Advances in Mathematical
Sciences and Applications
Vol. 25, No. 1 (2016), pp. 273-287

GAKKOTOSHO
TOKYO JAPAN

REGULARITY OF WEAK SOLUTIONS FOR A DIV-CURL SYSTEMS

Hong-Ming Yin
Department of Mathematics
Washington State University
Pullman, WA 99164, USA.
(Email Address: hyin@wsu.edu)

Abstract. In this paper we study the regularity for weak solution of Div-Curl system subject to a mixed normal-tangential boundary condition. It is shown that the weak solution is Hölder continuous under certain conditions on known data. This regularity result provides an estimate of a vector field in Hölder space in terms of its divergence and curl in suitable space. Moreover, further $C^{1+\alpha}$-regularity of the weak solution is also established if the known data have more regularity. The present result improves the classical inequality derived by W. von Wahl in 1992 in $W^{1,p}$-space. The main tool used in this paper is based on precise estimate of weak solution for Div-Curl system in Morrey-John-Nirenberg-Campanato space.

Communicated by R. Showalter; Received August 3, 2016.
AMS Subject Classification: 35J46, 35Q60, 35N25.
Keyworks:Div-Curl system, regularity of weak solution.

1 Introduction

Let Ω be a bounded domain in R^3 with boundary $S = \partial\Omega \in C^{2+\alpha}$ for some $\alpha \in (0,1)$. In this paper we study the regularity of weak solution for the following Div-Curl system subject to a mixed boundary condition: Find a vector field $\mathbf{V} : \Omega \to R^3$, which satisfies:

$$\nabla \times \mathbf{V} = \mathbf{F}, \qquad \text{in} \qquad \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{V} = g, \qquad \text{in} \qquad \Omega, \tag{1.2}$$

$$\mathbf{N} \times \mathbf{V} = 0, \text{on } \Gamma_1, \qquad \mathbf{N} \cdot \mathbf{V} = 0, \text{on } \Gamma_2, \tag{1.3}$$

where $\mathbf{N}$ is the outward unit normal on S, $\bar{\Gamma}_1 \bigcap \bar{\Gamma}_2 = \emptyset, \bar{\Gamma}_1 \bigcup \bar{\Gamma}_2 = S$.

The motivation of our study of div-curl system (1.1)-(1.2) comes from various applications in fluid mechanics and electromagnetic theory ([8, 15]). The importance of such systems is also found in elasticity theory and plasma physics ([16]) as well as other industrial applications ([20]). One of the interesting questions in those applications is that whether or not the derivative of a vector field $\mathbf{U}$ in some space can be estimated by its divergence, denoted by $Div\mathbf{U} = \nabla \cdot \mathbf{U}$ and its curl of $\mathbf{U}$, denoted by $Curl\mathbf{U} = \nabla \times \mathbf{U}$. An elementary calculation shows that for any $\mathbf{U} = (u_1, u_2, u_3) \in H(div, curl, R^3)$ (see definition in Section 2),

$$\sum_{i,j=1}^{3} \int_{R^3} |u_{ix_j}|^2 dx = \int_{R^3} [|\nabla \cdot \mathbf{U}|^2 + |\nabla \times \mathbf{U}|^2] dx.$$

However, the estimate is much more complicated for a bounded domain in R^3. It depends on the value of vector field on the boundary and the geometry of the domain through the first and second Betti numbers (see the definition in Section 2). In 1992, W. von Wahl [22] proved the following interesting inequality:

$$\sum_{i=1}^{3} ||\mathbf{U}_{x_i}||_{L^p(\Omega)} \leq C[||Div\mathbf{U}||_{L^p(\Omega)} + ||Curl\mathbf{U}||_{L^p(\Omega)}], \tag{1.4}$$

where $p \in (1,\infty)$, Ω is a bounded simply-connected domain in R^3 and $\mathbf{U}$ satisfies either the boundary condition $\mathbf{N} \times \mathbf{U} = 0$ or $\mathbf{N} \cdot \mathbf{U} = 0$ on $\partial\Omega$. The constant C depends only on p and Ω. A similar inequality with a mixed boundary condition is obtained by Auchmuty in 2004 ([2]). With this inequality, one can obtain by Sobolev's embedding that

$$||\mathbf{U}||_{L^{\frac{3p}{3-p}}} \leq C\left[||\nabla \cdot \mathbf{U}||_{L^p(\Omega)} + ||\nabla \times \mathbf{U}||_{L^p(\Omega)}\right], \text{if } 1 < p < 3, \tag{1.5}$$

$$||\mathbf{U}||_{C^\alpha(\bar{\Omega})} \leq C\left[||\nabla \cdot \mathbf{U}||_{L^p(\Omega)} + ||\nabla \times \mathbf{U}||_{L^p(\Omega)}\right], \text{if } p > 3. \tag{1.6}$$

An extension of the Hölder estimate for $\mathbf{V}$ in $C^{1+\alpha}$-space is established by Bolik and von Wahl [5] in 1997:

$$\sum_{j=1}^{3} ||\mathbf{U}_{x_j}||_{C^\alpha(\bar{\Omega})}$$
$$\leq C[||\nabla\cdot\mathbf{U}||_{C^\alpha(\bar{\Omega})} + ||\nabla\times\mathbf{U}||_{C^\alpha(\bar{\Omega})} + ||\mathbf{N}\times\mathbf{U}||_{C^{1+\alpha}(\partial\Omega)} + \sum_{i=1}^{m}|E_i|]; \qquad (1.7)$$
$$\sum_{j=1}^{3} ||\mathbf{U}_{x_j}||_{C^\alpha(\bar{\Omega})}$$
$$\leq C[||\nabla\cdot\mathbf{U}||_{C^\alpha(\bar{\Omega})} + ||\nabla\times\mathbf{U}||_{C^\alpha(\bar{\Omega})} + ||\mathbf{N}\cdot\mathbf{U}||_{C^{1+\alpha}(\partial\Omega)} + \sum_{i=1}^{n}|F_i|], \qquad (1.8)$$

where

$$E_i := -\int_{\partial\hat{\Omega}_i} \mathbf{N}\cdot\mathbf{U}ds, \qquad i = 1,\cdots,m;$$
$$F_i = -\int_{\partial\hat{\Omega}} (\mathbf{N}\times\mathbf{U})\cdot\hat{z}_i ds, \qquad i = 1,\cdots,n,$$

$n =$ the first Betti number and $m =$ the second Betti number, $\partial\hat{\Omega}_i$ is the boundary of one of bounded simply connected domains of $\Omega = R^3\backslash\Omega$, the functions $z_1,\cdots,z_n$ form a basis of the Neumann field on Ω.

The validity of inequality (1.4) for the case $p = 1$ was an open question. One would hope that the inequality (1.4) holds for $p = 1$, which would imply the $L^{3/2}-$estimate for $\mathbf{U}$ in terms of $Div\mathbf{U}$ and $Curl\mathbf{U}$, due to the Sobolev embedding from $W^{1,1}(\Omega)$ to $L^{3/2}(\Omega)$ in dimension 3. However, it turns out that the inequality (1.4) does not hold when $p = 1$. A counterexample is constructed in [23] by X. Xiang. It seems no hope to obtain the $L^{3/2}$-estimate for $\mathbf{U}$ in terms of $Div\mathbf{U}$ and $Curl\mathbf{U}$. A rather surprising result is that in R^3 the estimate is proved by J. Bougain and H. Brezis in 2004 ([6]). Namely,

$$||\mathbf{U}||_{L^{3/2}(R^3)} \leq C[||\nabla\cdot\mathbf{U}||_{L^1(R^3)} + ||\nabla\times\mathbf{U}||_{L^1(R^3)}], \qquad (1.9)$$

Some generalizations are obtained in half-space by C. Amrouche and H.H. Nguyen in 2011 ([1]). More recently, H. Kozono and Y. Yanagisawa in 2009 ([14]) extended the inequality (1.4) to a more general domain with nonhomogeneous boundary conditions. For a boundary domain in R^2, a similar estimate is established by D. Mitrea in 2005 ([17]) and an additional estimate is obtained by X. Xiang in 2013 ([23]) in R^3 with a smooth boundary.

It will be seen that the inequality (1.4) depends on the precise estimate of the solution of Div-Curl system (1.1)-(1.2). The L^p-solvability for the above Div-Curl system (1.1)-(1.3) is established by G. Auchmuty and J. C. Alexander for a bounded domain in R^2 in 2001 ([3]) and for the case in R^3 in 2004 ([4]). For an exterior of a bounded domain, the well-posedness of div-curl problem is studied by Neudert and von Wahl in 2001 ([9]). Some regularity results for a domain in R^2 with Lipschitz boundary is investigated by Mitrea in 2005 ([17], also see [18]). In the present paper our focus will be on the regularity of weak solution for the Div-Curl system (1.1)-(1.3) in Hölder space and $C^{1+\alpha}$-space. It is shown that the weak solution $\mathbf{V}$ is of class C^α under certain conditions for the known data $\mathbf{F}$ and g. Moreover, the $C^{1+\alpha}$-estimate is also established if more regularity assumption on known data is in force. Those regularity results are optimal in the sense that the smoothness conditions on known functions $\mathbf{F}$ and g are necessary. Moreover, we will derive the precise Hölder and $C^{1+\alpha}$-estimates in term of $\mathbf{F}$ and $g(x)$. By using these estimates we improve the well known inequalities (1.4). More specifically, we prove that

$$||\mathbf{V}||_{C^\alpha(\bar{\Omega})} \leq C[||Div\mathbf{V}||_{L^{2,\mu}(\Omega)} + ||Curl\mathbf{V}||_{L^{2,\mu}(\Omega)}],$$

where $\alpha = \frac{\mu-1}{2}$ for some $\mu \in (1,3)$, C depends only on Ω and μ. The norm $||\cdot||_{L^{2,\mu}(\Omega)}$ is the norm of Morrey-John-Nirenbery-Campanato space $L^{2,\mu}(\Omega)$ (see definition in Section 2 or [21]).

The main challenge for the regularity of weak solution is that the boundary condition (1.3) is coupled in a nonclassical form. The method used in this paper is based on the techniques developed in [24, 25, 26] for Maxwell's systems, which is quite different from those used in [5, 22]. We would like to point out that the Div-Curl system (1.1)-(1.2) is closely related to Maxwell's equations (see, for example, [7, 13, 16, 11] etc.). Particularly, an L^s-regularity for some $s > 2$ was established by F. Jochmann in 1999 ([11]).

The paper is organized as follows. In Section 2, some basic spaces are reviewed and some elementary propositions are recalled. In Section 3, the regularity of weak solution is investigated. The precise estimates in Hölder space and $C^{1+\alpha}$-space are derived. Some concluding remarks are presented in Section 4.

2 Some Notation of Spaces and Elementary Properties

For the reader's convenience, we recall some basic Banach spaces related to Div and Curl for vector functions. A vector function $\mathbf{U}$ in $L^2(\Omega; R^3)$ or Sobolev space $H^1(\Omega; R^3) = W^{1,2}(\Omega; R^3)$ simply means that each component of $\mathbf{U}$ belongs to $L^2(\Omega)$ or $H^1(\Omega)$.

$$H(curl, div, \Omega) = \{\mathbf{U} \in L^2(\Omega) : Div\mathbf{U}, Curl\mathbf{U} \in L^2(\Omega)\}.$$

It is well-known that $H(Curl, Div, \Omega)$ is a Hilbert space equipped with the following inner product:

$$< \mathbf{U}, \mathbf{V} >= \int_\Omega [\mathbf{U} \cdot \mathbf{V} + (Curl\mathbf{U}) \cdot (Curl\mathbf{V}) + (Div\mathbf{U})(Div\mathbf{V})]dx.$$

In study of the div-curl system with a mixed boundary condition, we need the following spaces:

$$H_{\Gamma_1,0}(curl, div, \Omega) = \{\mathbf{U} \in H(curl, div, \Omega) : \mathbf{N} \times \mathbf{U} = 0 \,\text{on}\, \Gamma_1\};$$
$$H_{\Gamma_2,0}(curl, div, \Omega) = \{\mathbf{U} \in H(curl, div, \Omega) : \mathbf{N} \cdot \mathbf{U} = 0 \,\text{on}\, \Gamma_2\};$$
$$H_0(curl, div, \Omega) = H_{\Gamma_1,0}(curl, div, \Omega) \bigcap H_{\Gamma_2,0}(curl, div, \Omega).$$

Proposition 2.1([2]): Both $H_{\Gamma_1,0}(curl, div, \Omega)$ and $\mathrm{H}_{\Gamma_2,0}(curl, div, \Omega)$ are Hilbert spaces with the following inner product:

$$< \mathbf{U}, \mathbf{V} >= \int_\Omega [\mathbf{U} \cdot \mathbf{V} + (Curl\mathbf{U}) \cdot (Curl\mathbf{V}) + (Div\mathbf{U})(Div\mathbf{V})]dx.$$

Moreover, if Ω is simply connected, then $H_0(curl, div, \Omega)$ is a Hilbert space equipped with the inner product

$$< \mathbf{U}, \mathbf{V} >= \int_\Omega [(Curl\mathbf{U}) \cdot (Curl\mathbf{V}) + (Div\mathbf{U})(Div\mathbf{V})]dx.$$

It is well-known that the Morrey-John-Nirenberg-Campanato space plays an important role in study of regularity of solution for elliptic equations. We recall the definition here. The interested reader may find more details in [21].

Let $Z_0 = (x_0, y_0, z_0) \in \Omega \subset R^3$ and

$$B_\rho(Z_0) = \{Z = (x, y, z) \in \Omega : |Z - Z_0| < \rho\}.$$

The average of a function f over a ball $B_\rho(Z_0)$ is defined

$$(f)_{Z_0} = \frac{1}{|B_\rho(Z_0)|} \int_{B_\rho(Z_0)} f(Z)dZ.$$

Let $\mu > 0$. Define

$$[f]_{2,\mu,\Omega} = \left(\sup_{Z_0 \in \Omega, 0<\rho<\infty} \frac{1}{\rho^\mu} \int_{B_\rho(Z_0)} |f - (f)_{Z_0}|^2 dZ \right)^{\frac{1}{2}}.$$

The Morrey-John-Nirenberg-Campanato space, denoted by $L^{2,\mu}(\Omega)$, is defined

$$L^{2,\mu}(\Omega) = \{f \in L^2(\Omega) : ||f||_{2,\mu,\Omega} < \infty, \}$$

where

$$||f||_{2,\mu,\Omega} = ||f||_{L^2(\Omega)} + [f]_{2,\mu,\Omega}.$$

One should always use $B_\rho(Z_0) \bigcap \Omega$ to replace $B_\rho(Z_0)$ in the above definition when $B_\rho(Z_0)$ is not a subset of Ω..

$L^{2,\mu}(\Omega)$ is equivalent to the Morrey space if $\mu \in (0, 3)$, to the John-Nirenberg space if $\mu = 3$ and to the Campanato space if $\mu \in (3, 5)$. An interesting property is the following

Proposition 2.2: For $\mu \in (3, 5)$, the space $L^{2,\mu}(\Omega)$ is isomorphic algebraically and topologically to the classical Hölder space $C^\alpha(\bar{\Omega})$ with $\alpha = \frac{\mu-3}{2}$.

Let us also recall the definition of Betti numbers for a domain in R^3.

Definition 2.3: For a bounded domain $\Omega \in R^3$ with boundary $\partial\Omega$, the first Betti number of Ω is the minimum number of cuts such that after cuts, the domain is simply connected. The second of the Betti number of Ω is the number of connected components $\partial\Omega$ except one component which connects infinity. Namely,

$$\partial\Omega = \sum_{i=0}^m \Gamma_i,$$

each Γ_i is connected, $0 \leq i \leq m$, $\Gamma_i \bigcap \Gamma_j =$ if $i \neq j$,Γ_0 is the boundary $\Omega^c = R^3 \backslash \Omega$.

Proposition 2.4([13]): (a) The first Betti number N is equal to the dimension of the space $H_{1,0}(Curl, Div, \Omega)$:

$$H_{1,0}(Curl, Div, \Omega) = \{\mathbf{U} \in L^2(\Omega) : Div\mathbf{U} = 0, Curl\mathbf{U} = 0, x \in \Omega, \mathbf{N} \cdot \mathbf{U} = 0, x \in \partial\Omega.\}$$

(b) The second Betti number m is equal to the dimension of the space $H_{2,0}(Curl, Div, \Omega)$:

$$H_{2,0}(\Omega) = \{\mathbf{U} \in L^2(\Omega) : Div\mathbf{U} = 0, Curl\mathbf{U} = 0, x \in \Omega, \mathbf{N} \times \mathbf{U} = 0, x \in \Omega\}.$$

Moreover, $H_{2,0}(\Omega)$ can be characterized as a gradient field:

$$H_{2,0}(\Omega) = \{\mathbf{U} = grad\psi : \Delta\psi = 0, \psi = constant_i \,\text{on}\, \partial\Omega, i = 1, 2, \cdots, m\}.$$

A well-known fact is that $H_{1,0}(curl, div, \Omega)$ and $H_{2,0}(curl, div, \Omega)$ are trivial space $m = n = 0$ if Ω is simply connected domain in R^3(see [13]).

Other classical Sobolev spaces such as $W^{k,p}$ and $H^1(\Omega) = W^{1,2}$ are as usual ([21]).

3 Regularity of Weak Solution

We begin with some basic assumptions for the known data.

H(3.1): Let $\mathbf{F} \in H(Curl, Div, \Omega)$ and $g \in L^2(\Omega)$ with

$$Div\mathbf{F} = 0, \qquad \text{in}\, \Omega$$

in the weak sense. Namely,

$$\int_\Omega (\nabla \cdot \mathbf{F})\psi dx = 0$$

for any $\psi \in C_0^\infty(\Omega)$.

H(3.2): Let $\mathbf{F} = (f_1, f_2, f_3), g \in L^{2,\mu}(\Omega)$ with some $\mu \in (1, 3)$.

H(3.3): $\partial\Omega = \bar{\Gamma}_1 \bigcup \bar{\Gamma}_2$ with $dist\{\Gamma_1, \Gamma_2\} > 0$.

Theorem 3.1: Under the hypotheses H(3.1)-H(3.3), the weak solution $\mathbf{V} \in H_0(Div, Curl, \Omega)$ of the system (1.1)-(1.3) is Hölder continuous on $\bar{\Omega}$. Moreover,

$$||\mathbf{V}||_{C^\alpha(\bar{\Omega})} \leq C[||\mathbf{F}||_{2,\mu,\Omega} + ||g||_{2,\mu,\Omega} + ||\mathbf{V}||_{H^1(\Omega)}], \tag{3.1}$$

where $\alpha = \frac{\mu-1}{2}$, C depends only on μ and Ω.

Proof: Since we will derive apriori estimates of solution in terms of known functions $\mathbf{F}$ and g, without loss of generality, we may assume that $\mathbf{V}$ is smooth in $\bar{\Omega}$.

First of all, we apply the curl operator to the Curl-system (1.1) to obtain:

$$\nabla \times \nabla \times \mathbf{V} = \nabla \times \mathbf{F}, \qquad (x, y, z) \in \Omega.$$

Note that

$$\nabla \times \nabla \times \mathbf{V} = -\Delta \mathbf{V} + \nabla(\nabla \cdot \mathbf{V}),$$

by using Eq.(1.2) we find that $\mathbf{V} = (v_1, v_2, v_3)$ satisfies the following elliptic system:

$$-\Delta \mathbf{V} = \nabla \times \mathbf{F} - \nabla g, \qquad (x, y, z) \in \Omega. \tag{3.2}$$

Let $\delta > 0$ and

$$\Omega_\delta = \{Z = (x, y, z) \in \Omega : dist(Z, \partial\Omega) > \delta\}.$$

By using interior estimate for scalar elliptic equations ([21]), we see that there exists a constant C such that

$$\sum_{i=1}^{3} ||\nabla v_i||_{2,\mu,\Omega_\delta} \leq C[||\mathbf{F}||_{2,\mu,\Omega} + ||g||_{2,\mu,\Omega} + ||\mathbf{V}||_{H^1(\Omega)}],$$

where C depends only on μ, δ and diameter of Ω.

It follows by Poincare's inequality that

$$\sum_{i=1}^{3} ||v_i||_{2,\mu+2,\Omega_\delta} \leq C[||\mathbf{F}||_{2,\mu,\Omega} + ||g||_{2,\mu,\Omega} + ||\mathbf{V}||_{H^1(\Omega)}],$$

which yields the Hölder's estimate by Proposition 2.1:

$$\sum_{i=1}^{3} ||v_i||_{C^\alpha(\bar{\Omega}_\delta)} \leq C[||\mathbf{F}||_{2,\mu,\Omega} + ||g||_{2,\mu,\Omega} + ||\mathbf{V}||_{H^1(\Omega)}],$$

where C depends only μ, δ and Ω.

The C^α-estimate for $\mathbf{V}$ near $S = \partial\Omega$ is much more complicated. To illustrate the idea, we begin with a simple case. Let $P(x_0, y_0, z_0) \in S$. Assume that in a neighborhood $N_r(P)$ of P with $r > 0$ sufficiently small, S is flat. Without loss of generality, we may assume that

$$N_r(P) \bigcap \partial\Omega = \{(x, y, z) \in \partial\Omega \bigcap N_r(P) : z = d > 0.\}$$

Moreover, the domain Ω is located below the hyperplane $z = d$. Then the outward unit normal $\mathbf{N}$ on $N_r(x_0, y_0, z_0) \bigcap \partial\Omega$ is equal to $\mathbf{N} =< 0, 0, 1 >$.

Case 1: $P \in \Gamma_1$.

From the boundary condition (1.3), we see

$$v_1(x, y, d) = v_2(x, y, d) = 0, \qquad (x, y, z) \in N_r(P) \bigcap \Gamma_1.$$

Now with the boundary condition for $v_1 = v_2 = 0$ in $N_r(P) \bigcap \Gamma_1$ we can apply the regularity theory for scalar elliptic equations (3.2) to conclude that $v_1(x, y, z), v_2(x, y, z)$ are Hölder continuous in the neighborhood of $N_0 = N_{\frac{r}{2}}(P) \bigcap \Omega$ ([21], Theorem 2.19). Moreover, there exists a constant C such that

$$\sum_{i=1}^{2} ||v_i||_{C^\alpha(\bar{N_0})} \leq C[||\mathbf{F}||_{2,\mu,N_r(P)\bigcap\Omega} + ||g||_{2,\mu,N_r(P)\bigcap\Omega} + ||\mathbf{V}||_{H^1(\Omega)}],$$

where $\alpha = \frac{\mu-1}{2}$ and C depends only on r and μ.

To obtain the regularity for v_3 in N_0, we note from Eq.(1.2) that

$$\nabla \cdot \mathbf{V} = g(x, y, z), \qquad \text{in} \qquad \Omega.$$

Since $\mathbf{V}$ is assumed to be smooth up to the boundary $\partial\Omega$, we find

$$v_{1x} + v_{2y} + v_{3z} = 0, \qquad \text{on}\, \Gamma_1 \bigcap N_r(P).$$

Since $v_1(x, y, d) = v_2(x, y, d) = 0$ on $\Gamma_1 \cap N_r(P)$, it follows that

$$v_{3z}(x, y, d) = 0, \qquad \text{on}\, \Gamma_1 \bigcap N_r(P).$$

Again, we use the regularity theory for elliptic equations with a Neumann boundary condition ([21]) to conclude the v_3 is Hölder continuous in $N_r(P) \cap \Omega$. Moreover, we have the following estimate:

$$||v_3||_{C^\alpha(\bar{N_0})} \leq C[||\mathbf{F}||_{2,\mu,N_r(P)\bigcap\Omega} + ||g||_{2,\mu,N_r(P)\bigcap\Omega} + ||\mathbf{V}||_{H^1(\Omega)}],$$

where C depends only on r and μ.

Case 2: $P \in \Gamma_2$.

From the boundary condition (1.3): $\mathbf{N} \cdot \mathbf{V} = 0$ on Γ_2, we see

$$v_3(x, y, d) = 0, \qquad (x, y, z) \in N_r(P) \bigcap \Gamma_2.$$

We use the same method as above for v_1 and v_2 in case 1 to conclude that v_3 is Hölder continuous in $N_{\frac{r}{2}}(P) \cap \Omega$. Moreover,

$$||v_3||_{C^\alpha(\bar{N_0})} \leq C[||\mathbf{F}||_{2,\mu,\Omega} + ||g||_{2,\mu,\Omega} + ||\mathbf{V}||_{H^1(\Omega)}],$$

where $\alpha = \frac{\mu-1}{2}$ and C depends only on μ and $N_r(P) \cap \Omega$.

To obtain the regularity for v_1 and v_2, we examine the system (1.1)-(1.2). Since $\mathbf{V}$ is assumed to be smooth in $\bar{\Omega}$, from the system (1.1) we see that $\mathbf{V}$ satisfies system (1.1) up to the boundary $\partial\Omega$. Hence, on $\partial\Omega$, we have

$$\begin{aligned} v_{3y} - v_{2z} &= f_1(x, y, z), \qquad && (x, y, z) \in S, \\ v_{1z} - v_{3x} &= f_2(x, y, z), \qquad && (x, y, z) \in S, \\ v_{2x} - v_{1y} &= f_2(x, y, z), \qquad && (x, y, z) \in S. \end{aligned}$$

On the other hand, since the boundary $\partial\Omega \cap N_r(P)$ is flat and $v_3(x, y, d) = 0$ on this part of the boundary, we see

$$v_{3x}(x, y, d) = v_{3y}(x, y, d) = 0, \qquad \text{on}\, N_r(P) \bigcap \Gamma_2.$$

It follows that

$$v_{1z}(x, y, d) = f_2, v_{2z}(x, y, d) = -f_1, \qquad \text{on}\, N_r(P) \bigcap \Gamma_2.$$

Now we can use the regularity theory of elliptic equations ([12]) to conclude that v_1 and v_2 are Hölder continuous in $N_0 := N_{\frac{r}{2}}(P) \bigcap \Omega$. Moreover, there exists a constant C such that

$$||\nabla v_i||_{2,\mu,N_0} \leq C[||\mathbf{F}||_{2,\mu,\Omega} + ||g||_{2,\mu,\Omega} + ||\mathbf{V}||_{H^1(\Omega)}],\ i = 1, 2,$$

where C depends only on r and μ.

Thus, by using Poincare's inequality and Proposition 2.1, we have

$$||v_i||_{C^\alpha(\bar{N}_0)} \leq C[||\mathbf{F}||_{2,\mu,\Omega} + ||g||_{2,\mu,\Omega} + ||\mathbf{V}||_{H^1(\Omega)}],\ i = 1, 2,$$

where $\alpha = \frac{\mu-1}{2}$ and C depends only on r and μ.

Now we consider the general case where $N_r(P) \bigcap \partial\Omega$ is not flat. Since $\partial\Omega \in C^2$ we assume that the boundary $N_r(P) \bigcap \partial\Omega$ can be expressed by

$$z = \phi(x, y), (x, y) \in N_r(P) \bigcap \partial\Omega,$$

where $\phi(x, y)$ is of class C^2 for both variables.

Now introduce new variables:

$$x' = x,\ y' = y,\ z' = z - \phi(x, y),$$

Under the new variables (x', y', z'), the boundary $N_r(P) \bigcap \Omega$ becomes flat:

$$z' = 0, (x', y', z') \in N_0^*(r) := N_r(P)^* \bigcap \Omega^*,$$

where $N_r(P)^* \bigcap \Omega^*$ represents the boundary $N_r(P) \bigcap \Omega$ under the new coordinate.

Define

$$\mathbf{U}(x', y', z') = \mathbf{V}(x, y, z), (x, y, z) \in N_r(P) \bigcap \Omega.$$

Then,

$$\mathbf{U}_{x'} = \mathbf{V}_x + \mathbf{V}_z \phi_x, \mathbf{U}_{y'} = \mathbf{V}_y + \mathbf{V}_z \phi_y.$$

For convenience, we assume that $N_0^*(r)$ is located on the upper part of the boundary $z' = 0$.

Then the unit normal on $N_r(P) \bigcap \Omega$ is

$$\mathbf{N} = \left\{ \frac{\phi_x}{\sqrt{1 + \phi_x^2 + \phi_y^2}}, \frac{\phi_y}{\sqrt{1 + \phi_x^2 + \phi_y^2}}, \frac{-1}{\sqrt{1 + \phi_x^2 + \phi_y^2}} \right\}.$$

The boundary condition

$$\mathbf{N} \times \mathbf{V} = 0, \qquad \text{on } \Gamma_1$$

is equivalent to

$$v_1 = -\phi_x v_3,\ v_2 = -\phi_y v_3, \qquad (x, y, z) \in \Gamma_1, \tag{3.3}$$

while the boundary condition

$$\mathbf{N} \cdot \mathbf{V} = 0, \qquad \text{on } \Gamma_2$$

is equivalent to

$$v_3 = \phi_x v_1 + \phi_y v_2,\ (x,y,z) \in \Gamma_2. \tag{3.4}$$

In terms of new variables, we see $\mathbf{U} = (u_1, u_2, u_3)$ satisfies

$$u_1 = -\phi_x u_3, u_2 = -\phi_y u_3, \qquad (x',y',z') \in \Gamma_1^* \bigcap N_r(P)^*, \tag{3.5}$$

$$u_3 = \phi_x u_1 + \phi_y u_2, \qquad (x',y',z') \in \Gamma_2^* \bigcap N_r(P)^*. \tag{3.6}$$

We first consider the regularity of $\mathbf{U}(x',y',z')$ in a neighborhood of $P^* \in \Gamma_1^*$, denoted by $N_0^*(r) := N_r(P)^* \bigcap \Omega^*$.

Since $\mathbf{V}$ is smooth up to the boundary $\partial\Omega$, we have

$$\nabla \cdot \mathbf{V} = v_{1x} + v_{2y} + v_{3z} = g, (x,y,z) \in \partial\Omega.$$

We see

$$v_{3z} = g - v_{1x} - v_{2y}, \qquad (x,y,z) \in \partial\Omega,$$

which is equivalent in the new coordinate to

$$u_{3z'} = g - [u_{1x'} - u_{1z'}\phi_{x'}] - [u_{2y'} - u_{2z'}\phi_{y'}].$$

Note that on the boundary $z' = 0$,

$$u_{1x'} = \phi_{xx}u_3 + \phi_x[u_{3x'} - u_{3z'}\phi_x], u_{2y} = \phi_{yy}u_3 + \phi_y[u_{3y'} - u_{3z'}\phi_y].$$

It follows that

$$\begin{aligned}&(1 - \phi_x^2 - \phi_y^2)u_{3z'} + \phi_x u_{3x'} + \phi_y u_{3y'} + (\phi_{xx} + \phi_{yy})u_3 \\ &= 0, \qquad (x',y',z') \in N_p^* \bigcap \partial\Omega^*.\end{aligned}$$

From Eq. (3.2), we know that $\mathbf{V}$ satisfies

$$\Delta\mathbf{V} = \nabla \times \mathbf{F} - \nabla g, \qquad (x,y,z) \in \Omega.$$

Since

$$\mathbf{V}(x,y,z) = \mathbf{U}(x',y',z') = \mathbf{U}(x,y,z-\phi(x,y)),$$

it is easy to see that $\mathbf{U}$ satisfies an elliptic equation in divergence form (see [21], page 124), where the coefficients are differentiable since $z = \phi(x,y)$ is of class C^2.

It follows by the elliptic theory with a general Neumann-type of boundary condition ([12]) to conclude that $u_3(x',y',z')$ is Hölder continuous in $N_0^*(\frac{r}{2}) := N_{\frac{r}{2}}(P)^* \bigcap \Omega^*$. Moreover, by using $L^{2,\mu}$-theory for elliptic equations (see [21]) that

$$||\nabla u_3||_{2,\mu,N_0^*(\frac{r}{2})} \le C[||\mathbf{F}||_{2,\mu,N_0^*(r)} + ||g||_{2,\mu,N_0^*(r)} + ||\mathbf{U}||_{H^1(\Omega)}],$$

where C depends only on r, $W^{1,\infty}$-norm of ϕ and $1 < \mu < 3$. By Poincare's inequality, we see

$$||u_3||_{2,\mu+2,N_0^*(\frac{r}{2})} \le C[||\mathbf{F}||_{2,\mu,N_0^*(r)} + ||g||_{2,\mu,N_0^*(r)} + ||\mathbf{U}||_{H^1(\Omega)}].$$

Proposition 2.1 yields that

$$||u_3||_{C^\alpha(\bar{N}_1(\frac{r}{2}))} \leq C[||\mathbf{F}||_{2,\mu,N_0^*(r)} + ||g||_{2,\mu,N_0^*(r)} + ||\mathbf{U}||_{H^1(\Omega)},]$$

where C depends only on r and μ.

To derive the Hölder estimate for u_1 and u_2, we use the boundary condition (3.5):

$$u_1 = \phi_x u_3, \qquad u_2 = \phi_y u_3, \qquad \text{on } N_r(P)^* \bigcap \partial\Omega^*$$

Since u_3 is Hölder continous and ϕ_x, ϕ_y are differentiable, we use regularity theory for elliptic equations to conclude that u_1 and u_2 are Hölder continuous in $N_0^*(\frac{r}{2})$. Moreover,

$$\begin{aligned}&||u_1||_{C^\alpha(\bar{N}_0^*(\frac{r}{2})} + ||u_2||_{C^\alpha(\bar{N}_0^*(\frac{r}{2})}\\ &\leq C[||\mathbf{F}||_{2,\mu,N_1(r)} + ||g||_{2,\mu,N_1(r)} + ||\mathbf{U}||_{H^1(\Omega)},]\end{aligned}$$

where C depends only on r and μ.

We now consider the regularity of $\mathbf{U}$ where $P^* \in \Gamma_2$. Note that

$$\mathbf{N}\cdot\mathbf{V} = 0, \qquad \text{on} \qquad \Gamma_2$$

is equivalent to

$$v_3 = v_1\phi_x + v_2\phi_y, \qquad \text{on} \qquad \Gamma_2.$$

It follows that on Γ_2

$$\begin{aligned}v_{3x} &= v_{1x}\phi_x + v_1\phi_{xx} + v_{2x}\phi_y + v_2\phi_{xy},\\ v_{3y} &= v_{1y}\phi_x + v_1\phi_{xy} + v_{2y}\phi_y + v_2\phi_{yy}.\end{aligned}$$

On the other hand, since $\mathbf{V}$ is differentiable up to the boundary of Ω, from the system (1.1) we see that on Γ_2

$$v_{1z} = f_2 + v_{3x} = f_2 + v_{1x}\phi_x + v_1\phi_{xx} + v_{2x}\phi_y + v_2\phi_{xy}, \tag{3.7}$$

$$v_{2z} = -f_1 + v_{3y} = -f_1 + v_{1y}\phi_x + v_1\phi_{xy} + v_{2y}\phi_y + v_2\phi_{yy}, \tag{3.8}$$

$$v_{2x} - v_{1y} = f_3(x). \tag{3.9}$$

We use (3.7)-(3.8) to obtain

$$(-\phi_x)v_{1x} + (-\phi_y)v_{1y} + v_{1z} = f_2 + v_1\phi_{xx} + v_2\phi_{xy} + f_3\phi_y, \text{on}\,\Gamma_2; \tag{3.10}$$

$$(-\phi_x)v_{2x} + (-\phi_y)v_{2y} + v_{2z} = -f_1 + v_1\phi_{xy} + v_2\phi_{yy} + f_3\phi_x, \text{on}\,\Gamma_2. \tag{3.11}$$

We know that the weak solution $\mathbf{V} \in H(curl, div, \Omega)$, it follows by trace estimate that

$$||v_i||_{L^4(\partial\Omega)} \leq C||\mathbf{V}||_{H^1(\Omega)}, \qquad i = 1, 2.$$

The boundary conditions (3.10)-(3.11) can be written as

$$\begin{aligned}\nabla_{\mathbf{N}} v_1 &= h_1(x,y,z), \qquad (x,y,z) \in N_0(r)\bigcap\Gamma_2,\\ \nabla_{\mathbf{N}} v_2 &= h_2(x,y,z), \qquad (x,y,z) \in N_0(r)\bigcap\Gamma_2,\end{aligned}$$

where

$$h_1(x,y,z) = \frac{f_2 + v_1\phi_{xx} + v_2\phi_{xy} + f_3\phi_y}{\sqrt{1+\phi_x^2+\phi_y^2}},$$
$$h_2(x,y,z) = \frac{-f_1 + v_1\phi_{xy} + v_2\phi_{yy} + f_3\phi_x}{\sqrt{1+\phi_x^2+\phi_y^2}},$$

and

$$||h_1||_{L^4(\partial\Omega)} + ||h_2||_{L^4(\partial\Omega)} \leq C[||\mathbf{F}||_{L^4(\Omega)} + ||\mathbf{V}||_{H^1(\Omega)}].$$

Now we can apply the result of [12] to conclude that v_1 and v_2 are Hölder continuous on $N_0(r)$. Moreover,

$$||v_1||_{C^\alpha(\bar{N}_0(\frac{r}{2}))} + ||v_2||_{C^\alpha(\bar{N}_0(\frac{r}{2}))} \leq C[||\mathbf{F}||_{2,\mu,N_0(r)} + ||g||_{2,\mu,N_0(r)}],$$

where C depends only on $W^{2,\infty}$-norm of $\phi(x,y)$ and r.

The boundary condition (1.3) yields that v_3 is also Hölder continuous in $N_0(r)$ and

$$||v_3||_{C^\alpha(\bar{N}_0(\frac{r}{2}))} \leq C[||\mathbf{F}||_{2,\mu,N_0(r)} + ||g||_{2,\mu,N_0(r)}],$$

where C has the same dependency as that for v_1 and v_2.

Since $\partial\Omega$ is compact, we can use a finite covering and a partition of unit ([21], page 125) to obtain the C^α-global estimates (3.1) and (3.2) for $\mathbf{V}$ on $\bar{\Omega}$.

Q.E.D.

If we assume more regularity on known data, then we can derive $C^{1+\alpha}$-estimate for the weak solution of (1.1)-(1.3).

H(3.4): Let $\mathbf{F}, g \in W^{2,p}(\Omega)$ with $p > 3$. Moreover, the boundary $S \in C^{3+\alpha}$.

Theorem 3.2: Under the hypotheses H(3.1)-H(3.4), the weak solution $\mathbf{V}$ of the system (1.1)-(1.3) is of class $C^{1+\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. Moreover,

$$||\mathbf{V}||_{C^{1+\alpha}(\bar{\Omega})} \leq C[\sum_{i=1}^{3} ||\nabla f_i||_{W^{1,p}(\Omega)} + ||\nabla g||_{W^{1,p}(\Omega)} + ||\mathbf{V}||_{H^1(\Omega)}], \tag{3.12}$$

where $\alpha = \frac{p-3}{2}$, C depends only on p and Ω.

The proof of Theorem 3.2 follows the same idea as that for Theorem 3.1. With the assumption H(3.3) we can derive the similar estimate in $W^{1,p}$-space for $\mathbf{W}_i = \nabla v_i, i = 1,2,3$. We shall not repeat those steps here.

Q.E.D.

4 Some Applications and Concluding Remarks

As an application, we derive an C^α-estimate of $\mathbf{V}$ in terms of its divergence and its curl. For convenience, instead using (x,y,z) we use (x_1,x_2,x_3) to represent a point in R^3.

Theorem 4.1: Suppose Ω is a simply-connected and bounded domain in R^3, either $\mathbf{N} \times \mathbf{V} = 0$ or $\mathbf{N} \cdot \mathbf{V} = 0$ on $\partial\Omega$. Then

$$||\mathbf{V}||_{C^{\alpha}(\bar{\Omega})} \leq C[||Curl\mathbf{V}||_{L^{2,\mu}(\Omega)} + ||Div\mathbf{V}||_{L^{2,\mu}(\Omega)}], \tag{4.1}$$

$$||\mathbf{V}||_{C^{1+\alpha}(\bar{\Omega})} \leq C[\sum_{i=1}^{3}[||Curl\mathbf{V}_{x_i}||_{L^{2,\mu}(\Omega)} + ||Div\mathbf{V}_{x_i}||_{L^{2,\mu}(\Omega)}] \tag{4.2}$$

where $\mu \in (1, 3)$ and C depends only on Ω and μ.

Proof: when Ω is simply connected and bounded in R^3 with either $\mathbf{N}\times\mathbf{V} = 0$ or $\mathbf{N}\cdot\mathbf{V} = 0$ on $\Gamma = \partial\Omega$, then by [10] we have

$$||\mathbf{V}||_{H^1(\Omega)} \leq C[||Curl\mathbf{V}||_{L^2(\Omega)} + ||Div\mathbf{V}||_{L^2(\Omega)}],$$

where C depends only on Ω.

Since $L^{2,\mu}(\Omega)$ is a subspace of $L^2(\Omega)$, by Theorem 3.1 we obtain the inequality (4.1). As for the estimate (4.2), we just take derivative with respect to x_i and then for $\mathbf{V}_{x_i}, i = 1, 2, 3$, we apply Theorem (3.2) to conclude the desired estimate (4.2).

Q.E.D.

Remark 4.1: For electromagnetic fields in an anisotropic medium, one often encounters a Div-Curl system with a weight:

$$\nabla \times \mathbf{V} = \mathbf{F}, \qquad \text{in} \qquad \Omega, \tag{4.3}$$

$$\nabla \cdot (A\mathbf{V}) = g, \qquad \text{in} \qquad \Omega, \tag{4.4}$$

where $A = (a_{ij}(x))_{3\times 3}$ is a positive-definite matrix. It would be of a great interest to derive similar estimates as the inequality (1.3) where $Div\mathbf{V}$ is replaced by $Div(A\mathbf{V})$. Along this direction, L^s-regularity for some $s > 2$ was obtained by F. Jochmann in [11]. However, the Hölder regularity of the weak solution is still a open question.

Acknowledgement: The author would like to thank Professor F. Jochmann and Professor Giles Auchmuty for many discussions during the preparation of the paper. The part of work is done when the author is visiting the Chinese University of Hong Kong. His deep gratitude is given to the Department of Mathematics at Chinese University of Hong Kong for its hospitality and excellent working environment.

References

[1] C. Amrouche and H.H. Nguyen, New estimates for the div-curl-grad operators and elliptic problems with L^1-data in the whole space and in the half-space, J. of Differential Equations, 250(2011), 3150-3195.

[2] G. Auchmuty, The main inequality of 3D vector analysis, Math. Models Methods Appl. Sci. 14 (2004), no. 1, 79-103.

[3] G. Auchmuty and J. C. Alexander, L^2-well-posedness of 3D div-curl boundary value problems, Quarterly of Applied Math., 63(2005), 479-508.

[4] G. Auchmuty and J. C. Alexander, Finite energy solutions of mixed 3D div-curl systems, Quarterly of Applied Math., 64(2006), 335-357.

[5] J. Bolik and W. von Wahl, Estimating ∇u in terms of $divu, curlu$, either (ν, u) or $\nu \times u$ and the topology, Mathematical Methods in the Applied Sciences, 20(1997), 737-744.

[6] J. Bourgain and H. Brezis, New estimates for the Laplacian, the div-curl and related Hodge systems, C.R. Math. Acad. Sci. Paris, 338(2005), 539-543.

[7] D. Costable, A coercive bilinear form for Maxwell's equations, J. Math. Anal. Appl., 157(1991), 527-541.

[8] R. Dautray and J.L. Lions, Mathematical Analysis and Numeical Methods for Sciences and Technology, vol. 3, Spring-Verlag, New York, 1990.

[9] Michael Neudert and Wolf von Wahl Asymptotic behaviour of the div-curl problem in exterior domains. (English summary) Adv. Differential Equations, 6(2001), no. 11,1347-1376.

[10] F. Jochmann, A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions, Applicable Analysis, 66(1997), 189-203.

[11] F. Jochmann, Regularity of weak solutions of Maxwell's equations with mixed boundary-conditions. Math. Methods Appl. Sci. 22 (1999), no. 14, 1255-1274.

[12] Gary M. Lieberman, Oblique derivative problems for elliptic equations, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[13] C. Hazard and M. Lenoir, On the solution of time-Harmonic scattering problems for Maxwell's equations, SIAM J. Math. Anal., 27(1996), 1597-1630.

[14] H. Kozono and T. Yanagisawa, L^r-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domain, Indiana University Math. Journal, 58(2009), 1853-1920.

[15] L. D. Landau and E.M. Lifshitz, Electrodynamics of Continuous Media, Pergamon Press, New York, 1960.

[16] R. Leis, Initial Boundary Value Problems in Mathematical Physics, John Wiley & Sons, New York, 1986.

[17] Dorina Mitrea, Integral Equation Methods for Div-Curl problems for planar vector fields in nonsmooth domains, Differential and Integral Equations, 18(2005),1039-1054.

[18] Irina Mitrea and Marius Mitrea, A remark on the regularity of the Div-Curl system, Proceedings of the American Mathematical society, 137(2009), 1729-1733.

[19] Michael Neudert and W. von Wahl, Asymptotic behaviour of the div-curl problem in exterior domains. Adv. Differential Equations 6 (2001), no. 11, 1347–1376.

[20] A.C. Metaxas, Foundations of Electroheat, a unified approach, John Wiley & Sons, New York, 1996.

[21] G. M. Troianiello, Elliptic Differential Equations and Obstacle Problems, Plenum Press, New York, 1987.

[22] W. von Wahl, Estimating ∇u by $divu$ and $curlu$, Math. Methods in Applied Sciences, 15(1992), 123-143.

[23] Xingfei Xiang, $L^{3/2}$-estimates of vector fields with L^1-curl in a bounded domain, Calculus Variation and Partial Differential Equations, 46(2013), 55-74.

[24] H.M. Yin, Regularity of Solutions to Maxwell's System in Quasi-Stationary Electromagnetic Fields and Applications, Communication in P.D.E., 22 (1997), No. 7 & 8, 1029-1053

[25] H.M. Yin, Regularity of Weak Solution to Maxwell's Equations and Applications to Microwave Heating, Journal of Differential Equations, 200(2004), 137-161.

[26] H.M. Yin and W. Wei,Regularity of weak solution for a coupled system arising from a microwave heating model. European Journal of Applied Mathematics, 24(2013), No.6, 117-131.