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CONTINUOUS DEPENDENCE OF A SOLUTION OF A FREE BOUNDARY PROBLEM DESCRIBING ADSORPTION PHENOMENON FOR A GIVEN DATA

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Abstract. In this paper we consider a one dimensional free boundary problem as a mathematical model describing adsorption phenomenon in one hole of a porous media. This model is proposed by Sato-Aiki-Murase-Shirakawa [7, 8] and consists of a partial differential equation for the relative humidity in the hole and an ordinary differential equation of the front of water region which represents the growth rate for water region. For this model, Sato-Aiki-Murase-Shirakawa [8] proved the existence and uniqueness of a time local solution, and Aiki-Murase[5] showed the existence of a solution globally in time and the convergence to a solution of a steady state problem as a large time behavior of solutions. In this paper, we consider this mathematical model in each hole for each position x of the porous media with respect to $\Omega \subset \mathbb{R}^3$, and prove the continuous dependance of the solution of this problem with respect to $x \in \Omega$.

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1 Introduction

In this paper, we consider an adsorption phenomenon in a porous material. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain occupied by the material and $\omega(x)$ be a hole in the material at $x \in \Omega$. Here, for each $x \in \Omega$ we denote the degree of saturation and the distribution of the relative humidity in $\omega(x)$ by $s(x)$ and $u(x)$, respectively.

More precisely, $s(x)$ is a function on $[0, T]$ for $T > 0$ and $x \in \Omega$ so that $s(x) = s(x)(t)$ for $t \in [0, T]$ and we require $s(x) \in C([0, T])$. Also, we put

$$Q_{s(x)}(T) := \{(t, z) : 0 < t < T,\ s(x)(t) < z < L\}.$$

Moreover, $u(x) = u(x)(t, z)$ is a function on $Q_{s(x)}(T)$. Sometimes in this paper, we omit the parameter x for simplicity as follows : $u = u(x) = u(t, z) = u(x)(t, z)$ and $s = s(x) = s(t) = s(x)(t)$. Now, we suppose that $s(x)$ and $u(x)$ satisfies the following free boundary problem for each $x \in \Omega$:

$$\rho_v u_t - k u_{zz} = 0 \text{ for } t \in [0, T] \text{ and } z \in (s(t), L), \tag{1.1}$$
$$u(t, L) = h(x, t) \text{ on } [0, T], \tag{1.2}$$
$$k u_z(t, s(t)) = (\rho_w - \rho_v u(t, s(t)))\, s_t(t) \text{ in } [0, T], \tag{1.3}$$
$$s_t(t) = a\,(u(t, s(t))) - \varphi(s(t)) \text{ in } [0, T], \tag{1.4}$$
$$s(0) = s_0(x), u(0, z) = u_0(x, z) \text{ for } z \in [s_0(x), L], \tag{1.5}$$

where L, ρ_v, k and a are given positive constants, h is a given function on $\Omega \times (0, T)$, φ is a given continuous function on $\mathbb{R}$ and s_0 and u_0 are also given functions on Ω, and on $Q_{s_0}(\Omega) := \{(x, z) : x \in \Omega, s_0(x) < z < L\}$, respectively.

This model is proposed by Sato-Aiki-Murase-Shirakawa [7, 8] and represents the relationship between the relative humidity and the degree of saturation in the porous material. In this model, they suppose that all holes are given by the closed interval, that is, $\omega(x) = [0, L]$ for $x \in \Omega$. Also, for $x \in \Omega$ and $t > 0$, $[0, s(t)]$ and $[s(t), L]$ indicate the water drop region and the air region in the hole $\omega(x)$, respectively. The equation (1.1) is derived from the mass conservation law of the vapor, and ρ_v represents the density of vapor. The condition (1.2) means that the hole is exposed to air at the end of hole, and (1.3) is given by the mass conservation law of the water on the free boundary, where ρ_w is the density of water. The ordinary differential equation (1.4) represents the growth rate of the water drop region. For the detail derivation of each equation, we refer to [7, 8].

As a mathematical result, Sato-Aiki-Murase-Shirakawa [8] prove the existence and uniqueness of a time local solution (s, u) of the above problem in each hole. Based on this result, Aiki-Murase [5] show the existence and uniqueness of a time global solution, and the convergence of the solution to a solution of a steady state problem as time goes to infinity.

In this paper, we consider that the adsorption phenomenon occurs in each hole for any position $x \in \overline{\Omega}$, where Ω is supposed to have a smooth boundary $\partial\Omega$. For this, we impose the condition (1.2) for each $x \in \overline{\Omega}$, and consider the above model $\{$(1.1)-(1.5)$\}$ denoted by $\mathrm{P}(x) := \mathrm{P}_{h,s_0,u_0}(x)$.

In order to show a purpose of this paper, we introduce the notation $\tilde{u}(t, y) = u(t, (1 - y)s(t) + yL)$ for $y \in [0, 1]$ and consider the following problem in a cylindrical domain instead of P(x):

$$
\begin{aligned}
&\rho_v \tilde{u}_t - \frac{k}{(L-s(t))^2}\tilde{u}_{yy} = \frac{\rho_v(1-y)s_t(t)}{L-s(t)}\tilde{u}_y \text{ on } Q(T) := (0,T)\times(0,1),\\
&\tilde{u}(t,1) = h(x,t) \text{ on } [0,T],\\
&\frac{k}{L-s(t)}\tilde{u}_y(t,0) = (\rho_w - \rho_v\tilde{u}(t,0))s_t(t) \text{ on } [0,T],\\
&s_t(t) = a(\tilde{u}(t,0) - \varphi(s(t))) \text{ on } [0,T],\\
&s(0) = s_0(x) \text{ in } \Omega,\\
&\tilde{u}(0,y) = \bar{u}(0,(1-y)s(0)+yL) \text{ on } [0,1].
\end{aligned}
$$

The purpose of this paper is to show that the solution $(s,\tilde{u}) = (s(x),\tilde{u}(x))$ is a continuous in $\mathbb{R}\times L^2(Q(T))$ with respect to $x\in\overline{\Omega}$. From this continuity, we infer that s and $\tilde{u}$ are measurable on $\Omega\times[0,T]$, and on $\Omega\times[0,T]\times(0,1)$, respectively. By using this property, in near future, we can consider h as the relative humidity in macroscopic domain Ω and consider a two scale problem coupled by a partial differential equation for h in Ω which was studied in [1, 2, 3, 4] and the free boundary problem P(x) in each hole as a mathematical model for moisture transport appearing concrete carbonation process. We refer to [6] for modeling of the two scale problem.

This paper is organized as follows: In section 2, we note the assumptions and the main result concerning about the existence and continuous dependance results. In section 3, we consider an approximation problem of P(x), and prove the existence of a solution thereof. Also, we obtain the uniform estimate and the continuous dependance for a solution of the approximation problem of P(x) with respect to $x\in\Omega$. At the end of section 3, we prove our main theorem by the limiting process for the solution of the approximation problem of P(x).

2 Our main results

In this paper we use the following notations. In general, for a Banach space X we denote by $|\cdot|_X$ its norm. Also, for $D\subset\mathbb{R}^N$ for $N=1$ and $N=3$, $H^1(D)$, $H^1_0(D)$ and $H^2(D)$ are the usual Sobolev spaces.

Throughout this paper, we assume the following conditions:

(A1) Ω is a open bounded connected domain of $\mathbb{R}^3$ which has the boundary $\partial\Omega$ in the class of C^2.

(A2) k and a are positive constants.

(A3) $h\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))$ with $0\le h\le h^* < 1$ on $\Omega\times(0,T)$ where h^* is a positive constant and $h_t\in L^\infty(\Omega\times(0,T))\cap L^2(0,T;H^2(\Omega))$.

(A4) $\varphi\in C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$, $\varphi=0$ on $(-\infty,0]$, $\varphi\le 1$ on $\mathbb{R}$, $\varphi'>0$ on $(0,L]$ and $\varphi(L)-h^*>0$ where h^* is the same constant as in (A3). Also, we denote by $\hat{\varphi}$ the primitive function of φ with $\hat{\varphi}(0)=0$ and put $C_\varphi = |\varphi'|_{L^\infty(\mathbb{R})}$.

(A5) Two positive constants ρ_w and ρ_v satisfy

$$\rho_w > 2\rho_v, \quad \rho_w \ge \rho_v(C_\varphi+2), \quad 9aL\rho_v^2 \le k\rho_w.$$

(A6) $s_0 \in C(\overline{\Omega})$ such that $0 < s_0(x) < L$ for $x \in \overline{\Omega}$, and $u_0 \in C(\overline{Q_{s_0}(\Omega)})$ such that $u_0(x) \in H^1(s_0(x), L)$ and $u_0(x, L) = h(0, x)$ for $x \in \overline{\Omega}$ and $0 \le u_0 \le 1$ on $Q_{s_0}(\Omega)$.

Next, for $x \in \overline{\Omega}$ we define solutions of $\mathrm{P}(x)$ on $[0, T]$ in the following way:

Definition 1.1 Let $x \in \overline{\Omega}$, and s and u be functions on $[0, T]$ and $Q_{s(x)}(T)$, respectively, for $T > 0$. We call that a pair $(s, u) = (s(x), u(x))$ is a solution of $\mathrm{P}(x)$ on $[0, T]$ if the conditions (S1)-(S6) hold:

(S1) $s(x) \in W^{1,\infty}(0, T)$, $0 \le s(x) < L$ a.e. on $[0, T]$, $u(x) \in L^\infty(Q_{s(x)}(T))$, $u_t(x)$, $u_{zz}(x) \in L^2(Q_{s(x)}(T))$ and $|u_z(x)(\cdot)|_{L^2(s(x)(\cdot), L)} \in L^\infty(0, T)$.

(S2) $\rho_v u_t - k u_{zz} = 0$ for a.e. $(t, z) \in Q_{s(x)}(T)$.

(S3) $u(x)(t, L) = h(x, t)$ a.e. on $[0, T]$.

(S4) $k u_z(t, s(t)) = (\rho_w - \rho_v u(t, s(t)))\, s_t(t)$ a.e. in $[0, T]$.

(S5) $s_t(t) = a\,(u(t, s(t))) - \varphi(s(t))$ a.e. in $[0, T]$.

(S6) $s(x)(0) = s_0(x)$, $u(x)(0, z) = u_0(x, z)$ for $z \in [s_0(x), L]$.

In order to handle the problem $\mathrm{P}(x)$, we consider the following problem in a cylindrical domain obtained by changes of variables:

$$\tilde{u}(t, y) := u(t, (1 - y)s(t) + yL) \text{ for } (t, y) \in [0, T] \times [0, 1], \tag{2.1}$$

and

$$\begin{cases} \rho_v \tilde{u}_t - \frac{k}{(L - s(t))^2} \tilde{u}_{yy} = \frac{\rho_v (1-y) s_t}{L - s(t)} \tilde{u}_y \text{ a.e. on } Q(T), \\ \tilde{u}(t, 1) = h(x, t) \text{ a.e. on } [0, T], \\ \frac{k}{L - s(t)} \tilde{u}_y(t, 0) = (\rho_w - \rho_v \tilde{u}(t, 0)) s_t(t) \text{ a.e. on } [0, T], \\ s_t(t) = a(\tilde{u}(t) - \varphi(s(t))) \text{ a.e. on } [0, T], \\ s(0) = s_0(x) \text{ in } \Omega, \\ \tilde{u}(0, y) = u(0, (1 - y)s(0) + yL) =: \tilde{u}_0(y) \text{ on } [0, 1]. \end{cases}$$

For the above problem $\tilde{\mathrm{P}}(x) := \tilde{\mathrm{P}}_{h, s_0, \tilde{u}_0}(x)$, we call that a pair $(s, \tilde{u})$ is a solution of $\tilde{\mathrm{P}}(x)$ on $[0, T]$ if the following (S) and each equation and condition of $\tilde{\mathrm{P}}(x)$ hold:

$$\text{(S)} \begin{cases} s(x) \in W^{1,\infty}(0, T), 0 \le s(x) < L \text{ a.e. on } [0, T], \\ \tilde{u}(x) \in W^{1,2}(0, T; L^2(0, 1)) \cap L^\infty(0, T; H^1(0, 1)) \cap L^\infty(Q(T)) \\ \cap L^2(0, T; H^2(0, 1)). \end{cases}$$

The first result is concerned about the existence of a solution of $\tilde{\mathrm{P}}(x)$ for $x \in \overline{\Omega}$.

Theorem 1. *If (A1) $\sim$ (A6) hold, then for any $T > 0$ and $x \in \overline{\Omega}$ there exists a unique solution $(s, \tilde{u}) = (s(x), \tilde{u}(x))$ of $\tilde{P}(x)$ on $[0, T]$ such that $0 \le \tilde{u}(x) \le 1$ a.e. $Q(T)$ and $0 \le s(x) \le s^* < L$ a.e. $[0, T]$, where s^* is a positive constant which does not depend on x.*

Clearly, (A1) $\sim$ (A6) imply the assumptions in Aiki-Murase [5]. Then, Theorem 1 is a direct consequence of them. In this paper, we give the detail of the proof in order to obtain uniform estimates with respect to $x \in \overline{\Omega}$. From Theorem 1 by putting $u(x)(t, z) = \tilde{u}(x)\left(t, \frac{z - s(x)}{L - s(x)}\right)$ for $(t, z) \in Q_{s(x)}(T)$ we see that $(s, u) = (s(x), u(x))$ is a unique solution of $\mathrm{P}(x)$ for $x \in \overline{\Omega}$. Now, we state our main theorem of this paper.

Theorem 2. *Under the same assumptions as in Theorem 1, let* $(s, \tilde{u})$ *be a solution of* $\tilde{P}(x)$ *on* $[0, T]$ *for* $x \in \overline{\Omega}$ *and* $T > 0$. *Then,* $\tilde{u} \in C(\overline{\Omega}; L^2(Q(T)))$ *and* $s \in C(\overline{\Omega}; C([0, T]))$.

3 Approximation problem

In order to prove Theorems 1 and 2, we consider an approximation problem of $\tilde{\mathrm{P}}(x)$ for $x \in \overline{\Omega}$.

Now, we take $\{h_n\} \subset C^\infty(\overline{\Omega \times (0,T)})$ such that $0 \le h_n \le 1$ on $\Omega \times (0, T)$, $h_n \to h$ in $W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$, $h_{nt} \to h_t$ in $L^2(0, T; H^2(\Omega))$ and $h_n(0) \to h(0)$ in $C(\overline{\Omega})$ as $n \to \infty$, and $\{h_{nt}\}$ is bounded in $L^\infty(\Omega \times (0, T))$. Also, we put $s_{0n} = s_0$, $\bar{u}_0 = \tilde{u}_0 - h(0)$ and $\tilde{u}_{0n} = \min\{\max\{0, \bar{u}_0 + h_n(0)\}, 1\}$. By using h_n, s_{0n}, u_{0n}, we consider the following problem $\tilde{\mathrm{P}}_n(x) := \tilde{\mathrm{P}}_{h_n, s_{0n}, \tilde{u}_{0n}}(x)$ for $x \in \overline{\Omega}$:

$$
\begin{aligned}
&\rho_v \tilde{u}_t - \frac{k}{(L-s)^2}\tilde{u}_{yy} = \frac{\rho_v(1-y)s_t}{L-s}\tilde{u}_y \text{ in } Q(T), && (3.1)\\
&\tilde{u}(t, 1) = h_n(x, t) \text{ for } t \in (0, T),\\
&\frac{k}{L - s(t)}\tilde{u}_y(t, 0) = (\rho_w - \rho_v \tilde{u}(t, 0))s_t(t) \text{ for } t \in (0, T),\\
&s_t(t) = a(\tilde{u}(t, 0) - \varphi(s(t))) \text{ for } t \in (0, T), && (3.2)\\
&s(0) = s_{0n}(x) \text{ in } \Omega,\\
&\tilde{u}(0, y) = \tilde{u}_{0n}(y) \text{ for } y \in [0, 1].
\end{aligned}
$$

Obviously, it holds that $h_n(x) \in W^{1,2}(0, T)$, $0 \le s_{0n} < L$, $\tilde{u}_{0n} \in H^1(0, 1)$, $0 \le \tilde{u}_{0n} \le 1$ and $\tilde{u}_{0n}(x, 1) = h_n(x, 0)$ for each $n \in \mathbb{N}$. Therefore, by Sato-Aiki-Murase-Shirakawa [8] we see that there exists $T' < T$ such that $\tilde{\mathrm{P}}_n(x)$ has a solution $(s_n, \tilde{u}_n) = (s_n(x), \tilde{u}_n(x))$ on $[0, T']$ for $x \in \overline{\Omega}$ and $n \in \mathbb{N}$. This means that by putting $u_n(t, z) = \tilde{u}_n\left(t, \frac{z - s_n(x)}{L - s_n(x)}\right)$ for $t \in [0, T]$, $x \in \overline{\Omega}$ and $z \in [s_n(x), L]$, (s_n, u_n) is a solution of $\mathrm{P}_n(x)$ for $x \in \overline{\Omega}$ and $n \in \mathbb{N}$. Next, by using the results of Aiki-Murase [5], we can extend the solution (s_n, u_n) of $\mathrm{P}_n(x) := \mathrm{P}_{h_n, s_{0n}, u_{0n}}(x)$ on the whole interval $[0, T]$ for $x \in \overline{\Omega}$ and $n \in \mathbb{N}$ such that $0 \le u_n(x) \le 1$ a.e. on $Q_{s(x)}(T)$ and $0 \le s_n(x) \le s^*(x) < L$ a.e. on $[0, T]$, where $s^*(x)$ is a positive constant depending on x and n. First, we show the uniform estimate of the solution (s_n, u_n) of $\mathrm{P}_n(x)$ for $x \in \overline{\Omega}$.

Lemma 1. *Let* $(s_n, u_n) = (s_n(x), u_n(x))$ *be a solution of* $P(x)$ *for* $x \in \overline{\Omega}$ *and each* $n \in \mathbb{N}$. *Then, there exist* $M_1 > 0$ *and* $M_2 > 0$ *independent of* n *such that*

$$
\int_{s_n(t_1)}^{L} |u_n(t_1, z)|^2 dz \le M_1, \quad \int_0^{t_1}\int_{s_n(t)}^{L} |u_{nz}(t, z)|^2 dz dt \le M_1,
$$

and

$$
\int_0^{t_1}\int_{s_n(t)}^{L} |u_{nt}|^2 dz dt \le M_2, \quad \int_{s_n(t_1)}^{L} |u_{nz}(t_1)|^2 dz \le M_2 \text{ for } t_1 \in [0, T] \text{ and } x \in \overline{\Omega}.
$$

Proof. By multiplying $u_n - h_n$ and $\frac{\rho_w}{a} s_{nt}$ to (3.1) and (3.2) and using the same idea of the proof of Lemma 3.1 in Aiki-Murase [8] we can obtain

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}\int_{s_n(t)}^{L} |u_n(t) - h_n(x,t)|^2 dz + k\int_{s_n(t)}^{L} |u_{nz}(t)|^2 dx + \frac{\rho_w}{a}|s_{nt}(t)|^2 + \rho_w \frac{d}{dt}\hat{\varphi}(s_n(t)) \\
\le& \rho_w(1+h^*)L|h_{nt}(x,t)| + \rho_w s_{nt}(t) h_n(x,t) \\
&+ \left(\frac{\rho_v}{2}(1+h^*) + \rho_v h^*\right) |u_n(t, s_n(t)) - h_n(x,t)||s_{nt}(t)| \text{ for a.e. } t \in [0,T],
\end{aligned} \tag{3.3}$$

where $\hat{\varphi}$ is the primitive function of φ and h^* is the same positive constant as in (A3). Since $|u_n(t, s_n(t)) - h_n(x,t)| \le \sqrt{L}\left(\int_{s_n(t)}^{L} |u_{nz}(t)|^2 dz\right)^{1/2}$ for $t \in [0,T]$ we have

$$\begin{aligned}
&\left(\frac{\rho_v}{2}(1+h^*) + \rho_v h^*\right) |u_n(t, s_n(t)) - h_n(x,t)||s_{nt}(t)| \\
\le& \left(\frac{\rho_v}{2}(1+h^*) + \rho_v h^*\right) \sqrt{L}\left(\int_{s_n(t)}^{L} |u_{nz}(t)|^2 dz\right)^{1/2} |s_{nt}(t)| \\
\le& \frac{k}{2}\int_{s_n(t)}^{L} |u_{nz}(t)|^2 dz + \frac{L\rho_v^2(1+2h^*)^2}{2k}|s_{nt}(t)|^2 \text{ for } x \in \overline{\Omega} \text{ and a.e. } t \in [0,T].
\end{aligned} \tag{3.4}$$

By applying (3.4) to (3.3) and using (A5) and $0 \le h_n \le 1$ on $\overline{(0,1) \times \Omega}$, we have

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}\int_{s_n(t)}^{L} |u_n(t) - h_n(x,t)|^2 dz + \frac{k}{2}\int_{s_n(t)}^{L} |u_{nz}(t)|^2 dx + \rho_w \frac{d}{dt}\hat{\varphi}(s_n(t)) \\
\le& \rho_w(1+h^*)L|h_{nt}(x,t)| + \frac{\rho_w a}{2} \text{ for } x \in \overline{\Omega} \text{ and a.e. } t \in [0,T].
\end{aligned} \tag{3.5}$$

On account of the boundedness of $\{h_{nt}\}$ in $L^\infty(\Omega \times (0,T))$, there exists $C > 0$ such that

$$|h_{nt}(x,t)| \le C \text{ for } x \in \overline{\Omega} \text{ and } t \in [0,T]. \tag{3.6}$$

Therefore, by integrating $[0, t_1]$ for $t_1 \in [0,T]$ we have

$$\begin{aligned}
&\frac{\rho_v}{2}\int_{s_n(t_1)}^{L} |u_n(t) - h_n(x,t)|^2 dz + \frac{k}{2}\int_0^{t_1}\int_{s_n(t)}^{L} |u_{nz}(t)|^2 dx \\
\le& \frac{\rho_v}{2}|u_n(0) - h_n(x,0)|^2_{L^2(s_n(0),L)} + \rho_w\hat{\varphi}(s_n(0)) + \rho_w(1+h^*)L\sqrt{T C} + \frac{\rho_w a}{2}T \text{ for } x \in \overline{\Omega}.
\end{aligned} \tag{3.7}$$

Next, by the same calculation as in the proof of Lemma 3.2 in [5] we can obtain that

$$\begin{aligned}
&\frac{\rho_v}{2}\int_0^{t_1}\int_{s_n(t)}^{L} |u_{nt}(t)|^2 dz dt + \frac{k}{2}\int_{s_n(t_1)}^{L} |u_{nz}(t_1)|^2 dz \\
\le& \frac{k}{2}\int_{s_{0n}}^{L} |u_{0nz}|^2 dz + \frac{k}{2}\int_0^{t_1} s_{nt}(t)|u_{nz}(t, s_n(t))|^2 dt \\
&+ \tilde{C}_1\int_0^{t_1} (|s_{nt}(t)|^2 + |h_{nt}(x,t)|^2) dt + \tilde{C}_1 \text{ for } x \in \overline{\Omega} \text{ and } t_1 \in [0,T],
\end{aligned} \tag{3.8}$$

where $\tilde{C}_1$ is a positive constant depending on ρ_a, ρ_g, L, φ and k. By using $|s_{nt}(t)| \le 2a$ and $|u_{nz}(t, s_n(t))| \le (\rho_w + \rho_v)2a$ for a.e. $t \in [0,T]$ and (3.4), it holds that

$$\begin{aligned}&\frac{\rho_v}{2}\int_0^{t_1}\int_{s_n(t)}^L |u_{nt}(t)|^2 dzdt + \frac{k}{2}\int_{s_n(t_1)}^L |u_{nz}(t_1)|^2 dz \\ \le&\frac{k}{2}\int_{s_{0n}}^L |u_{0nz}|^2 dz + 2ka^2(\rho_w+\rho_v)T + \tilde{C}_1 T(4a^2 + C) + \tilde{C}_1 \text{ for } x \in \overline{\Omega} \text{ and } t_1 \in [0,T].\end{aligned} \tag{3.9}$$

Therefore, (3.7) and (3.9) imply that Lemma 1 holds. ◇

Here, we put

$$s^* := L - \left(\frac{\varphi(L) - h^*}{2(\sqrt{M_2} + C_\varphi \sqrt{L})}\right)^2,$$

where M_2 is the same positive constant as in Lemma 1. By the same proof as in Proposition 3.4 of [5] we see that for the solution $(s_n, u_n) = (s_n(x), u_n(x))$ of $\mathrm{P}_n(x)$ for $x \in \overline{\Omega}$ it holds that

$$0 \le s_n(x)(t) < s^* \text{ for } x \in \overline{\Omega} \text{ and } t \in [0,T]. \tag{3.10}$$

Moreover, the following uniform estimate of the solution $(s_n, \tilde{u}_n)$ of $\tilde{\mathrm{P}}_n(x)$ for $x \in \overline{\Omega}$ holds.

Lemma 2. *Let $T > 0$ and $(s_n, \tilde{u}_n)$ be a solution of $\tilde{P}_n(x)$ on $[0,T]$ for $x \in \overline{\Omega}$ and $n \in \mathbb{N}$. Then, $\{\tilde{u}_n(x) : x \in \overline{\Omega}, n \in \mathbb{N}\}$ is bounded in $W^{1,2}(0,T; L^2(0,1)) \cap L^\infty(0,T; H^1(0,1)) \cap L^2(0,T; H^2(0,1))$ and $\{s_n(x) : x \in \overline{\Omega}, n \in \mathbb{N}\}$ is bounded in $W^{1,\infty}(Q(T))$.*

Proof. By using (2.1) and $z = (1-y)s_n(t) + yL$ for $(t,y) \in [0,T] \times [0,1]$ we can calculate that

$$\int_0^{t_1}\int_0^1 |\tilde{u}_n(t,y)|^2 dydt = \int_0^{t_1}\int_{s_n(t)}^L \left(\frac{1}{L - s_n(t)}\right) |u_n(t,z)|^2 dzdt$$

and

$$\begin{aligned}&\int_0^{t_1}\int_0^1 |\tilde{u}_{nt}(t,y)|^2 dydt \\ =&\int_0^{t_1}\int_{s_n(t)}^L \left(\frac{1}{L - s_n(t)}\right) \left|u_{nt}(t,z) + u_{nz}(t,z)(L - s_n(t))\right|^2 dzdt \\ \le&\int_0^{t_1}\int_{s_n(t)}^L \left(\frac{1}{L - s_n(t)}\right) |u_{nt}(t,z)|^2 dzdt + \int_0^{t_1}\int_{s_n(t)}^L 2|u_{nt}(t,z)||u_{nz}(t,z)| dzdt \\ &+ \int_0^{t_1}\int_{s_n(t)}^L (L - s_n(t))|u_{nz}(t,z)|^2 dzdt.\end{aligned}$$

Therefore, by Lemma 1 and (3.10) we have

$$|\tilde{u}_n(x)|^2_{L^2(Q(T))} \le \frac{M_1}{L - s^*} \text{ for } x \in \overline{\Omega}, \tag{3.11}$$

and

$$|\tilde{u}_{nt}(x)|^2_{L^2(Q(T))} \le \left(\frac{1}{L-s^*}+1\right) M_2 + (1+L)TM_2 \text{ for } x \in \overline{\Omega}. \tag{3.12}$$

Similarly to the derivation of (3.11) and (3.12), we have

$$\begin{aligned} &\int_0^1 |\tilde{u}_{ny}(t,y)|^2 dy \\ =&\int_{s_n(t)}^L \left(\frac{1}{L-s_n(x)}\right) |\tilde{u}_{nz}(t,z)(L-s_n(t))|^2 dz \le LM_2 \text{ for } t \in [0,T]. \end{aligned} \tag{3.13}$$

Moreover, by (3.1) we obtain

$$\begin{aligned} &|\tilde{u}_{nyy}(x)|^2_{L^2(0,1)} \\ \le&\frac{1}{k}\left(2a\rho_v L|\tilde{u}_{ny}(x)|_{L^2(0,1)}|\tilde{u}_{nyy}(x)|_{L^2(0,1)} + \rho_v L^2|\tilde{u}_{nt}(x)|_{L^2(0,1)}|\tilde{u}_{nyy}(x)|_{L^2(0,1)}\right) \\ \le&\left(\frac{2a\rho_v L}{k}\right)^2 |\tilde{u}_{ny}(x)|^2_{L^2(0,1)} + \left(\frac{\rho_v L^2}{k}\right)^2 |\tilde{u}_{nt}(x)|^2_{L^2(0,1)} + \frac{1}{2}|\tilde{u}_{nyy}(x)|^2_{L^2(0,1)}. \end{aligned} \tag{3.14}$$

Therefore, by (3.11), (3.12), (3.13) and (3.14), we conclude that $\{\tilde{u}_n(x) : x \in \overline{\Omega}, n \in \mathbb{N}\}$ is bounded in $W^{1,2}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^1(0,1)) \cap L^2(0,T;H^2(0,1))$. Since $0 \le s_n(x) < L$ and $|s_{nt}(x)| \le 2a$ a.e. on $[0,T]$ for $x \in \overline{\Omega}$, it is clear that $\{s_n(x) : x \in \overline{\Omega}, n \in \mathbb{N}\}$ is bounded in $W^{1,\infty}(0,T)$. $\diamond$

Now, we show that the function s_n and $\tilde{u}_n$ are continuous in the following sense for each $n \in \mathbb{N}$.

Lemma 3. *Let $(s_n, \tilde{u}_n)$ be a solution of $\tilde{P}_n(x)$ for $x \in \overline{\Omega}$ and each $n \in \mathbb{N}$. Then, it holds that $s_n(t) \in C(\overline{\Omega}; C([0,T]))$ and $\tilde{u}_n(t) \in C(\overline{\Omega}; C([0,T]; L^2(\Omega)))$.*

Proof. First, for x, $x' \in \overline{\Omega}$, let $(s_n(x), \tilde{u}_n(x))$ and $(s_n(x'), \tilde{u}_n(x'))$ be solutions of $\tilde{\mathrm{P}}_n(x)$ and $\tilde{\mathrm{P}}_n(x')$ for each $n \in \mathbb{N}$. In this proof, we denote $s_n(x)$, $\tilde{u}_n(x)$ and h_n by $s(x)$, $\tilde{u}(x)$ and h, and put $\bar{u}(x) = \tilde{u}(x) - h$. By (3.1), since

$$\begin{aligned} \rho_v(\tilde{u}(x) - \tilde{u}(x'))_t - \left[\frac{k}{(L-s(x))^2}\tilde{u}_{yy}(x) - \frac{k}{(L-s(x'))^2}\tilde{u}_{yy}(x')\right] \\ = \frac{\rho_v(1-y)s_t(x)}{L-s(x)}\tilde{u}_y(x) - \frac{\rho_v(1-y)s_t(x')}{L-s(x')}\tilde{u}_y(x'), \end{aligned}$$

we have

$$\begin{aligned} &\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}(x) - \bar{u}(x')|^2_{L^2(0,1)} + \frac{k}{L^2}|\bar{u}_y(x) - \bar{u}_y(x')|^2_{L^2(0,1)} \\ &\le \rho_v \int_0^1 (h_t(x,t) - h_t(x',t))(\bar{u}(x) - \bar{u}(x'))dy \\ &- \left(\frac{k}{L-s(x)}\tilde{u}_y(x)(t,0) - \frac{k}{L-s(x')}\tilde{u}_y(x')(t,0)\right)(\tilde{u}(x)(t,0) - \tilde{u}(x')(t,0)) \end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 \left(\frac{k}{(L-s(x))^2} - \frac{k}{(L-s(x'))^2}\right)\tilde{u}_y(x')(\tilde{u}_y(x)-\tilde{u}_y(x'))dy \\
&+\int_0^1 \left(\frac{\rho_v(1-y)s_t(x)}{L-s(x)} - \frac{\rho_v(1-y)s_t(x')}{L-s(x')}\right)\tilde{u}_y(x)(\bar{u}(x)-\bar{u}(x'))dy \\
&+\int_0^1 \frac{\rho_v(1-y)s_t(x')}{L-s(x')}(\tilde{u}_y(x)-\tilde{u}_y(x'))(\bar{u}(x)-\bar{u}(x'))dy. \qquad (3.15)
\end{aligned}
$$

Now, we put each term of the right hand side of (3.15) by I_i for $1 \le i \le 5$. For I_2, we have

$$
\begin{aligned}
I_2 &= \Big((\rho_w - \rho_v\tilde{u}(x)(t,0))s_t(x)(t) - (\rho_w - \rho_v\tilde{u}(x')(t,0))s_t(x')(t)\Big) \\
&\quad \times (\tilde{u}(x)(t,0) - \tilde{u}(x')(t,0)) \\
&\le (\rho_w+\rho_v)|s_t(x)(t) - s_t(x')(t)||\tilde{u}(x)(t,0)-\tilde{u}(x')(t,0)| \\
&\quad +2a\rho_v|\tilde{u}(x)(t,0)-\tilde{u}(x')(t,0)|^2 \\
&\le \left(\frac{\rho_w+\rho_v}{2}\right)|s_t(x)(t)-s_t(x')(t)|^2 + \left(\frac{\rho_w+\rho_v}{2}+2a\rho_v\right)|\tilde{u}(x)(t,0)-\tilde{u}(x')(t,0)|^2.
\end{aligned}
$$

Here, it holds that

$$
\begin{aligned}
&|\tilde{u}(x)(t,0)-\tilde{u}(x')(t,0)|^2 \\
&= \int_0^1 \frac{\partial}{\partial y}|\tilde{u}(x)(t,y)-\tilde{u}(x')(t,y)|^2 dy + |h(x,t)-h(x',t)|^2 \\
&\le 2\int_0^1 |\tilde{u}(x)(t,y)-\tilde{u}(x')(t,y)||\tilde{u}_y(x)(t,y)-\tilde{u}_y(t,y)|dy + |h(x,t)-h(x',t)|^2 \\
&\le \frac{\eta}{2}|\tilde{u}_y(x)(t)-\tilde{u}_y(x')(t)|^2_{L^2(0,1)} + \frac{2}{\eta}|\tilde{u}(x)(t)-\tilde{u}(x')(t)|^2_{L^2(0,1)} \qquad (3.16) \\
&\quad + |h(x,t)-h(x',t)|^2,
\end{aligned}
$$

where η is arbitrary positive constant. Therefore, we obtain that

$$
\begin{aligned}
I_2 &\le C_1|s_t(x)(t)-s_t(x')(t)|^2 \\
&\quad + C_2\left(\frac{\eta}{2}|\tilde{u}_y(x)(t)-\tilde{u}_y(x')(t)|^2_{L^2(0,1)} + \frac{2}{\eta}|\tilde{u}(x)(t)-\tilde{u}(x')(t)|^2_{L^2(0,1)}\right) \\
&\quad + C_2|h(x,t)-h(x',t)|^2,
\end{aligned}
$$

where $C_1 = (\rho_w+\rho_v)/2$ and $C_2 = (\rho_w+\rho_v)/2 + 2a\rho_v$. Next, for the term I_3, by using (3.10)

$$
\begin{aligned}
I_3 &\le \frac{k}{(L-s^*)^4}|s(x)-s(x')||2L-s(x)-s(x')||\tilde{u}_y(x')|_{L^2(0,1)}|\tilde{u}_y(x)-\tilde{u}_y(x')|_{L^2(0,1)} \\
&\le \frac{\eta}{2}|\tilde{u}_y(x)-\tilde{u}_y(x')|_{L^2(0,1)} + \left(\frac{4Lk}{(L-s^*)^4}\right)^2\frac{1}{2\eta}|s(x)-s(x')|^2|\tilde{u}_y(x')|^2_{L^2(0,1)}.
\end{aligned}
$$

Furthermore, we can calculate I_4 as follows.

$$\begin{aligned}
I_4 = &\int_0^1 \left(\rho_v(1-y)\left(\frac{1}{L-s(x)} - \frac{1}{L-s(x')}\right)s_t(x)(t)\right)\tilde{u}_y(x)(\bar{u}(x)-\bar{u}(x'))dy \\
&+ \int_0^1 \frac{\rho_v(1-y)}{L-s(x')}\left(s_t(x)(t)-s_t(x')(t)\right)\tilde{u}_y(x)(\bar{u}(x)-\bar{u}(x'))dy \\
\leq &\frac{2a\rho_v}{(L-s^*)^2}|s(x)-s(x')||\tilde{u}_y(x)|_{L^2(0,1)}|\bar{u}(x)-\bar{u}(x')|_{L^2(0,1)} \\
&+ \frac{\rho_v}{L-s^*}|s_t(x)(t)-s_t(x')(t)||\tilde{u}_y(x)|_{L^2(0,1)}|\bar{u}(x)-\bar{u}(x')|_{L^2(0,1)} \\
\leq &\frac{\eta}{2}|s(x)-s(x')|^2|\tilde{u}_y(x)|^2_{L^2(0,1)} + \frac{1}{2\eta}\left(\frac{2a\rho_v}{(L-s^*)^2}\right)^2|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)} \\
&+ \frac{\eta}{2}|s_t(x)(t)-s_t(x')(t)|^2|\tilde{u}_y(x)|^2_{L^2(0,1)} + \frac{1}{2\eta}\left(\frac{\rho_v}{L-s^*}\right)^2|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)}.
\end{aligned}$$

By substituting the above estimates into (3.15), we have

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)} + \left(\frac{k}{L^2} - \frac{(3+C_2)}{2}\eta\right)|\bar{u}_y(x)-\bar{u}_y(x')|^2_{L^2(0,1)} \\
&\leq \frac{\rho_v}{2}|h_t(x,t)-h_t(x',t)|^2 + C_2|h(x,t)-h(x',t)|^2 \\
&+ \left(\frac{\rho_v}{2} + \frac{2C_2}{\eta} + \frac{1}{\eta}\left(\frac{2a\rho_v}{(L-s^*)^2}\right)^2 + \frac{1}{2\eta}\left(\frac{\rho_v}{L-s^*}\right)^2\right)|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)} \\
&+ \left(\left(\frac{4Lk}{(L-s^*)^2}\right)^2\frac{1}{2\eta} + \frac{\eta}{2}\right)|s(x)-s(x')|^2|\tilde{u}_y(x')|^2_{L^2(0,1)} \\
&+ C_1|s_t(x)(t)-s_t(x')(t)|^2 + \frac{\eta}{2}|s_t(x)(t)-s_t(x')(t)|^2|\tilde{u}_y(x)|^2_{L^2(0,1)}. \qquad (3.17)
\end{aligned}$$

Here, let $C_3(\eta)$ and $C_4(\eta)$ be coefficients of the second and third term of the right hand side in (3.17). By using (3.13), we have

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)} + \left(\frac{k}{L^2} - \frac{(3+C_2)}{2}\eta\right)|\bar{u}_y(x)-\bar{u}_y(x')|^2_{L^2(0,1)} \\
&\leq \frac{\rho_v}{2}|h_t(x,t)-h_t(x',t)|^2 + C_2|h(x,t)-h(x',t)|^2 \\
&+ C_3(\eta)|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)} + C_4(\eta)M_2L|s(x)-s(x')|^2 \\
&+ \left(C_1 + \frac{\eta}{2}M_2L\right)|s_t(x)(t)-s_t(x')(t)|^2. \qquad (3.18)
\end{aligned}$$

Next, by (3.2) and (3.16) we get

$$\begin{aligned}
&\frac{1}{2}|s_t(x)(t)-s_t(x')(t)|^2\\
\le& a^2|\tilde{u}(x)(t,0)-\tilde{u}(x')(t,0)|^2+a^2|\varphi(s(x)(t))-\varphi(s(x')(t))|^2\\
\le& \frac{a^2\delta}{2}|\tilde{u}_y(x)(t)-\tilde{u}_y(x')(t)|^2_{L^2(0,1)}+\frac{2a^2}{\delta}|\tilde{u}(x)(t)-\tilde{u}(x')(t)|^2_{L^2(0,1)}\\
&+a^2|h(x,t)-h(x',t)|^2+C_\varphi^2 a^2|s(x)(t)-s(x')(t)|^2,
\end{aligned}\tag{3.19}$$

where δ is arbitrary positive constant. By substituting (3.19) into (3.18) we obtain

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)}\\
&+\left(\frac{k}{L^2}-\frac{(3+C_2)}{2}\eta-\left(C_1+\frac{\eta}{2}M_2L\right)a^2\delta\right)|\bar{u}_y(x)-\bar{u}_y(x')|^2_{L^2(0,1)}\\
\le&\frac{\rho_v}{2}|h_t(x,t)-h_t(x',t)|^2+\left[C_2+2a^2\left(C_1+\frac{\eta}{2}M_2L\right)\right]|h(x,t)-h(x',t)|^2\\
&+C_3(\eta)|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)}+\left(C_1+\frac{\eta}{2}M_2L\right)\frac{4a^2}{\delta}|\tilde{u}(x)(t)-\tilde{u}(x')(t)|^2_{L^2(0,1)}\\
&+\left[C_4(\eta)M_2L+2C_\varphi^2a^2\left(C_1+\frac{\eta}{2}M_2L\right)\right]|s(x)(t)-s(x')(t)|^2.
\end{aligned}\tag{3.20}$$

By $|\tilde{u}(x)(t)-\tilde{u}(x')(t)|^2_{L^2(0,1)}\le 2|\bar{u}(x)(t)-\bar{u}(x')(t)|^2_{L^2(0,1)}+2|h(x,t)-h(x',t)|^2$, we have

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}(x)-\bar{u}(x')|^2_{L^2(0,1)}\\
&+\left[\frac{k}{L^2}-\frac{(3+C_2)}{2}\eta-\left(C_1+\frac{\eta}{2}M_2L\right)a^2\delta\right]|\bar{u}_y(x)-\bar{u}_y(x')|^2_{L^2(0,1)}\\
\le&\frac{\rho_v}{2}|h_t(x,t)-h_t(x',t)|^2\\
&+\left[C_2+2a^2\left(C_1+\frac{\eta}{2}M_2L\right)+\left(C_1+\frac{\eta}{2}M_2L\right)\frac{8a^2}{\delta}\right]|h(x,t)-h(x',t)|^2\\
&+\left[C_3(\eta)+\left(C_1+\frac{\eta}{2}M_2L\right)\frac{8a^2}{\delta}\right]|\bar{u}(x)(t)-\bar{u}(x')(t)|^2_{L^2(0,1)}\\
&+\left[C_4(\eta)M_2L+2C_\varphi^2a^2\left(C_1+\frac{\eta}{2}M_2L\right)\right]|s(x)(t)-s(x')(t)|^2.
\end{aligned}\tag{3.21}$$

Similarly to the derivation of (3.19),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|s(x)(t) - s(x')(t)|^2 \\
\le&\frac{a^2}{2}|\tilde{u}(x)(t,0) - \tilde{u}(x')(t,0)|^2 + \frac{a^2}{2}|\varphi(s(x)(t)) - \varphi(s(x')(t))|^2 \\
&+ |s(x)(t) - s(x')(t)|^2 \\
\le&\frac{a^2\delta}{4}|\tilde{u}_y(x)(t) - \tilde{u}_y(x')(t)|^2_{L^2(0,1)} + \frac{a^2}{\delta}|\tilde{u}(x)(t) - \tilde{u}(x')(t)|^2_{L^2(0,1)} \\
&+ \frac{a^2}{2}|h(x,t) - h(x',t)|^2 + \frac{C_\varphi^2 a^2}{2}|s(x)(t) - s(x')(t)|^2 \\
&+ |s(x)(t) - s(x')(t)|^2.
\end{aligned} \tag{3.22}$$

By adding (3.21) and (3.22) we obtain

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}(x) - \bar{u}(x')|^2_{L^2(0,1)} + \frac{1}{2}\frac{d}{dt}|s(x)(t) - s(x')(t)|^2 \\
&+ \left[\frac{k}{L^2} - \frac{(3 + C_2)}{2}\eta - \left(C_1 + \frac{\eta}{2}M_2 L\right)a^2\delta - \frac{a^2\delta}{4}\right]|\bar{u}_y(x) - \bar{u}_y(x')|^2_{L^2(0,1)} \\
&\le \frac{\rho_v}{2}|h_t(x,t) - h_t(x',t)|^2 \\
&+ \left[C_2 + \left(C_1 + \frac{\eta}{2}M_2 L\right)\left(2a^2 + \frac{8a^2}{\delta}\right) + \frac{a^2}{2} + \frac{2a^2}{\delta}\right]|h(x,t) - h(x',t)|^2 \\
&+ \left[C_3(\eta) + \left(C_1 + \frac{\eta}{2}M_2 L\right)\frac{8a^2}{\delta} + \frac{2a^2}{\delta}\right]|\bar{u}(x)(t) - \bar{u}(x')(t)|^2_{L^2(0,1)} \\
&+ \left[C_4(\eta)M_2 L + 2C_\varphi^2 a^2\left(C_1 + \frac{\eta}{2}M_2 L\right) + \frac{C_\varphi^2 a^2}{2} + 1\right]|s(x)(t) - s(x')(t)|^2.
\end{aligned} \tag{3.23}$$

By taking a suitable number δ_0 after setting a suitable number η_0,

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}(x)(t) - \bar{u}(x')(t)|^2_{L^2(0,1)} + \frac{1}{2}\frac{d}{dt}|s(x)(t) - s(x')(t)|^2 \\
&+\frac{k}{2L^2}|\bar{u}_y(x) - \bar{u}_y(x')|^2_{L^2(0,1)} \\
&\le \frac{\rho_v}{2}|h_t(x,t) - h_t(x',t)|^2 + \tilde{C}_1(\eta_0,\delta_0)|h(x,t) - h(x',t)|^2 \\
&+\tilde{C}_2(\eta_0,\delta_0)|\bar{u}(x)(t) - \bar{u}(x')(t)|^2_{L^2(0,1)} + \tilde{C}_3(\eta_0)|s(x)(t) - s(x')(t)|^2,
\end{aligned} \tag{3.24}$$

where $\tilde{C}_1(\eta_0,\delta_0) = C_2 + (C_1 + \eta_0 M_2 L/2)(2a^2 + 8a^2/\delta_0) + a^2/2 + 2a^2/\delta_0$, $\tilde{C}_2(\eta_0,\delta) = C_3(\eta_0) + (C_1 + \eta_0 M_2 L/2)8a^2/\delta_0 + 2a^2/\delta_0$ and $C_3(\eta_0) = C_4 M_2 L + 2C_\varphi^2 a^2(C_1 + \eta_0 M_2 L/2) + C_\varphi^2 a^2/2 + 1$. Now, by setting

$$\begin{aligned}
I(t) = &\frac{\rho_v}{2}|\bar{u}(x)(t) - \bar{u}(x')(t)|^2_{L^2(0,1)} + \frac{1}{2}|s(x)(t) - s(x')(t)|^2 \\
&+ \frac{k}{2L^2}\int_0^t |\bar{u}_y(x) - \bar{u}_y(x')|^2_{L^2(0,1)}d\tau \text{ for } t \in [0,T],
\end{aligned}$$

Gronwall's inequality implies that for $t_1 \in [0,T]$,

$$
\begin{aligned}
&I(t_1)\\
\le& \left[I(0) + \frac{\rho_v}{2}\int_0^{t_1} |h_\tau(x,\tau) - h_\tau(x',\tau)|^2 d\tau + \tilde{C}_1(\eta_0,\delta_0)\int_0^{t_1} |h(x,\tau) - h(x',\tau)|^2 d\tau \right] e^{\left(\frac{2\tilde{C}_2}{\rho_v} + 2\tilde{C}_3\right)T}.
\end{aligned} \tag{3.25}
$$

Therefore, we finally obtain

$$
\begin{aligned}
&\frac{\rho_v}{2}|\bar{u}_n(x)(t) - \bar{u}_n(x')(t)|^2_{L^2(0,1)} + \frac{1}{2}|s_n(x)(t) - s_n(x')(t)|^2\\
&\le \left[\frac{\rho_v}{2}|\bar{u}_n(x)(0) - \bar{u}_n(x')(0)|^2_{L^2(0,1)} + \frac{1}{2}|s_{0n}(x) - s_{0n}(x')|^2 \right.\\
&\left. + \frac{\rho_v}{2}\int_0^t |h_{n\tau}(x) - h_{n\tau}(x')|^2 d\tau + \tilde{C}_1(\eta_0,\delta_0)\int_0^t |h_n(x) - h_n(x')|^2 d\tau \right] e^{(\tilde{C}_2 + \tilde{C}_3)T}.
\end{aligned} \tag{3.26}
$$

By the definition of $\bar{u}_n(x)(0)$ and h_n, we see that

$$
\begin{aligned}
&|\bar{u}_n(x)(0) - \bar{u}_n(x')(0)|^2_{L^2(0,1)}\\
\le& 4\left(|\tilde{u}_{0n}(x) - \tilde{u}_{0n}(x')|^2_{L^2(0,1)} + |h_n(0,x) - h_n(0,x')|^2 \right)\\
\le& 4M_1(n)|x - x'|^2,
\end{aligned}
$$

$$
\int_0^t |h_{n\tau}(x,\tau) - h_{n\tau}(x',\tau)|^2 d\tau \le M_2(n)T|x - x'|^2,
$$

and

$$
\int_0^t |h_n(x,\tau) - h_n(x',\tau)|^2 d\tau \le M_3(n)T|x - x'|^2 \text{ for } t \in [0,T]. \tag{3.27}
$$

where $M_i(n)(1 \le i \le 3)$ is a positive constant depending on n. By (3.26) and (3.27),

$$
\begin{aligned}
&\frac{\rho_v}{2}|\bar{u}_n(x)(t) - \bar{u}_n(x')(t)|^2_{L^2(0,1)} + \frac{1}{2}|s_n(x)(t) - s_n(x')(t)|^2\\
&\le \left[2\rho_v M_1(n)|x - x'|^2 + \frac{1}{2}|s_{0n}(x) - s_{0n}(x')|^2 \right.\\
&\left. + \frac{\rho_v}{2}M_2(n)T|x - x'|^2 + \tilde{C}_1(\eta_0,\delta_0)M_3(n)T|x - x'|^2 \right] e^{\left(\frac{2\tilde{C}_2}{\rho_v} + 2\tilde{C}_3\right)T} \text{ for } t \in [0,T].
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&|\bar{u}_n(x) - \bar{u}_n(x')|^2_{C([0,T];L^2(\Omega))} + |s_n(x) - s_n(x')|^2_{C([0,T])}\\
\le& C_n(|x - x'|^2 + |s_{0n}(x) - s_{0n}(x')|^2) \text{ for each } x, x' \in \overline{\Omega} \text{ and each } n,
\end{aligned}
$$

where C_n is a positive constant independent of x, $x' \in \overline{\Omega}$. Since s_{0n} is continuous on $\overline{\Omega}$ for each n, we conclude that $s_n \in C(\overline{\Omega}; C([0,T]))$ and $\tilde{u}_n \in C(\overline{\Omega}; C([0,T]; L^2(0,1)))$. ⋄

Lemma 4. *Let $(s_n, \tilde{u}_n)$ be a solution of $\tilde{P}_n(x)$ for $x \in \overline{\Omega}$ and $n \in \mathbb{N}$. Then, $\{\tilde{u}_n\}$ is a Cauchy sequence in $C(\overline{\Omega}; C([0,T]; L^2(0,1))) \cap C(\overline{\Omega}; L^2(0,T; H^1(0,1)))$ and $\{s_n\}$ is a Cauchy sequence in $C(\overline{\Omega}; C([0,T]))$.*

Proof. By the same argument of the proof of Lemma 3 and (3.18), we see that

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}_n(x) - \bar{u}_m(x)|^2_{L^2(0,1)} + \left(\frac{k}{L^2} - \frac{(3+C_2)}{2}\eta\right)|\bar{u}_{ny}(x) - \bar{u}_{my}(x)|^2_{L^2(0,1)} \\
&\le \frac{\rho_v}{2}|h_{nt}(x,t) - h_{mt}(x,t)|^2 + C_2|h_n(x,t) - h_m(x,t)|^2 \\
&+C_3(\eta)|\bar{u}_n(x) - \bar{u}_m(x)|^2_{L^2(0,1)} + C_4(\eta)M_2L|s_n(x) - s_m(x)|^2 \\
&+\left(C_1 + \frac{\eta}{2}M_2L\right)|s_{nt}(x) - s_{mt}(x)|^2 \text{ for } n \in N \text{ and } x \in \overline{\Omega}, \qquad (3.28)
\end{aligned}$$

where $C_i (1 \le i \le 4)$ is the same positive constant as in (3.15). Also the following inequalities hold (cf. (3.19), (3.22));

$$\begin{aligned}
&\max\left\{\frac{1}{2}|s_{nt}(x)(t) - s_{mt}(x)(t)|^2, \ \frac{1}{2}\frac{d}{dt}|s_n(x)(t) - s_m(x)(t)|^2\right\} \\
&\le \frac{a^2\delta}{4}|\tilde{u}_{ny}(x)(t) - \tilde{u}_{my}(x)(t)|^2_{L^2(0,1)} + \frac{a^2}{\delta}|\tilde{u}_n(x)(t) - \tilde{u}_m(x)(t)|^2_{L^2(0,1)} \\
&+ \frac{a^2}{2}|h_n(x,t) - h_m(x,t)|^2 + \frac{C_\varphi a^2}{2}|s_n(x)(t) - s_m(x)(t)|^2 \\
&+ |s_n(x)(t) - s_m(x)(t)|^2 \text{ for } t \in [0,T] \text{ and } x \in \overline{\Omega}. \qquad (3.29)
\end{aligned}$$

By using the derivation of (3.23), we obtain that

$$\begin{aligned}
&\frac{\rho_v}{2}\frac{d}{dt}|\bar{u}_n(x) - \bar{u}_m(x)|^2_{L^2(0,1)} + \frac{1}{2}\frac{d}{dt}|s_n(x)(t) - s_m(x)(t)|^2 \\
&+\left[\frac{k}{L^2} - \frac{(3+C_2)}{2}\eta - \left(C_1 + \frac{\eta}{2}M_2L\right)a^2\delta - \frac{a^2\delta}{4}\right]|\bar{u}_{ny}(x) - \bar{u}_{my}(x)|^2_{L^2(0,1)} \\
&\le \frac{\rho_v}{2}|h_{nt}(x,t) - h_{mt}(x,t)|^2 \\
&+\left[C_2 + \left(C_1 + \frac{\eta}{2}M_2L\right)\left(2a^2 + \frac{8a^2}{\delta}\right) + \frac{a^2}{2} + \frac{2a^2}{\delta}\right]|h_n(x,t) - h_m(x,t)|^2 \\
&+\left[C_3(\eta) + \left(C_1 + \frac{\eta}{2}M_2L\right)\frac{8a^2}{\delta} + \frac{2a^2}{\delta}\right]|\bar{u}_n(x)(t) - \bar{u}_m(x)(t)|^2_{L^2(0,1)} \\
&+\left[C_4(\eta)M_2L + 2C_\varphi^2a^2\left(C_1 + \frac{\eta}{2}M_2L\right) + \frac{C_\varphi^2a^2}{2} + 1\right]|s_n(x)(t) - s_m(x)(t)|^2. \qquad (3.30)
\end{aligned}$$

Now, we take suitable numbers η_0 and δ_0, and put each coefficients of the right hand side of (3.30) by $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$. By setting

$$\begin{aligned}
J(t) = &\frac{\rho_v}{2}|\bar{u}_n(x)(t) - \bar{u}_m(x)(t)|^2_{L^2(0,1)} + \frac{1}{2}|s_n(x)(t) - s_m(x)(t)|^2 \\
&+ \frac{k}{2L^2}\int_0^t |\bar{u}_{ny}(x) - \bar{u}_{my}(x)|^2_{L^2(0,1)} d\tau \text{ for } t \in [0,T],
\end{aligned}$$

it holds that

$$\frac{d}{dt}J(t) \le \frac{\rho_v}{2}|h_{nt}(x,t)-h_{mt}(x,t)|^2+\tilde{C}_1|h_n(x,t)-h_m(x,t)|^2+\left(\frac{2\tilde{C}_2}{\rho_v}+2\tilde{C}_3\right)J(t). \quad (3.31)$$

By applying Gronwall's lemma to (3.31) we have

$$\begin{aligned} &J(t_1)\\ \le &\left[\frac{\rho_v}{2}\int_0^{t_1}|h_{nt}(x,t)-h_{mt}(x,t)|^2dt+\tilde{C}_1\int_0^{t_1}|h_n(t,x)-h_m(t,x)|^2dt\right]e^{\tilde{C}_4}T \text{ for } t_1\in[0,T], \end{aligned} \quad (3.32)$$

where $\tilde{C}_4 = 2\tilde{C}_2/\rho_v + 2\tilde{C}_3$, and

$$\begin{aligned} &|\bar{u}_n(x)-\bar{u}_m(x)|^2_{C([0,T];L^2(0,1))}+|s_n(x)-s_m(x)|^2_{C([0,T])}+\int_0^T|\bar{u}_{ny}(x)-\bar{u}_{my}(x)|^2_{L^2(0,1)}dt\\ \le &C_5\left(\int_0^T|h_{nt}(x,t)-h_{mt}(x,t)|^2dt+\int_0^T|h_n(x,t)-h_m(x,t)|^2dt\right) \text{ for } x\in\overline{\Omega} \text{ and } n,m\in\mathbb{N}, \end{aligned}$$

where C_5 is a positive constant. Moreover, by the Sobolev embedding $H^2(\Omega)\subset C(\overline{\Omega})$ we have

$$\begin{aligned} &|\bar{u}_n(x)-\bar{u}_m(x)|^2_{C([0,T];L^2(0,1))}+|s_n(x)-s_m(x)|^2_{C([0,T])}+\int_0^T|\bar{u}_{ny}(x)-\bar{u}_{my}(x)|^2_{L^2(0,1)}dt\\ \le\ &C_6\left(\sup_{x\in\overline{\Omega}}\left|\int_0^T|h_{nt}(x,t)-h_{mt}(x,t)|^2dt\right|+\sup_{x\in\overline{\Omega}}\left|\int_0^T|h_n(x,t)-h_m(x,t)|^2dt\right|\right)\\ \le\ &C_6\left(\int_0^T|h_{nt}(\cdot,t)-h_{mt}(\cdot,t)|^2_{C(\overline{\Omega})}dt+\int_0^T|h_n(\cdot,t)-h_m(\cdot,t)|^2_{C(\overline{\Omega})}dt\right)\\ \le\ &C_7\left(\int_0^T|h_{nt}(\cdot,t)-h_{mt}(\cdot,t)|^2_{H^2(\Omega)}dt+\int_0^T|h_n(\cdot,t)-h_m(\cdot,t)|^2_{H^2(\Omega)}dt\right), \end{aligned}$$

where C_6 and C_7 are positive constants. By the definition of $\{h_n\}$, we conclude that Lemma 4 holds. $\diamond$

From now on, we prove Theorems 1 and 2. Let $(s_n, \tilde{u}_n)$ be a solution of $\tilde{\mathrm{P}}_n(x)$ for $x\in\overline{\Omega}$ and $n\in\mathbb{N}$. Then, $(s_n,\tilde{u}_n)=(s_n(x),\tilde{u}_n(x))$ satisfies that

$$\begin{cases} \rho_v\tilde{u}_{nt}-\frac{k}{(L-s_n)^2}\tilde{u}_{nyy}=\frac{\rho_v(1-y)s_{nt}}{L-s_n}\tilde{u}_{ny} \text{ a.e. in } Q(T),\\ \tilde{u}_n(t,1)=h_n(x,t) \text{ for a.e. } t\in(0,T),\\ \frac{k}{L-s_n}\tilde{u}_{ny}(t,0)=(\rho_w-\rho_v\tilde{u}_n(t,0))s_{nt}(t) \text{ for a.e. } t\in(0,T),\\ s_{nt}(t)=a(\tilde{u}_n(t,0)-\varphi(s_n(t))) \text{ for a.e. } t\in(0,T),\\ s_n(0)=s_{0n}(x),\\ \tilde{u}_n(0,y)=\tilde{u}_{0n}(y) \text{ for } y\in[0,1]. \end{cases}$$

By Lemma 4 there exist $\tilde{u} \in C(\overline{\Omega}; C([0,T]; L^2(0,1))) \cap C(\overline{\Omega}; L^2(0,T; H^1(0,1)))$ and $s \in C(\overline{\Omega}; C([0,T]))$ such that

$$\tilde{u}_n \to \tilde{u} \begin{cases} \text{in } C(\overline{\Omega}; C([0,T]; L^2(0,1))), \\ \text{in } C(\overline{\Omega}; L^2(0,T; H^1(0,1))), \end{cases}$$

and

$$s_n \to s \text{ in } C(\overline{\Omega}; C([0,T]))$$

as $n \to \infty$. Namely, for $x \in \overline{\Omega}$,

$$\tilde{u}_n(x) \to \tilde{u}(x) \text{ in } C([0,T]; L^2(0,1)) \text{ and } s_n(x) \to s(x) \text{ in } C([0,T]).$$

Here, we fix $x \in \overline{\Omega}$. Then, Lemma 2 implies that

$$\tilde{u}_n(x) \to \tilde{u}(x) \begin{cases} \text{weakly in } W^{1,2}(0,T; L^2(0,1)), \\ \text{weakly in } L^2(0,T; H^2(0,1)), \\ \text{weakly-* in } L^\infty(0,T; H^1(0,1)), \end{cases}$$

$$s_n(x) \to s(x) \text{ weakly in } W^{1,\infty}(0,T).$$

Also, by the compact embedding $H^1(0,1) \subset C([0,1])$ and Aubin's compactness theorem, we see that

$$\tilde{u}_n(x) \to \tilde{u}(x) \text{ in } C(\overline{(0,T) \times (0,1)}) \text{ as } n \to \infty.$$

From the above convergences, it is easy to see that $(s, \tilde{u}) = (s(x), \tilde{u}(x))$ satisfies

$$\rho_v \tilde{u}_t - \frac{k}{(L-s)^2}\tilde{u}_{yy} = \frac{\rho_v(1-y)s_t}{L-s}\tilde{u}_y \text{ a.e. in } Q(T), \tag{3.33}$$

$$\tilde{u}(t,1) = h(x,t) \text{ for } t \in [0,T],$$

$$s_t(t) = a(\tilde{u}(t,0) - \varphi(s(t))) \text{ for a.e. } t \in (0,T).$$

Next, Let $\phi \in C^\infty(\overline{Q(T)})$ with $\phi(t,1) = 0$ for $t \in [0,T]$. Then, it holds that

$$\begin{aligned} &\int_0^T \int_0^1 \rho_v \tilde{u}_{nt}(t,y)\phi(t,y)dydt + \int_0^T \int_0^1 \frac{k}{(L-s_n)^2}\tilde{u}_{ny}(t,y)\phi_y(t,y)dydt \\ &\quad + \int_0^T \frac{1}{(L-s_n)}(\rho_w - \rho_v \tilde{u}_n(t,0))s_{nt}(t)\phi(t,0)dt \\ &= \int_0^T \int_0^1 \frac{\rho_v(1-y)s_{nt}}{L-s_n}\tilde{u}_{ny}(t,y)\phi(t,y)dydt. \end{aligned} \tag{3.34}$$

From (3.34), by letting $n \to \infty$ and (3.33) we can see that $\frac{k}{L-s}\tilde{u}_y(t,0) = (\rho_w - \rho_v \tilde{u}(t,0))s_t(t)$ a.e. $t \in [0,T]$. Here, we note that $\tilde{u}_{0n} \to \tilde{u}_0$ uniformly on $\overline{\Omega}$ as $n \to \infty$. Therefore, we can conclude that the limit function $(s, \tilde{u})$ is a solution of $\mathrm{P}(x)$ for $x \in \overline{\Omega}$. The boundedness and uniqueness of a solution of $\mathrm{P}(x)$ can be proved by the argument of [5] and [8]. Finally, by Lemmas 3 and 4, $\{s_n\}$ is a Cauchy sequence in $C(\overline{\Omega}; C([0,T]))$ and $\{\tilde{u}_n\}$ is a Cauchy sequence in $C(\overline{\Omega}; L^2(Q(T)))$ and by the fact that $C(\overline{\Omega}; L^2(Q(T)))$ is a Banach space, we see that the limit function $(s, \tilde{u})$ satisfies that $s \in C(\overline{\Omega}; C([0,T]))$ and $\tilde{u} \in C(\overline{\Omega}; L^2(Q(T)))$. Thus, Theorem 1 and Theorem 2 are proved.

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