *If $L_1 < L$, then $\mathcal{E}(L) = E(L) \cup P(L)$, where $E(L)$ is defined by* (3.3) *and*

$$P(L) = \{(u^*, v^*, w^*)|\ u^* \text{ and } v^* \text{ are defined by (3.2) and } w^* \in P_w(L)\}.$$

Here $P_w(L)$ is a set defined in Proposition 4.3.

Proof. When L satisfies $L \leq L_1$, (SPW) has no solutions with positive cores: so that $\mathcal{E}(L) = E(L)$. By virtue of (3.3), Proposition 4.1 allows us to see $E(L) = \emptyset$ for $0 < L < L_0$ and $E(L) =\neq \emptyset$ for $L_0 \leq L \leq L_1$. Moreover, since $G^{-1}(w)$ is an increasing function for $w \geq 0$, (ii) of Proposition 4.1 together with (3.2) implies the existence of a maximal element and a minimal element in $E(L)$ provided that $L_0 < L < L_1$.

When L satisfies $L > L_1$, Proposition 4.3 gives the structure of all nontrivial and nonnegative solution of (SPW) and enables us to get assertion (iii) with use of (3.2). □

Remark 4.2. Let $m = 1$. If $0 < L < L_0$ with $L_0 = \min_{\mu \in (\mu_0, w^+)} X(\mu)$, then $\mathcal{E}(L) = E(L) = \emptyset$. If $L \geq L_0$, then $\mathcal{E}(L) = E(L)$ is a non-empty finite set. In particular, if $L > L_0$, then $E(L)$ contains at least two elements.

Remark 4.3. From Theorem 4.1, if L is sufficiently large, then (SP) has not only positive solutions but also a continuum of nonnegative solutions (u^*, v^*, w^*) such that u^*, v^* and w^* have compact supports in $(-L, L)$. From the view-point of ecology, such a solution with compact support will correspond to a patchy pattern in forests and is very interesting.

Remark 4.4. Although the stability properties of stationary solutions are important problems, they are still open. We will discuss nonstationary problem (1.1) elsewhere.

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