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CONVERGENCE OF NUMERICAL ALGORITHM FOR APPROXIMATING OPTIMAL CONTROL PROBLEMS OF PHASE FIELD SYSTEM WITH SINGULAR DIFFUSIVITY

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Abstract. We consider optimal control problems for one-dimensional phase field system with the singular diffusivity and the constraint. Our system consists of two parabolic PDEs: a heat equation and a singular kinetic equation for a nonconserved order parameter. Recently, we showed the necessary condition of the optimal pair by using the approximating optimal control problem. Due to the singular diffusivity and the constraint we observe that it is very hard to study the original control problem numerically. Therefore, in this paper we study the approximating optimal control problem from the view-point of numerical analysis. Then, we propose the numerical scheme to find the optimal control of the approximating problem, and show the convergence of the numerical algorithm proposed in this paper.

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1 Introduction

We consider the following phase field system with the singular diffusivity and the constraint:

$$[u+w]_t - u_{xx} = a_0 f(t,x) \quad \text{in } Q := (0,T)\times(0,L), \tag{1.1}$$

$$w_t - \kappa\Big(\frac{w_x}{|w_x|}\Big)_x + \partial I_{[-1,1]}(w) + \nu w^3 - w \ni u \quad \text{in } Q, \tag{1.2}$$

$$-u_x(t,0) + n_0(u(t,0)-b_1) = a_1 h(t), \quad t\in(0,T), \tag{1.3}$$

$$u_x(t,L) + n_0(u(t,L)-b_2) = a_2 \ell(t), \quad t\in(0,T), \tag{1.4}$$

$$w_x(t,0) = w_x(t,L) = 0, \quad t\in(0,T), \tag{1.5}$$

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x), \quad x\in(0,L), \tag{1.6}$$

where $0<T<\infty$ and $0<L<\infty$ are fixed positive constants, $\kappa>0$, $n_0>0$, $\nu\geq 0$, a_0, a_1, a_2, b_1, b_2 are given constants, f, h, ℓ are given functions, and u_0, w_0 are given initial data. Also, $\partial I_{[-1,1]}(\cdot)$ is the subdifferential of the indicator function $I_{[-1,1]}(\cdot)$ on the closed interval $[-1,1]$ defined by

$$I_{[-1,1]}(z) := \begin{cases} 0, & \text{if } z\in[-1,1], \\ \infty, & \text{otherwise.} \end{cases} \tag{1.7}$$

The system (P):={(1.1), (1.2), (1.3), (1.4), (1.5), (1.6)} is based on a mathematical model of solid-liquid phase transitions in a mesoscopic length scale proposed by Visintin [33]. In the physical context, the unknown function $u=u(t,x)$ is the relative temperature, and $w=w(t,x)$ is the nonconserved order parameter that indicates the physical phase of material. Note that the equation (1.2) is derived as the L^2-gradient flow of the free energy functional as follows:

$$\mathscr{F}_u(w) := \kappa\int_0^L |Dw| + \int_0^L \Big\{ I_{[-1,1]}(w) + \frac{\nu w^4}{4} - \frac{w^2}{2} - wu \Big\} dx, \quad w\in L^2(0,L),$$

where $\int_0^L |Dw|$ is the total variation of a function $w\in L^2(0,L)$. So, we can regard (1.2) as one kind of mathematical formulation of Gibbs-Thomson law.

Many mathematicians studied the singular diffusion equation (1.2) with or without constraint $\partial I_{[-1,1]}(w)$ (cf. [3, 4, 5, 12, 13, 17, 18, 20, 23, 24, 28, 30, 33]). For instance, Kenmochi–Shirakawa [17, 18] studied the detailed structure of steady-state solution and the characterization of asymptotics of solutions to (1.2) in one dimensional case of the spatial domain. For higher dimensional case, we refer to Shirakawa–Kimura [30]. Recently, Ohtsuka–Shirakawa–Yamazaki [23, 24, 25] considered the optimal control problem of (1.2) with respect to the temperature control u.

The system (P) was considered by Kenmochi–Shirakawa [19] and Shirakawa [29]. In particular, Kenmochi–Shirakawa [19] dealt with the structural and stability analysis for the steady-state solutions to (P).

Also, there is a vast amount of literature on optimal control problems to phase transitions. For instance, we refer to [1, 8, 10, 14, 26, 27, 32]. In particular, the authors [31] investigated optimal control problems of phase field system (P) with the singular diffusivity arising from the total variation $\int_0^L |Dw|$. In fact, the following optimal control problem (OP) was considered in [31]:

Problem (OP). Find a pair of functions (optimal control) $(f_*, h_*, \ell_*) \in \mathcal{U}_{ad}$ such that

$$J(f_*, h_*, \ell_*) = \inf_{(f,h,\ell)\in\mathcal{U}_{ad}} J(f, h, \ell).$$

Here, $\mathcal{U}_{ad} := L^2(0, T; L^2(0, L)) \times L^2(0, T) \times L^2(0, T)$ is the control space, and $J(f, h, \ell)$ is the cost functional defined by

$$\begin{aligned} J(f, h, \ell) := \frac{c_0}{2}\int_0^T |(u - u_d)(t)|^2_{L^2(0,L)}dt + \frac{c_1}{2}\int_0^T |(w - w_d)(t)|^2_{L^2(0,L)}dt \\ + \frac{m_0}{2}\int_0^T a_0^2|f(t)|^2_{L^2(0,L)}dt + \frac{m_1}{2}\int_0^T a_1^2|h(t)|^2dt + \frac{m_2}{2}\int_0^T a_2^2|\ell(t)|^2dt, \end{aligned} \tag{1.8}$$

where $(f, h, \ell) \in \mathcal{U}_{ad}$ is the control, a couple of functions (u, w) is a unique solution to the state problem (P) with the source term (f, h, ℓ), c_0, c_1, m_0, m_1, m_2 are nonnegative constants, and u_d, w_d are the given desired target profiles in $L^2(0, T; L^2(0, L))$.

Note that if $a_0 = 0$, then (OP) is the boundary valued control problem. Similarly, if the constant $a_1 = a_2 = 0$, then (OP) is the heat source control problem. Also, note that b_1 (resp., b_2) denotes the outside temperature at $x = 0$ (resp. $x = L$).

From the singular diffusion term and the constraint $\partial I_{[-1,1]}(w)$ in (1.2), we observe that it is very hard to show the necessary condition of (OP). Therefore, considering approximating problems of (P) and (OP), the necessary condition of (OP) was proved in [31, Theorem 3.5]. Indeed, for each $\varepsilon \in (0, 1]$, the following approximating problem of (OP), denoted by $(\mathrm{OP})^\varepsilon$, was considered (cf. [31, Theorem 3.2]):

Problem (OP)$^\varepsilon$. Find a pair of functions (optimal control) $(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) \in \mathcal{U}_{ad}$ such that

$$J^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) = \inf_{(f,h,\ell)\in\mathcal{U}_{ad}} J^\varepsilon(f, h, \ell).$$

Here, $J^\varepsilon(f, h, \ell)$ is the cost functional defined by

$$\begin{aligned} J^\varepsilon(f, h, \ell) := \frac{c_0}{2}\int_0^T |(u^\varepsilon - u_d)(t)|^2_{L^2(0,L)}\,dt + \frac{c_1}{2}\int_0^T |(w^\varepsilon - w_d)(t)|^2_{L^2(0,L)}\,dt \\ + \frac{m_0}{2}\int_0^T a_0^2|f(t)|^2_{L^2(0,L)}\,dt + \frac{m_1}{2}\int_0^T a_1^2|h(t)|^2\,dt + \frac{m_2}{2}\int_0^T a_2^2|\ell(t)|^2\,dt, \end{aligned} \tag{1.9}$$

where a couple of functions $(u^\varepsilon, w^\varepsilon)$ is a unique solution to the following approximating state problem, denoted by $(\mathrm{P})^\varepsilon$, with the source control term $(f, h, \ell) \in \mathcal{U}_{ad}$:

Problem (P)$^\varepsilon$.

$$[u^\varepsilon + w^\varepsilon]_t - u^\varepsilon_{xx} = a_0 f(t, x) \quad \text{in } Q, \tag{1.10}$$

$$w^\varepsilon_t - \kappa\left(\frac{w^\varepsilon_x}{\sqrt{|w^\varepsilon_x|^2 + \varepsilon^2}} + \varepsilon w^\varepsilon_x\right)_x + K^\varepsilon(w^\varepsilon) + \nu(w^\varepsilon)^3 - w^\varepsilon = u^\varepsilon \;\text{ in }\; Q, \tag{1.11}$$

$$-u^\varepsilon_x(t, 0) + n_0(u^\varepsilon(t, 0) - b_1) = a_1 h(t), \quad t \in (0, T), \tag{1.12}$$

$$u_x^\varepsilon(t,L) + n_0(u^\varepsilon(t,L) - b_2) = a_2\ell(t), \quad t \in (0,T), \tag{1.13}$$

$$w_x^\varepsilon(t,0) = w_x^\varepsilon(t,L) = 0, \quad t \in (0,T), \tag{1.14}$$

$$u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad w^\varepsilon(0,x) = w_0^\varepsilon(x), \quad x \in (0,L). \tag{1.15}$$

Here, $K^\varepsilon(\cdot)$ is a nondecreasing function on $\mathbb{R}$ defined by

$$K^\varepsilon(r) := \mathrm{sign}(r) \int_0^{|r|} \min\left\{ \frac{1}{\varepsilon}, \frac{[s-1]^+}{\varepsilon^2} \right\} ds \quad \text{for } r \in \mathbb{R}, \tag{1.16}$$

where $[\,\cdot\,]^+$ denotes the positive part of functions and $\mathrm{sign}(\cdot)$ is a signum function so that $\mathrm{sign}(0) = 0$.

The aim of the present paper is to study $(\mathrm{OP})^\varepsilon$ from the view-point of numerical analysis, since the numerical study of (OP) is very hard because of the singular diffusivity and the constraint in (1.2). The main novelties are the following:

(i) Taking into account the necessary condition of $(\mathrm{OP})^\varepsilon$, we propose the numerical scheme to find the optimal control of $(\mathrm{OP})^\varepsilon$.

(ii) We show the convergence of the numerical algorithm proposed in (i).

The plan of this paper is as follows. In Section 2, we briefly recall the fundamentals of the theory of BV-functions. In Section 3, we recall the known results of (OP) obtained in [31]. In Section 4, we propose the numerical scheme to find the optimal control of $(\mathrm{OP})^\varepsilon$, which corresponds to the item (i) listed in the above. Moreover, we mention the main theorem, denoted by Theorem 4.1, which corresponds to the item (ii) listed in the above. In Section 5, we prove Theorem 4.1.

1.1 Notations and basic assumptions

First, we mention the notations that are used throughout this paper.

For each dimension $n \in \mathbb{N}$, we denote by $\mathscr{L}^n$ the n-dimensional Lebesgue measure, and we use this measure unless otherwise specified.

For any reflexive Banach space B, we denote by $|\cdot|_B$ the norm of B, and denote by B' the dual space of B. Additionally, we denote by $\langle\cdot,\cdot\rangle_{B',B}$ the duality pairing between B' and B.

In particular, we put $H := L^2(0,L)$ with the usual real Hilbert structure, and denote by $(\cdot,\cdot)$ the inner product in H, for simplicity. Also, let X be the Sobolev space $H^1(0,L)$ with the norm

$$|z|_X := \left\{ |z_x|_H^2 + n_0\left(|z(0)|^2 + |z(L)|^2\right)\right\}^{1/2} \quad \text{for any } z \in X,$$

which is equivalent to the standard norm of $H^1(0,L)$. We denote by X' the dual space of X. Also, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X' and X. As usual, we have

$$X \subset H \subset X' \tag{1.17}$$

with dense and compact imbeddings, and $\langle v, z\rangle = (v, z)$ for $v \in H$ and $z \in X$. Furthermore, let $F : X \to X'$ be the duality mapping defined by

$$\langle Fv, z\rangle := (v_x, z_x) + n_0 \left(v(0)z(0) + v(L)z(L)\right) \quad \text{for all } v, z \in X. \tag{1.18}$$

Also, for given $f \in H$, $h \in \mathbb{R}$, $\ell \in \mathbb{R}$, $a_0 \in \mathbb{R}$, $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$, $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}$ and $n_0 \in \mathbb{R}$, an element $\widetilde{f} \in X'$ is uniquely determined by

$$\langle \widetilde{f}, z\rangle := (a_0 f, z) + (a_1 h + n_0 b_1)z(0) + (a_2\ell + n_0 b_2)z(L) \quad \text{for all } z \in X.$$

For this $\widetilde{f}$, it is easy to check that $Fv = \widetilde{f}$ in X' is formally equivalent to

$$\begin{cases} -v_{xx} = a_0 f \quad \text{in } (0, L), \\ -v_x(0) + n_0(v(0) - b_1) = a_1 h, \quad v_x(L) + n_0(v(L) - b_2) = a_2\ell. \end{cases} \tag{1.19}$$

Note that X' becomes a Hilbert space with inner product $(\cdot, \cdot)_{X'}$ given by

$$(v, z)_{X'} := \langle v, F^{-1}z\rangle \quad \text{for all } v, z \in X'.$$

Next, let us prepare some notations and definitions. For a proper (i.e., not identically equal to infinity), l.s.c. (lower semi-continuous) and convex function $\psi : H \to \mathbb{R} \cup \{\infty\}$, the effective domain $D(\psi)$ of ψ is defined by $D(\psi) := \{z \in H;\ \psi(z) < \infty\}$. We denote by $\partial\psi$ the subdifferential of ψ in the topology of H. In general, the subdifferential is a possibly multi-valued operator from H into itself, and for any $z \in H$, the value $\partial\psi(z)$ is defined as:

$$\partial\psi(z) := \{z^* \in H \ ;\ (z^*, y - z) \le \psi(y) - \psi(z) \quad \text{for all} \ \ y \in H\}.$$

Then, a set $D(\partial\psi) := \{z \in H \ ;\ \partial\psi(z) \neq \emptyset\}$ is called the domain of $\partial\psi$. We refer to the monograph by Brézis [9], for detailed properties and related notions of convex functions and their subdifferentials.

Also, we recall a notion of convergence for convex functions, developed by Mosco [22].

Definition 1.1 (cf. [22])**.** Let ψ, ψ_n $(n \in \mathbb{N})$ be proper, l.s.c. and convex functions on H. Then, we say that ψ_n converges to ψ on H in the sense of Mosco [22] as $n \to \infty$, if the following two conditions are satisfied:

(i) for any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \to z$ weakly in H as $k \to \infty$, then

$$\liminf_{k\to\infty} \psi_{n_k}(z_k) \ge \psi(z);$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in H such that

$$z_n \to z \text{ in } H \text{ as } n \to \infty \quad \text{ and } \quad \lim_{n\to\infty} \psi_n(z_n) = \psi(z).$$

Now, let us give some assumptions on data. Throughout this paper, we assume the following conditions (A1)–(A2).

(A1) $T > 0$, $L > 0$, $\kappa > 0$, $n_0 > 0$, $\nu \ge 0$, $c_0 \ge 0$, $c_1 \ge 0$, $m_0 \ge 0$, $m_1 \ge 0$, $m_2 \ge 0$ are fixed constants. Also, a_0, a_1, a_2, b_1, b_2 are fixed real numbers.

(A2) u_d and w_d are the given desired target profiles in $L^2(0, T; H)$.

Finally, throughout this paper, N_i, $i = 1, 2, 3, \cdots$, denotes positive (or nonnegative) constants depending only on its argument(s).

2 Preliminary

In this section, we recall the fundamentals concerned with the total variation and functions of bounded variation. These notions are rigorously defined as follows.

Definition 2.1. **(I)** Let $f \in L^1(0, L)$. Then, f is called a function of bounded variation, or simply a BV-function, on $(0, L)$, if and only if:

$$V_0(f) := \sup \left\{ \int_0^L f \varphi_x dx; \begin{array}{l} \varphi \in C^1[0, L] \text{ with a compact support on } (0, L), \\ |\varphi| \leq 1 \text{ on } [0, L] \end{array} \right\} < \infty.$$

Here, we call $V_0(f)$ the total variation of f.
(II) We denote by $BV(0, L)$ the space of all BV-functions on $(0, L)$.

Here are listed usual properties of BV-functions and the space $BV(0, L)$, in forms of some propositions and remarks.

Proposition 2.1 (cf. [11, Chapter 5]). *Let $f \in BV(0, L)$. Then, there exist a Radon measure $|Df|$ on $(0, L)$, and $|Df|$-measurable function $\sigma_f : (0, L) \to \mathbb{R}$ such that*

(i) $V_0(f) = \int_0^L |Df|$, *and* $|\sigma_f| = 1$, $|Df|$*-a.e. on* $(0, L)$;

(ii) $\int_0^L f \varphi_x dx = -\int_0^L \varphi \, \sigma_f |Df|$ *for any* $\varphi \in C^1[0, L]$ *with a compact support on* $(0, L)$.

Proposition 2.2 (cf. [7, Chapter 10], [11, Chapter 5]). **(I)** *The functional $z \in L^1(0, L) \mapsto V_0(z)$ forms a proper, l.s.c. and convex function on $L^1(0, L)$.*
(II) *The space $BV(0, L)$ is a Banach space with the norm:*

$$|z|_{BV(0,L)} := |z|_{L^1(0,L)} + V_0(z) \quad \text{for all } z \in BV(0, L).$$

Proposition 2.3 (cf. [2, Corollary 3.49], [7, Chapter 10]). *$BV(0, L)$ is continuously embedded in $L^\infty(0, L)$, and compactly embedded in $L^p(0, L)$ for any $1 \leq p < \infty$.*

Next, let us set a proper, l.s.c. and convex functional $\mathcal{I}_{[-1,1]}$ on H, by putting:

$$\mathcal{I}_{[-1,1]}(z) := \int_0^L I_{[-1,1]}(z(x)) dx \quad \text{for all } z \in H.$$

Then, we define the following total variation functional V with a constraint by the indicator function $I_{[-1,1]}$:

$$V(z) = V_0(z) + \mathcal{I}_{[-1,1]}(z) \quad \text{for all } z \in H. \tag{2.1}$$

Clearly, V is proper, l.s.c. and convex on H, and its effective domain is formulated by:

$$D(V) = \{ z \in BV(0, L) \; ; \; |z| \leq 1, \text{ a.e. on } (0, L) \}.$$

Finally, we recall the decomposition result of the subdifferential ∂V of V. For the detailed proof, we refer to [30, Theorem 3.1].

Proposition 2.4 (cf. [30, Theorem 3.1])**.** *The subdifferential ∂V of V is decomposed into the following form:*

$$\partial V(z) = \partial \left(V_0|_H \right)(z) + \partial \mathcal{I}_{[-1,1]}(z) \ \textit{in } H \qquad \textit{for all } z \in H,$$

where $V_0|_H$ denotes the restriction of V_0 onto H.

3 Known results

We begin by defining the notion of solutions to (P). To do so, for given $f \in L^2(0,T;H)$, $h \in L^2(0,T)$ and $\ell \in L^2(0,T)$, we define $\widetilde{f} \in L^2(0,T;X')$ by putting

$$\langle \widetilde{f}(t), z \rangle := (a_0 f(t), z) + (a_1 h(t) + n_0 b_1) z(0) + (a_2 \ell(t) + n_0 b_2) z(L) \\ \text{for all } z \in X \text{ and a.e. } t \in (0,T). \tag{3.1}$$

Definition 3.1. Let $u_0 \in X'$ and $w_0 \in H$. Then, a couple of functions (u, w) is called a solution to (P), or (P;u_0, w_0, f, h, ℓ) when the data are specified, on $[0,T]$, if the following conditions are satisfied:

(S1) $u \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H)$.

(S2) $w \in W^{1,2}(0,T;H)$ with $V(w) \in L^\infty(0,T)$.

(S3) For all $z \in X$ and a.e. $t \in (0,T)$,

$$\langle u'(t), z \rangle + (w'(t), z) + \langle Fu(t), z \rangle = \langle \widetilde{f}(t), z \rangle.$$

(S4) There is a function $w^* \in L^2(0,T;H)$ such that $w^*(t) \in \partial V(w(t))$ and

$$w'(t) + \kappa w^*(t) + \nu w^3(t) - w(t) = u(t) \ \text{ in } H \ \text{ a.e. } t \in (0,T).$$

(S5) $u(0) = u_0$ in X' and $w(0) = w_0$ in H.

Remark 3.1. By Proposition 2.4, the condition (S4) of Definition 3.1 is equivalent to the following condition (S4)$'$:

(S4)$'$ There are a function $w_0^* \in L^2(0,T;H)$ and a function $\xi \in L^2(0,T;H)$ such that

$$w_0^*(t) \in \partial \left(V_0|_H \right)(w(t)) \ \text{ in } H, \qquad \xi(t) \in \partial \mathcal{I}_{[-1,1]}(w(t)) \ \text{ in } H, \\ w'(t) + \kappa w_0^*(t) + \xi(t) + \nu w^3(t) - w(t) = u(t) \ \text{ in } H$$

for a.e. $t \in (0,T)$.

Remark 3.2. It follows from (S4) of Definition 3.1 that the equation (1.2) is equivalent to the following variational inequality:

$$\begin{gathered}(w'(t) + \nu w^3(t) - w(t) - u(t), w(t) - z) + \kappa V(w(t)) - \kappa V(z) \le 0 \\ \text{for any } z \in D(V) \text{ and a.e. } t \in (0,T).\end{gathered} \tag{3.2}$$

Remark 3.3. Usually, the expressions of subdifferentials are obtained by computing the first variations of corresponding convex functions. In this light, the function $w_0^* \in L^2(0,T;H)$ as in (S4)$'$ of Remark 3.1 somehow links to the first variation of the total variation functional $V_0|_H$. Also, as is well-known (cf. [9, Proposition 2.16]),

$$\partial \mathcal{I}_{[-1,1]}(z) = \left\{ \xi \in H; \xi \in \partial I_{[-1,1]}(z), \text{ a.e. on } (0,L) \right\}$$

for any $z \in D(\partial \mathcal{I}_{[-1,1]})$. Hence, taking account of Remark 3.1, we see that the subdifferential ∂V corresponds to the rigorous expression of the singular term $-(\frac{w_x}{|w_x|})_x + \partial I_{[-1,1]}(w)$ as in (1.2), and the variational inequality (3.2) implicitly includes the homogeneous Neumann type boundary condition.

Here, we recall the known results of the existence-uniqueness of solutions to (P) and the existence of optimal control to (OP).

Proposition 3.1 (cf. [19, Section 2], [31, Theorem 3.1])**.** *Assume* (A1)*. Let* $u_0 \in H$ *and* $w_0 \in D(V)$.
(I) *For each* $f \in L^2(0,T;H)$, $h \in L^2(0,T)$ *and* $\ell \in L^2(0,T)$, *there is a unique solution* (u,w) *to* $(\mathrm{P}; u_0, w_0, f, h, \ell)$ *on* $[0,T]$.
(II) *Furthermore, assume* (A2)*. Then, the problem* (OP) *has at least one optimal control* $(f_*, h_*, \ell_*) \in \mathcal{U}_{ad}$ *so that*

$$J(f_*, h_*, \ell_*) = \inf_{(f,h,\ell) \in \mathcal{U}_{ad}} J(f,h,\ell).$$

Remark 3.4. Note that **(II)** of Proposition 3.1 does not cover the uniqueness of optimal controls. Although Hoffmann–Jiang [14] reported the uniqueness of optimal controls for a regular Fix–Caginalp system, their technique is not applicable to the problem (OP) because of the singular diffusivity and the constraint in (P). Therefore, the uniqueness question of optimal controls to (OP) is still open.

Next, we recall the known results of the existence-uniqueness of solutions to $(\mathrm{P})^\varepsilon$ for each $\varepsilon \in (0,1]$.

Proposition 3.2 (cf. [31, Proposition 3.2])**.** *Assume* (A1)*. Let* $\varepsilon \in (0,1]$, $u_0^\varepsilon \in H$ *and* $w_0^\varepsilon \in X$. *Then, for each* $f \in L^2(0,T;H)$, $h \in L^2(0,T)$, $\ell \in L^2(0,T)$, *there is a unique pair of functions* $(u^\varepsilon, w^\varepsilon)$, *called a solution to* $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$ *on* $[0,T]$, *which solves the equations* (1.10)–(1.15) *in the following sense:*

(AS1) $u^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H)$;

(AS2) $w^\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$;

(AS3) *For all* $z \in X$ *and a.e.* $t \in (0,T)$,

$$\langle (u^\varepsilon)'(t), z\rangle + ((w^\varepsilon)'(t), z) + \langle F u^\varepsilon(t), z\rangle = \langle \widetilde{f}(t), z\rangle,$$

where $\widetilde{f}$ *is the function in* $L^2(0,T;X')$ *defined by* (3.1);

(AS4) (1.11) *holds in the variational sense, i.e.,*

$$\begin{aligned}((w^\varepsilon)'(t), z) + \kappa \left(\frac{w_x^\varepsilon(t)}{\sqrt{|w_x^\varepsilon(t)|^2 + \varepsilon^2}} + \varepsilon w_x^\varepsilon(t), z_x \right) + (K^\varepsilon(w^\varepsilon(t)), z) \\ + \left(\nu (w^\varepsilon(t))^3 - w^\varepsilon(t), z\right) = (u^\varepsilon(t), z)\end{aligned}$$

$$\text{for all } z \in X \text{ and a.e. } t \in (0,T);$$

(AS5) $u^\varepsilon(0) = u_0^\varepsilon$ *in* X' *and* $w^\varepsilon(0) = w_0^\varepsilon$ *in* H.

Next, we recall the known results of the existence and necessary condition of optimal control to $(\mathrm{OP})^\varepsilon$ for each $\varepsilon \in (0,1]$.

Proposition 3.3 (cf. [31, Theorems 3.2 and 3.4])**.** *Assume* (A1)–(A2)*. Let* $\varepsilon \in (0,1]$, $u_0^\varepsilon \in H$ *and* $w_0^\varepsilon \in X$*. Then:*
(I) *The approximating problem* $(\mathrm{OP})^\varepsilon$ *has at least one optimal control* $(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) \in \mathcal{U}_{ad}$ *so that*

$$J^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) = \inf_{(f,h,\ell)\in \mathcal{U}_{ad}} J^\varepsilon(f,h,\ell).$$

(II) *Let* $(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) \in \mathcal{U}_{ad}$ *be the optimal control of* $(\mathrm{OP})^\varepsilon$ *obtained in* **(I)***. Also, let* $(u_*^\varepsilon, w_*^\varepsilon)$ *be the unique solution to* $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)^\varepsilon$ *on* $[0,T]$*. Then, there exists a unique solution* $(p^\varepsilon, q^\varepsilon)$ *to the adjoint equation on* $[0,T]$ *as follows:*

$$p^\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X); \tag{3.3}$$

$$q^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H); \tag{3.4}$$

$$-(p^\varepsilon)' - p_{xx}^\varepsilon - q^\varepsilon = c_0(u_*^\varepsilon - u_d) \quad \text{in } Q; \tag{3.5}$$

$$\begin{aligned}&\int_0^T (-(p^\varepsilon)'(\tau), \zeta(\tau))d\tau + \int_0^T \langle -(q^\varepsilon)'(\tau), \zeta(\tau)\rangle d\tau \\ &\quad + \kappa \int_0^T ((a^\varepsilon)'((w_*^\varepsilon)_x(\tau)) q_x^\varepsilon(\tau), \zeta_x(\tau))\, d\tau \\ &\quad + \int_0^T ((K^\varepsilon)'(w_*^\varepsilon(\tau)) q^\varepsilon(\tau), \zeta(\tau))\, d\tau \\ &\quad + \int_0^T \left([3\nu (w_*^\varepsilon(\tau))^2 - 1] q^\varepsilon(\tau), \zeta(\tau)\right) d\tau \\ &= c_1 \int_0^T (w_*^\varepsilon(\tau) - w_d(\tau), \zeta(\tau))\, d\tau \qquad \text{for all } \zeta \in L^2(0,T;X);\end{aligned} \tag{3.6}$$

$$-p^\varepsilon_x(t,0)+n_0p^\varepsilon(t,0)=p^\varepsilon_x(t,L)+n_0p^\varepsilon(t,L)=0,\quad t\in(0,T), \tag{3.7}$$

$$p^\varepsilon(T,x)=q^\varepsilon(T,x)=0,\quad x\in(0,L), \tag{3.8}$$

where $(a^\varepsilon)'(\cdot)$ *is the derivative of* $a^\varepsilon(\cdot)$ *defined by*

$$a^\varepsilon(r)=\frac{r}{\sqrt{|r|^2+\varepsilon^2}}+\varepsilon r \quad \text{for any} \ \ r\in\mathbb{R}. \tag{3.9}$$

Moreover, p^ε *satisfies the following equations:*

$$a_0(p^\varepsilon+m_0a_0f^\varepsilon_*)=0 \ \ \text{in } L^2(0,T;H), \tag{3.10}$$

$$a_1(p^\varepsilon(\cdot,0)+m_1a_1h^\varepsilon_*)=0 \ \ \text{in } L^2(0,T), \tag{3.11}$$

$$a_2(p^\varepsilon(\cdot,L)+m_2a_2\ell^\varepsilon_*)=0 \ \ \text{in } L^2(0,T). \tag{3.12}$$

Here, for any $\varepsilon\in(0,1]$, let us set:

$$V^\varepsilon(z):=\begin{cases}\displaystyle\int_0^L\sqrt{|z_x|^2+\varepsilon^2}\,dx+\frac{\varepsilon}{2}\int_0^L|z_x|^2dx+\frac{1}{\kappa}\int_0^L\widehat{K}^\varepsilon(z)\,dx, & \text{if } z\in X,\\ \infty, & \text{otherwise,}\end{cases} \tag{3.13}$$

where $\widehat{K}^\varepsilon$ is a primitive of K^ε such that

$$\widehat{K}^\varepsilon(0)=0 \quad \text{and} \quad \widehat{K}^\varepsilon(r)\ge 0 \quad \text{for all } r\in\mathbb{R}.$$

Clearly, each functional V^ε ($\varepsilon\in(0,1]$) forms a proper, l.s.c. and convex functional on H. Moreover, we observe that (cf. [31, Lemma 5.1])

$$V^\varepsilon(\cdot)\to V(\cdot) \ \text{ on } H \text{ in the sense of Mosco [22] as } \varepsilon\to 0. \tag{3.14}$$

Now, we recall the following result of continuous dependence between (P) and $(\mathrm{P})^\varepsilon$ as $\varepsilon\to 0$.

Proposition 3.4 (cf. [31, Proposition 5.1])**.** *Assume* (A1)*. Let* $(f,h,\ell)\in\mathcal{U}_{ad}$*,* $u_0\in H$ *and* $w_0\in D(V)$*. Also, let* $\varepsilon\in(0,1]$*,* $\{(f^\varepsilon,h^\varepsilon,\ell^\varepsilon)\}\subset\mathcal{U}_{ad}$*,* $\{u^\varepsilon_0\}\subset H$ *and* $\{w^\varepsilon_0\}\subset X$*. Furthermore, suppose that*

$$f^\varepsilon\to f \ \text{ weakly in } L^2(0,T;H),$$
$$h^\varepsilon\to h \ \text{ weakly in } L^2(0,T),$$
$$\ell^\varepsilon\to \ell \ \text{ weakly in } L^2(0,T),$$
$$u^\varepsilon_0\to u_0 \ \text{in } H,\quad w^\varepsilon_0\to w_0 \ \text{in } H \quad \text{and} \quad V^\varepsilon(w^\varepsilon_0)\to V(w_0)$$

as $\varepsilon\to 0$*. Then, the unique solution* $(u^\varepsilon,w^\varepsilon)$ *to* $(\mathrm{P};u^\varepsilon_0,w^\varepsilon_0,f^\varepsilon,h^\varepsilon,\ell^\varepsilon)^\varepsilon$ *converges to the solution* (u,w) *to* $(\mathrm{P};u_0,w_0,f,h,\ell)$ *in the following sense:*

$$(u^\varepsilon,w^\varepsilon)\to(u,w) \quad \text{in } L^2(0,T;H)\times C([0,T];H) \ \text{as } \varepsilon\to 0.$$

Also, we recall the result of the relationship between (OP) and $(\mathrm{OP})^\varepsilon$ ($\varepsilon \in (0,1]$).

Proposition 3.5 (cf. [31, Theorem 3.3])**.** *Assume* (A1)–(A2)*,* $u_0 \in H$*,* $\{u_0^\varepsilon\} \subset H$*,* $w_0 \in D(V)$*,* $\{w_0^\varepsilon\} \subset X$*,*

$$u_0^\varepsilon \to u_0 \text{ in } H,\ w_0^\varepsilon \to w_0 \text{ in } H \ \text{ and } \ V^\varepsilon(w_0^\varepsilon) \to V(w_0) \text{ as } \varepsilon \to 0.$$

Let $(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) \in \mathcal{U}_{ad}$ *be the optimal control of approximating problem* $(\mathrm{OP})^\varepsilon$*. Also, let* $(u_*^\varepsilon, w_*^\varepsilon)$ *be the unique solution to* $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)^\varepsilon$ *on* $[0,T]$*. Then, there exist a subsequence* $\{\varepsilon_k\} \subset \{\varepsilon\}$*, the pair of functions* $(f_{**}, h_{**}, \ell_{**}) \in \mathcal{U}_{ad}$ *and the unique solution* (u_{**}, w_{**}) *to* $(\mathrm{P};u_0, w_0, f_{**}, h_{**}, \ell_{**})$ *on* $[0,T]$ *such that* $(f_{**}, h_{**}, \ell_{**})$ *is the optimal control of* (OP)*,* $\varepsilon_k \to 0$*,*

$$\begin{aligned} f_*^{\varepsilon_k} &\to f_{**} \quad \text{weakly in } L^2(0,T;H),\\ h_*^{\varepsilon_k} &\to h_{**} \quad \text{weakly in } L^2(0,T),\\ \ell_*^{\varepsilon_k} &\to \ell_{**} \quad \text{weakly in } L^2(0,T),\\ (u_*^{\varepsilon_k}, w_*^{\varepsilon_k}) &\to (u_{**}, w_{**}) \quad \text{in } L^2(0,T;H) \times C([0,T];H) \end{aligned}$$

as $k \to \infty$*.*

Taking into account the approximating problems $(\mathrm{OP})^\varepsilon$, we get the optimality condition to (OP). Now, let us recall the result concerning the necessary condition of the optimal control to (OP).

Proposition 3.6 (cf. [31, Theorem 3.5])**.** *Suppose* (A1)–(A2)*,* $u_0 \in H$ *and* $w_0 \in D(V)$*. Let* $(f_{**}, h_{**}, \ell_{**})$ *be the optimal control to* (OP) *obtained in Proposition 3.5. Let* (u_{**}, w_{**}) *be the unique solution to* $(\mathrm{P};u_0,\ w_0, f_{**}, h_{**}, \ell_{**})$ *on* $[0,T]$*. Additionally, let us set:*

$$W := \{z \in H^1(Q)\ ;\ z(0,x) = 0,\ \text{a.e. } x \in (0,L)\}.$$

Then, there are the functions $p \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$*,* $q \in L^\infty(0,T;H)$ *and an element* $\mu \in W'$ *satisfying the following:*

$$\begin{aligned} -p' - p_{xx} - q = c_0(u_{**} - u_d) \quad \text{in } Q,\\ \int_0^T (-p'(\tau), z(\tau))\,d\tau + \int_0^T (q(\tau), z'(\tau))d\tau + \langle \mu, z\rangle_{W',W}\\ + \int_0^T ([3\nu w_{**}^2(\tau) - 1]q(\tau), z(\tau))\,d\tau \\ = \ c_1 \int_0^T (w_{**}(\tau) - w_d(\tau), z(\tau))\,d\tau \quad \text{for all } z \in W,\\ -p_x(t,0) + n_0 p(t,0) = p_x(t,L) + n_0 p(t,L) = 0, \quad t \in (0,T),\\ p(T,x) = 0, \quad x \in (0,L). \end{aligned} \tag{3.15}$$

Moreover, p *satisfies the following equations:*

$$\begin{aligned} a_0(p + m_0 a_0 f_{**}) = 0 \ \text{ in } L^2(0,T;H),\\ a_1(p(\cdot,0) + m_1 a_1 h_{**}) = 0 \ \text{ in } L^2(0,T),\\ a_2(p(\cdot,L) + m_2 a_2 \ell_{**}) = 0 \ \text{ in } L^2(0,T). \end{aligned}$$

Remark 3.5. Proposition 3.6 was proved through the limiting observation of the approximating situations shown in **(II)** of Proposition 3.3. Also, the identities (3.6) and (3.15) can be regarded as some variational forms of

$$-p_t^\varepsilon - q_t^\varepsilon - \kappa\left((a^\varepsilon)'((w_*^\varepsilon)_x)q_x^\varepsilon\right)_x + (K^\varepsilon)'(w_*^\varepsilon)q^\varepsilon + [3\nu(w_*^\varepsilon)^2 - 1]q^\varepsilon = c_1(w_*^\varepsilon - w_d)$$

and

$$-p_t - q_t + \mu + [3\nu w_{**}^2 - 1]q = c_1(w_{**} - w_d)$$

in the distribution sense, respectively.

4 Main Result

We observe from the singular diffusivity and the constraint in (P) and Proposition 3.6 that it is very hard to study (P) and (OP) numerically. However, we note from Propositions 3.4 and 3.5 that $(\mathrm{P})^\varepsilon$ and $(\mathrm{OP})^\varepsilon$ ($\varepsilon \in (0,1]$) are the approximating problems of (P) and (OP), respectively. Therefore, it is worthy considering $(\mathrm{OP})^\varepsilon$ from the view-point of numerical analysis.

Now, we fix the constant $\varepsilon \in (0,1]$ and the initial data $(u_0^\varepsilon, w_0^\varepsilon) \in H \times X$. Then, throughout this paper, we use the following notations.

Definition 4.1. (I) We denote by $\Lambda^\varepsilon : \mathcal{U}_{ad} \to L^2(0,T;H) \times L^2(0,T;H)$ a solution operator to $(\mathrm{P})^\varepsilon$ that assigns to any control $(f,h,\ell) \in \mathcal{U}_{ad}$ the unique solution $(u^\varepsilon, w^\varepsilon) := \Lambda^\varepsilon(f,h,\ell)$ to the state system $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$ on $[0,T]$.
(II) We denote by $\Lambda_{ad}^\varepsilon : \mathcal{U}_{ad} \to L^2(0,T;H) \times L^2(0,T;H)$ a solution operator to the adjoint system $\{(3.5)\text{–}(3.8)\}$ that assigns to any control $(f,h,\ell) \in \mathcal{U}_{ad}$ the unique solution $(p^\varepsilon, q^\varepsilon) := \Lambda_{ad}^\varepsilon(f,h,\ell)$ to the adjoint system $\{(3.5)\text{–}(3.8)\}$ on $[0,T]$ under $(u^\varepsilon, w^\varepsilon) := \Lambda^\varepsilon(f,h,\ell)$.

Note that the control space $\mathcal{U}_{ad} := L^2(0,T;H) \times L^2(0,T) \times L^2(0,T)$ is a Hilbert space with the inner product

$$((z,y_0,y_L),(\bar{z},\bar{y}_0,\bar{y}_L))_{\mathcal{U}_{ad}} := (z,\bar{z})_{L^2(0,T;H)} + (y_0,\bar{y}_0)_{L^2(0,T)} + (y_L,\bar{y}_L)_{L^2(0,T)}$$
$$\text{for all } (z,y_0,y_L),(\bar{z},\bar{y}_0,\bar{y}_L) \in \mathcal{U}_{ad},$$

and the norm

$$\|(z,y_0,y_L)\|_{\mathcal{U}_{ad}}^2 := |z|_{L^2(0,T;H)}^2 + |y_0|_{L^2(0,T)}^2 + |y_L|_{L^2(0,T)}^2 \quad \text{for all } (z,y_0,y_L) \in \mathcal{U}_{ad}.$$

Now, we study the problem $(\mathrm{OP})^\varepsilon$ from the view-point of numerical analysis.

For a moment, we often omit the subscript $\varepsilon \in (0,1]$. Then, taking into account the necessary conditions (3.10)–(3.12) of $(\mathrm{OP})^\varepsilon$ mentioned in **(II)** of Proposition 3.3 (cf. [31, Theorem 3.4]), we propose the numerical algorithm, denoted by (NA), to find the optimal control of $(\mathrm{OP})^\varepsilon$ numerically, as follows.

Numerical Algorithm (NA) of $(\mathrm{OP})^\varepsilon$

(Step 0) Give the stop parameter $\mu \in (0,1)$;

(Step 1) Choose the initial function $(f,h,\ell) \in \mathcal{U}_{ad}$, and put $f_n = f$, $h_n = h$ and $\ell_n = \ell$;

(Step 2) Solve the problem $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f_n, h_n, \ell_n)^\varepsilon$, and let $(u_n, w_n) := \Lambda^\varepsilon(f_n, h_n, \ell_n)$;

(Step 3) Solve the adjoint equation $\{(3.5)–(3.8)\}$ for n, and let $(p_n, q_n) := \Lambda_{ad}^\varepsilon(f_n, h_n, \ell_n)$;

(Step 4) Test: Put $f_n^{p_n} := a_0(p_n + m_0 a_0 f_n)$, $h_n^{p_n} := a_1(p_n(\cdot,0) + m_1 a_1 h_n)$ and $\ell_n^{p_n} := a_2(p_n(\cdot,L) + m_2 a_2 \ell_n)$. If

$$\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}} < \mu,$$

then, STOP; Otherwise, go to (Step 5);

(Step 5) Put $f_{n+1} = f_n - \rho_n f_n^{p_n}$, $h_{n+1} := h_n - \rho_n h_n^{p_n}$ and $\ell_{n+1} := \ell_n - \rho_n \ell_n^{p_n}$, where ρ_n is some appropriate constant found by using a line search. More precisely, given $\beta \in (0,1)$ and $\mu \in (0,1)$, find the minimal constant $j_n \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} & J^\varepsilon\left(f_n - \beta^{j_n} f_n^{p_n}, h_n - \beta^{j_n} h_n^{p_n}, \ell_n - \beta^{j_n} \ell_n^{p_n}\right) - J^\varepsilon(f_n, h_n, \ell_n) \\ \leq\ & -\mu \beta^{j_n} \|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}}^2, \end{aligned}$$

and put the constant $\rho_n := \beta^{j_n}$;

(Step 6) Set $n = n+1$, and go to (Step 2).

Now, we mention the main result in this paper, which is concerned with the convergence of the numerical algorithm (NA).

Theorem 4.1. *Assume* (A1)–(A2), $\varepsilon \in (0,1]$ *and* $(u_0^\varepsilon, w_0^\varepsilon) \in H \times X$. *Let* $\{(f_n, h_n, \ell_n)\}$ *be a sequence in* $\mathcal{U}_{ad}$ *defined by the numerical algorithm* (NA). *Also, let* $(p_n, q_n) := \Lambda_{ad}^\varepsilon(f_n, h_n, \ell_n)$. *Then:*

(I) $\lim_{n\to\infty} J^\varepsilon(f_n, h_n, \ell_n)$ *exists.*

(II)

$$\lim_{n\to\infty} a_0(p_n + m_0 a_0 f_n) = 0 \quad in\ L^2(0,T;H), \tag{4.1}$$

$$\lim_{n\to\infty} a_1(p_n(\cdot,0) + m_1 a_1 h_n) = 0 \quad in\ L^2(0,T), \tag{4.2}$$

$$\lim_{n\to\infty} a_2(p_n(\cdot,L) + m_2 a_2 \ell_n) = 0 \quad in\ L^2(0,T). \tag{4.3}$$

(III) *There are functions* $(f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon) \in \mathcal{U}_{ad}$ *and* $(p_{**}^\varepsilon, q_{**}^\varepsilon) \in L^2(0,T;X) \times L^2(0,T;H)$, *and a subsequence* $\{n_k\} \subset \{n\}$ *such that* $(p_{**}^\varepsilon, q_{**}^\varepsilon) = \Lambda_{ad}^\varepsilon(f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon)$, *i.e.,* $(p_{**}^\varepsilon, q_{**}^\varepsilon)$ *is a unique solution to the adjoint equation for* $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon)^\varepsilon$,

$$\lim_{k\to\infty} (f_{n_k}, h_{n_k}, \ell_{n_k}) = (f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon) \quad in\ \mathcal{U}_{ad}, \tag{4.4}$$

$$a_0(p_{**}^\varepsilon + m_0 a_0 f_{**}^\varepsilon) = 0 \quad in\ L^2(0,T;H), \tag{4.5}$$

$$a_1(p_{**}^\varepsilon(\cdot,0) + m_1 a_1 h_{**}^\varepsilon) = 0 \quad in\ L^2(0,T), \tag{4.6}$$

$$a_2(p_{**}^\varepsilon(\cdot,L) + m_2 a_2 \ell_{**}^\varepsilon) = 0 \quad in\ L^2(0,T). \tag{4.7}$$

Hence, $D_{(\tilde f,\tilde h,\tilde \ell)} J^\varepsilon(f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon) = 0$ *for any direction* $(\tilde f, \tilde h, \tilde \ell) \in \mathcal{U}_{ad}$.

In the next Section 5, we show Theorem 4.1.

5 Proof of Theorem 4.1

In this section, we prove the main Theorem 4.1. To this end, we need some lemmas.

Now, we suppose that all the assumptions of Theorem 4.1 hold. We also fix the constant $\varepsilon \in (0,1]$. Then, the following first lemma is concerned with the continuous dependence of solutions to $(\mathrm{P})^\varepsilon$.

Lemma 5.1 (cf. [31, Proposition 5.1]). *Suppose that all the assumptions of Theorem 4.1 hold. Let $(u_0, w_0) \in H \times X$, $(f,h,\ell) \in \mathcal{U}_{ad}$, and let $(u,w) = \Lambda^\varepsilon(f,h,\ell)$, namely, let (u,w) be a unique solution to $(\mathrm{P};u_0,w_0,f,h,\ell)^\varepsilon$ on $[0,T]$. Then, there is a positive constant N_1, dependent only on T, n_0 and independent of ε, such that the following bounded estimate holds:*

$$\begin{aligned} & |u'|^2_{L^2(0,T;X')} + |u|^2_{C([0,T];H)} + |u|^2_{L^2(0,T;X)} + |w'|^2_{L^2(0,T;H)} + |w|^2_{C([0,T];H)} \\ & \qquad + \kappa \sup_{0\le t\le T} V^\varepsilon(w(t)) + \sup_{0\le t\le T} \int_0^L \widehat{g}(w(t,x))\,dx \\ \le \; & N_1 \biggl(|u_0|^2_H + |w_0|^2_H + \kappa V^\varepsilon(w_0) + \int_0^L \widehat{g}(w_0(x))\,dx + a_0^2 |f|^2_{L^2(0,T;H)} \\ & \qquad + a_1^2 |h|^2_{L^2(0,T)} + a_2^2 |\ell|^2_{L^2(0,T)} + b_1^2 + b_2^2 \biggr), \end{aligned} \tag{5.1}$$

where V^ε is a functional defined in (3.13), and $\widehat{g}$ is a non-negative primitive of $g(w) := \nu w^3 - w$.

Furthermore, assume $\{(f_n,h_n,\ell_n)\} \subset \mathcal{U}_{ad}$, $(f,h,\ell) \in \mathcal{U}_{ad}$ and

$$f_n \to f \text{ weakly in } L^2(0,T;H),$$

$$h_n \to h \text{ weakly in } L^2(0,T),$$

$$\ell_n \to \ell \text{ weakly in } L^2(0,T)$$

as $n \to \infty$. Then, $(u_n,w_n) = \Lambda^\varepsilon(f_n,h_n,\ell_n)$ converges to $(u,w) = \Lambda^\varepsilon(f,h,\ell)$ in the sense that

$$(u_n,w_n) \to (u,w) \quad \text{in } L^2(0,T;H) \times C([0,T];H) \quad \text{as } n \to \infty. \tag{5.2}$$

The proof of Lemma 5.1 is a slight modification of that as in [31, Proposition 5.1]. Therefore, we omit the detailed proof.

The next lemma is concerned with the Gâteaux differentiability of Λ^ε and J^ε, which was already proved by [31, Proposition 6.1]. Therefore, we omit the detailed proof of the following lemma.

Lemma 5.2 (cf [31, Proposition 6.1]). *Assume the same conditions in Theorem 4.1. Then, the following two statements hold.*

- **(I)** *The solution operator Λ^ε admits the Gâteaux derivative at any $(f,h,\ell) \in \mathcal{U}_{ad}$. More precisely, for arbitrary $(f,h,\ell) \in \mathcal{U}_{ad}$, there exists a pair of functions $(\theta,\chi) \in L^2(0,T;H) \times L^2(0,T;H)$ such that:*

$$D_{(\tilde{f},\tilde{h},\tilde{\ell})}\Lambda^\varepsilon(f,h,\ell) := \lim_{\lambda\to 0} \frac{\Lambda^\varepsilon(f+\lambda\tilde{f}, h+\lambda\tilde{h}, \ell+\lambda\tilde{\ell}) - \Lambda^\varepsilon(f,h,\ell)}{\lambda}$$

$$= (\theta, \chi) \quad in\ L^2(0,T;H) \times L^2(0,T;H) \tag{5.3}$$

$$for\ all\ direction\ (\tilde{f}, \tilde{h}, \tilde{\ell}) \in \mathcal{U}_{ad},$$

$$\theta \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \tag{5.4}$$

$$\chi \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H) \tag{5.5}$$

and (θ, χ) *solves the following linear system:*

$$\begin{aligned} &\langle \theta'(t), z \rangle + \langle \chi'(t), z \rangle + (\theta_x(t), z_x) + n_0\,(\theta(t,0)z(0) + \theta(t,L)z(L)) \\ &\qquad = (a_0 \tilde{f}(t), z) + a_1 \tilde{h}(t) z(0) + a_2 \tilde{\ell}(t) z(L), \\ &\qquad a.e.\ t \in (0,T), \ \ for\ all\ z \in X; \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\langle \chi'(t), z \rangle + \kappa\,((a^\varepsilon)'(w_x(t))\chi_x(t), z_x) + ((K^\varepsilon)'(w(t))\chi(t), z) \\ &\qquad + ([3\nu w^2(t) - 1]\chi(t), z) = (\theta(t), z), \\ &\qquad a.e.\ t \in (0,T), \ \ for\ all\ z \in X; \end{aligned} \tag{5.7}$$

$$\theta(0,x) = \chi(0,x) = 0, \quad a.e.\ x \in (0,L). \tag{5.8}$$

(II) *The cost function* J^ε *admits the Gâteaux derivative at any* $(f,h,\ell) \in \mathcal{U}_{ad}$*. More precisely,*

$$D_{(\tilde{f},\tilde{h},\tilde{\ell})} J^\varepsilon(f,h,\ell) := \lim_{\lambda \to 0} \frac{J^\varepsilon(f + \lambda \tilde{f}, h + \lambda \tilde{h}, \ell + \lambda \tilde{\ell}) - J^\varepsilon(f,h,\ell)}{\lambda}$$

$$\begin{aligned} = \ & c_0 \int_0^T ((u - u_d)(t), \theta(t))dt + c_1 \int_0^T ((w - w_d)(t), \chi(t))dt \\ & + m_0 a_0^2 \int_0^T (f(t), \tilde{f}(t))dt + m_1 a_1^2 \int_0^T h(t)\tilde{h}(t)dt + m_2 a_2^2 \int_0^T \ell(t)\tilde{\ell}(t)dt \end{aligned} \tag{5.9}$$

for any $(f,h,\ell) \in \mathcal{U}_{ad}$ *and any direction* $(\tilde{f}, \tilde{h}, \tilde{\ell}) \in \mathcal{U}_{ad}$*, where* $(u,w) = \Lambda^\varepsilon(f,h,\ell)$ *is the solution to* $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$ *on* $[0,T]$*,* u_d *and* w_d *are the given target profiles in* $L^2(0,T;H)$*, and* $(\theta,\chi)(= D_{(\tilde{f},\tilde{h},\tilde{\ell})}\Lambda^\varepsilon(f,h,\ell))$ *is the pair of functions obtained in the assertion* **(I)**.

Now we show the continuity of the Gâteaux derivative of the solution operator Λ^ε $(\varepsilon \in (0,1])$.

Lemma 5.3. *Assume the same conditions as in Theorem 4.1. Let* $\xi \in [-1,1] \backslash \{0\}$*. Then, the Gâteaux derivative of the solution operator* Λ^ε *is continuous in the following sense:*

$$\begin{aligned} & (\theta_\xi, \chi_\xi) = D_{(\tilde{f},\tilde{h},\tilde{\ell})} \Lambda^\varepsilon(f + \xi z, h + \xi y_0, \ell + \xi y_L) \\ \to \ & (\theta, \chi) = D_{(\tilde{f},\tilde{h},\tilde{\ell})} \Lambda^\varepsilon(f,h,\ell) \quad in\ L^2(0,T;H) \times L^2(0,T;H) \end{aligned} \tag{5.10}$$

for all $(f,h,\ell) \in \mathcal{U}_{ad}$*,* $(z, y_0, y_L) \in \mathcal{U}_{ad}$ *and any direction* $(\tilde{f}, \tilde{h}, \tilde{\ell}) \in \mathcal{U}_{ad}$

as $\xi \to 0$, *where*

$$D_{(\tilde{f},\tilde{h},\tilde{\ell})}\Lambda^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$$
$$:= \lim_{\lambda\to 0} \frac{\Lambda^\varepsilon(f+\xi z+\lambda\tilde{f}, h+\xi y_0+\lambda\tilde{h}, \ell+\xi y_L+\lambda\tilde{\ell}) - \Lambda^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)}{\lambda}$$
$$\text{in } L^2(0,T;H)\times L^2(0,T;H)$$

and

$$D_{(\tilde{f},\tilde{h},\tilde{\ell})}\Lambda^\varepsilon(f,h,\ell) := \lim_{\lambda\to 0}\frac{\Lambda^\varepsilon(f+\lambda\tilde{f}, h+\lambda\tilde{h}, \ell+\lambda\tilde{\ell}) - \Lambda^\varepsilon(f,h,\ell)}{\lambda}$$
$$\text{in } L^2(0,T;H)\times L^2(0,T;H).$$

Proof. For any $(f,h,\ell)\in\mathcal{U}_{ad}$, $(z,y_0,y_L)\in\mathcal{U}_{ad}$ and $\xi\in[-1,1]\setminus\{0\}$, we put $(u_\xi,w_\xi) := \Lambda^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$ and $(u,w) := \Lambda^\varepsilon(f,h,\ell)$. Then, from the convergence result of solutions to $(\mathrm{P})^\varepsilon$ (cf. (5.2)), we observe that

$$(u_\xi, w_\xi)\to(u,w) \quad \text{in } L^2(0,T;H)\times C([0,T];H) \quad \text{as } \xi\to 0. \tag{5.11}$$

Also, by the second equation (1.11) in $(\mathrm{P})^\varepsilon$ and the quite standard calculation, we get the following estimate:

$$\sup_{t\in[0,T]}|w_\xi(t)-w(t)|_H^2 + 2\varepsilon\kappa\int_0^T |(w_\xi-w)_x(t)|_H^2 dt \le e^{3T}|u_\xi - u|^2_{L^2(0,T;H)}. \tag{5.12}$$

Thus, we infer from (5.11) and (5.12) that

$$w_\xi \to w \text{ strongly in } L^2(0,T;X) \text{ as } \xi\to 0. \tag{5.13}$$

Now, we show (5.10). Note from the definitions $a^\varepsilon(\cdot)$ and $K^\varepsilon(\cdot)$ (cf. (3.9) and (1.16)) that $a^\varepsilon(\cdot)\in C^2(\mathbb{R})$, $K^\varepsilon(\cdot)\in C^1(\mathbb{R})$,

$$\varepsilon \le (a^\varepsilon)'(r) \le \frac{1}{\varepsilon}+\varepsilon \quad \text{for any } r\in\mathbb{R} \tag{5.14}$$

and

$$0\le (K^\varepsilon)'(r)\le \frac{1}{\varepsilon} \quad \text{for any } r\in\mathbb{R}. \tag{5.15}$$

Now, we give the uniform estimate of $(\theta_\xi,\chi_\xi) = D_{(\tilde{f},\tilde{h},\tilde{\ell})}\Lambda^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$ with respect to $\xi\in[-1,1]\setminus\{0\}$. Note from **(I)** of Lemma 5.2 that $(\theta_\xi,\chi_\xi) = D_{(\tilde{f},\tilde{h},\tilde{\ell})}\Lambda^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$ satisfies the following linear system:

$$\begin{aligned}&\langle\theta_\xi'(\tau), z\rangle + \langle\chi_\xi'(\tau), z\rangle + ((\theta_\xi)_x(\tau), z_x) + n_0\left(\theta_\xi(\tau,0)z(0) + \theta_\xi(\tau,L)z(L)\right)\\ &\qquad = (a_0\tilde{f}(\tau), z) + a_1\tilde{h}(\tau)z(0) + a_2\tilde{\ell}(\tau)z(L),\\ &\qquad\qquad \text{a.e. } \tau\in(0,T), \ \text{for all } z\in X;\end{aligned} \tag{5.16}$$

$$\begin{aligned}&\langle\chi_\xi'(\tau), z\rangle + \kappa\left((a^\varepsilon)'((w_\xi)_x(\tau))(\chi_\xi)_x(\tau), z_x\right) + ((K^\varepsilon)'(w_\xi(\tau))\chi_\xi(\tau), z)\\ &\qquad +([3\nu w_\xi^2(\tau)-1]\chi_\xi(\tau), z) = (\theta_\xi(\tau), z),\\ &\qquad\qquad \text{a.e. } \tau\in(0,T), \ \text{for all } z\in X;\end{aligned} \tag{5.17}$$

$$\theta_\xi(0,x) = \chi_\xi(0,x) = 0, \quad \text{a.e. } x \in (0,L). \tag{5.18}$$

Assigning $\theta_\xi(\tau)$ to z in (5.16), applying the Schwarz inequality and integrating in τ over $[0,t]$, we get:

$$\begin{aligned} &\frac{1}{2}|\theta_\xi(t)|_H^2 + \int_0^t \langle \chi'_\xi(\tau), \theta_\xi(\tau)\rangle d\tau + \int_0^t |(\theta_\xi)_x(\tau)|_H^2 d\tau \\ &\quad + \frac{n_0}{2}\int_0^t |\theta_\xi(\tau,0)|^2 d\tau + \frac{n_0}{2}\int_0^t |\theta_\xi(\tau,L)|^2 d\tau \\ &\leq \frac{1}{2}\int_0^t |\theta_\xi(\tau)|_H^2 d\tau + \frac{a_0^2}{2}\int_0^t |\tilde{f}(\tau)|_H^2 d\tau + \frac{a_1^2}{2n_0}\int_0^t |\tilde{h}(\tau)|^2 d\tau + \frac{a_2^2}{2n_0}\int_0^t |\tilde{\ell}(\tau)|^2 d\tau \\ &\qquad \text{for all } t \in [0,T]. \end{aligned} \tag{5.19}$$

Next, assigning $\chi_\xi(\tau)$ to z in (5.17), we have:

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}|\chi_\xi(\tau)|_H^2 + \kappa\left((a^\varepsilon)'((w_\xi)_x(\tau))(\chi_\xi)_x(\tau), (\chi_\xi)_x(\tau)\right) \\ &\quad + ((K^\varepsilon)'(w_\xi(\tau))\chi_\xi(\tau), \chi_\xi(\tau)) \\ &\quad + ([3\nu w_\xi^2(\tau) - 1]\chi_\xi(\tau), \chi_\xi(\tau)) = (\theta_\xi(\tau), \chi_\xi(\tau)), \\ &\qquad \text{a.e. } \tau \in (0,T). \end{aligned} \tag{5.20}$$

Note that:

$$([3\nu w_\xi^2(\tau) - 1]\chi_\xi(\tau), \chi_\xi(\tau)) \geq -|\chi_\xi(\tau)|_H^2 \quad \text{for all } \tau \in [0,T].$$

Therefore, taking into account (5.14), (5.15) and the inequality as above, applying the Schwarz inequality and integrating in τ over $[0,t]$, it follows from (5.20) that:

$$\begin{aligned} &\frac{1}{2}|\chi_\xi(t)|_H^2 + \varepsilon\kappa\int_0^t |(\chi_\xi)_x(\tau)|_H^2 d\tau \leq \frac{3}{2}\int_0^t |\chi_\xi(\tau)|_H^2 d\tau + \frac{1}{2}\int_0^t |\theta_\xi(\tau)|_H^2 d\tau \\ &\qquad \text{for all } t \in [0,T]. \end{aligned} \tag{5.21}$$

Note from (5.1) and **(II)** of Proposition 2.2 that

$$\begin{aligned} &\sup_{t\in[0,T]} |w_\xi(t)|_{BV(0,L)} \\ \leq\ & N_2\Bigg(|u_0^\varepsilon|_H^2 + |w_0^\varepsilon|_H^2 + \kappa V^\varepsilon(w_0^\varepsilon) + \int_0^L \widehat{g}(w_0^\varepsilon(x))\,dx + a_0^2|f + \xi z|_{L^2(0,T;H)}^2 \\ &\quad + a_1^2|h + \xi y_0|_{L^2(0,T)}^2 + a_2^2|\ell + \xi y_L|_{L^2(0,T)}^2 + b_1^2 + b_2^2\Bigg), \end{aligned}$$

where $N_2 > 0$ is some positive constant independent of $\xi \in [-1,1] \setminus \{0\}$. Therefore, from the continuous imbedding $BV(0,L) \hookrightarrow L^\infty(0,L)$ (cf. Proposition 2.3), we observe that

$$\sup_{t\in[0,T]} |w_\xi(t)|_{L^\infty(0,L)} \leq N_3, \tag{5.22}$$

where $N_3 > 0$ is some positive constant independent of $\xi \in [-1,1] \setminus \{0\}$.

Now, assigning $\theta_\xi(\tau)$ to z in (5.17), and integrating in τ over $[0,t]$, we observe from (5.14)–(5.15) that

$$\begin{aligned}
&\left|\int_0^t \langle \chi'_\xi(\tau), \theta_\xi(\tau)\rangle d\tau\right| \\
\le\ &\kappa\left(\frac{1}{\varepsilon}+\varepsilon\right)\int_0^t |(\chi_\xi)_x(\tau)|_H|(\theta_\xi)_x(\tau)|_H d\tau + \frac{1}{\varepsilon}\int_0^t |\chi_\xi(\tau)|_H|\theta_\xi(\tau)|_H d\tau \\
&\quad + \int_0^t |[3\nu w_\xi^2(\tau)-1]\chi_\xi(\tau)|_H|\theta_\xi(\tau)|_H d\tau + \int_0^t |\theta_\xi(\tau)|_H^2 d\tau \\
&\text{for all } t\in[0,T].
\end{aligned} \tag{5.23}$$

Applying the Schwarz inequality and using the bounded estimate (5.22), it follows from (5.23) that

$$\begin{aligned}
&\left|\int_0^t \langle \chi'_\xi(\tau), \theta_\xi(\tau)\rangle d\tau\right| \\
\le\ &\frac{1}{2}\int_0^t |(\theta_\xi)_x(\tau)|_H^2 d\tau + \frac{1}{2}\kappa^2\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\int_0^t |(\chi_\xi)_x(\tau)|_H^2 d\tau + \frac{1}{2\varepsilon^2}\int_0^t |\chi_\xi(\tau)|_H^2 d\tau \\
&\quad + \frac{(3\nu N_3^2-1)^2}{2}\int_0^t |\chi_\xi(\tau)|_H^2 d\tau + 2\int_0^t |\theta_\xi(\tau)|_H^2 d\tau \\
&\text{for all } t\in[0,T].
\end{aligned} \tag{5.24}$$

Therefore, adding (5.19) to (5.21)$\times\kappa\left(1/\varepsilon+\varepsilon\right)^2/\varepsilon$, and using (5.24), we get:

$$\begin{aligned}
&\frac{1}{2}|\theta_\xi(t)|_H^2 + \frac{1}{2\varepsilon}\kappa\left(\frac{1}{\varepsilon}+\varepsilon\right)^2|\chi_\xi(t)|_H^2 + \frac{1}{2}\int_0^t |(\theta_\xi)_x(\tau)|_H^2 d\tau + \frac{n_0}{2}\int_0^t |\theta_\xi(\tau,0)|^2 d\tau \\
&\quad + \frac{n_0}{2}\int_0^t |\theta_\xi(\tau,L)|^2 d\tau + \frac{1}{2}\kappa^2\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\int_0^t |(\chi_\xi)_x(\tau)|_H^2 d\tau \\
&\le \left(\frac{5}{2}+\frac{1}{2\varepsilon}\kappa\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\right)\int_0^t |\theta_\xi(\tau)|_H^2 d\tau \\
&\quad + \left(\frac{1}{2\varepsilon^2}+\frac{(3\nu N_3^2-1)^2}{2}+\frac{3}{2\varepsilon}\kappa\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\right)\int_0^t |\chi_\xi(\tau)|_H^2 d\tau \\
&\quad + \frac{a_0^2}{2}\int_0^t |\tilde{f}(\tau)|_H^2 d\tau + \frac{a_1^2}{2n_0}\int_0^t |\tilde{h}(\tau)|^2 d\tau + \frac{a_2^2}{2n_0}\int_0^t |\tilde{\ell}(\tau)|^2 d\tau
\end{aligned} \tag{5.25}$$

for all $t\in[0,T]$. Hence, applying the Gronwall lemma to (5.25), we observe that the following inequality holds:

$$\begin{aligned}
&\int_0^t |\theta_\xi(\tau)|_H^2 d\tau + \frac{1}{\varepsilon}\kappa\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\int_0^t |\chi_\xi(\tau)|_H^2 d\tau \\
&\le N_4\left(a_0^2|\tilde{f}|_{L^2(0,T;H)}^2 + \frac{a_1^2}{n_0}|\tilde{h}|_{L^2(0,T)}^2 + \frac{a_2^2}{n_0}|\tilde{\ell}|_{L^2(0,T)}^2\right) \quad \text{for all } t\in[0,T],
\end{aligned} \tag{5.26}$$

where N_4 is some positive constant independent of $\xi\in[-1,1]\setminus\{0\}$. Thus, it follows from

(5.25) and (5.26) that

$$
\begin{aligned}
&|\theta_\xi(t)|_H^2 + \frac{1}{\varepsilon}\kappa\left(\frac{1}{\varepsilon}+\varepsilon\right)^2|\chi_\xi(t)|_H^2 + \int_0^t |(\theta_\xi)_x(\tau)|_H^2 d\tau + n_0\int_0^t |\theta_\xi(\tau,0)|^2 d\tau \\
&\quad + n_0\int_0^t |\theta_\xi(\tau,L)|^2 d\tau + \kappa^2\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\int_0^t |(\chi_\xi)_x(\tau)|_H^2 d\tau \\
&\leq N_5\left(a_0^2|\tilde{f}|^2_{L^2(0,T;H)} + \frac{a_1^2}{n_0}|\tilde{h}|^2_{L^2(0,T)} + \frac{a_2^2}{n_0}|\tilde{\ell}|^2_{L^2(0,T)}\right) \quad \text{for all } t\in[0,T],
\end{aligned} \tag{5.27}
$$

where N_5 is some positive constant independent of $\xi \in [-1,1]\setminus\{0\}$.

Note from (5.14), (5.15) and (5.17) that the following inequality holds (cf. (5.23)):

$$
\begin{aligned}
&\left|\int_0^T \langle \chi'_\xi(\tau), z(\tau)\rangle d\tau\right| \\
\leq\ & \kappa\left(\frac{1}{\varepsilon}+\varepsilon\right)|(\chi_\xi)_x|_{L^2(0,T;H)}|z_x|_{L^2(0,T;H)} + \frac{1}{\varepsilon}|\chi_\xi|_{L^2(0,T;H)}|z|_{L^2(0,T;H)} \\
&\quad + |3\nu N_3^2 - 1||\chi_\xi|_{L^2(0,T;H)}|z|_{L^2(0,T;H)} + |\theta_\xi|_{L^2(0,T;H)}|z|_{L^2(0,T;H)} \\
&\text{for all } z\in L^2(0,T;X).
\end{aligned} \tag{5.28}
$$

Therefore, we observe from (5.22), (5.27) and (5.28) that

$$
|\chi'_\xi|_{L^2(0,T;X')} \leq N_6\left(|a_0||\tilde{f}|_{L^2(0,T;H)} + \frac{|a_1|}{\sqrt{n_0}}|\tilde{h}|_{L^2(0,T)} + \frac{|a_2|}{\sqrt{n_0}}|\tilde{\ell}|_{L^2(0,T)}\right) \tag{5.29}
$$

for some positive constant $N_6 > 0$, dependent on $\varepsilon, T, \kappa, n_0$ and independent of $\xi \in [-1,1]\setminus\{0\}$.

Similarly, we infer from (5.16), (5.27) and (5.29) that

$$
|\theta'_\xi|_{L^2(0,T;X')} \leq N_7\left(|a_0||\tilde{f}|_{L^2(0,T;H)} + \frac{|a_1|}{\sqrt{n_0}}|\tilde{h}|_{L^2(0,T)} + \frac{|a_2|}{\sqrt{n_0}}|\tilde{\ell}|_{L^2(0,T)}\right) \tag{5.30}
$$

for some positive constant $N_7 > 0$, dependent on $\varepsilon, T, \kappa, n_0$ and independent of $\xi \in [-1,1]\setminus\{0\}$.

By the uniform estimates (5.27), (5.29) and (5.30), there are a subsequence $\{\xi_n\}\subset\{\xi\}$ and functions $\overline{\theta}\in W^{1,2}(0,T;X')\cap L^2(0,T;X)\cap L^\infty(0,T;H)$ and $\overline{\chi}\in W^{1,2}(0,T;X')\cap L^2(0,T;X)\cap L^\infty(0,T;H)$ such that $\xi_n\to 0$,

$$
\left.
\begin{aligned}
\theta_{\xi_n}\to\overline{\theta}\quad & \text{weakly in } W^{1,2}(0,T;X'),\\
& \text{weakly in } L^2(0,T;X),\\
& \text{in } C([0,T];X'),\\
& \text{in } L^2(0,T;H),\\
& \text{weakly-}*\text{ in } L^\infty(0,T;H)
\end{aligned}
\right\} \tag{5.31}
$$

and

$$
\left.
\begin{aligned}
\chi_{\xi_n}\to\overline{\chi}\quad & \text{weakly in } W^{1,2}(0,T;X'),\\
& \text{weakly in } L^2(0,T;X),\\
& \text{in } C([0,T];X'),\\
& \text{in } L^2(0,T;H),\\
& \text{weakly-}*\text{ in } L^\infty(0,T;H)
\end{aligned}
\right\} \tag{5.32}
$$

as $n \to \infty$.

Here, from (5.13)–(5.15), (5.22), Lipschitz continuity of functions $(a^\varepsilon)'$ and $(K^\varepsilon)'$, and Lebesgue's dominated convergence theorem, we note that:

$$\begin{cases} (a^\varepsilon)'((w_{\xi_n})_x) \to (a^\varepsilon)'(w_x), \\ (K^\varepsilon)'(w_{\xi_n}) \to (K^\varepsilon)'(w), \qquad \text{strongly in } L^2(0,T;H), \text{ as } n \to \infty. \\ 3\nu w_{\xi_n}^2 - 1 \to 3\nu w^2 - 1, \end{cases} \tag{5.33}$$

Thus, taking a subsequence if necessary, we see from (5.13), (5.14) and (5.31)–(5.33) that:

$$\begin{cases} (a^\varepsilon)'((w_{\xi_n})_x)(\chi_{\xi_n})_x \to (a^\varepsilon)'(w_x)\overline{\chi}_x, \\ (K^\varepsilon)'(w_{\xi_n})\chi_{\xi_n} \to (K^\varepsilon)'(w)\overline{\chi}, \qquad \text{weakly in } L^2(0,T;H), \text{ as } n \to \infty. \\ [3\nu w_{\xi_n}^2 - 1]\chi_{\xi_n} \to [3\nu w^2 - 1]\overline{\chi}, \end{cases} \tag{5.34}$$

Assigning $z(\tau)$ to z in (5.16) and (5.17), integrating in τ over $[0,T]$ and taking the limits as $n \to \infty$, we observe from (5.31)–(5.34) that $(\overline{\theta}, \overline{\chi})$ satisfies the following system:

$$\begin{aligned} &\int_0^T \langle \overline{\theta}'(\tau), z(\tau)\rangle d\tau + \int_0^T \langle \overline{\chi}'(\tau), z(\tau)\rangle d\tau + \int_0^T \left(\overline{\theta}_x(\tau), z_x(\tau)\right) d\tau \\ &\qquad + n_0 \int_0^T \overline{\theta}(\tau,0) z(\tau,0) d\tau + n_0 \int_0^T \overline{\theta}(\tau,L) z(\tau,L) d\tau \\ &= \int_0^T (a_0 \tilde{f}(\tau), z(\tau)) d\tau + a_1 \int_0^T \tilde{h}(\tau) z(\tau,0) d\tau + a_2 \int_0^T \tilde{\ell}(\tau) z(\tau,L) d\tau \\ &\qquad \text{for all } z \in L^2(0,T;X) \text{ and any direction } (\tilde{f}, \tilde{h}, \tilde{\ell}) \in \mathcal{U}_{ad}; \end{aligned} \tag{5.35}$$

$$\begin{aligned} &\int_0^T \langle \overline{\chi}'(\tau), z(\tau)\rangle d\tau + \kappa \int_0^T ((a^\varepsilon)'(w_x(\tau))\overline{\chi}_x(\tau), z_x(\tau)) d\tau \\ &\qquad + \int_0^T ((K^\varepsilon)'(w(\tau))\overline{\chi}(\tau), z(\tau)) d\tau \\ &+ \int_0^T ([3\nu w^2(\tau) - 1]\overline{\chi}(\tau), z(\tau)) d\tau = \int_0^T (\overline{\theta}(\tau), z(\tau)) d\tau \\ &\qquad \text{for all } z \in L^2(0,T;X) \text{ and any direction } (\tilde{f}, \tilde{h}, \tilde{\ell}) \in \mathcal{U}_{ad}; \end{aligned} \tag{5.36}$$

$$\overline{\theta}(0,\cdot) = \lim_{n\to\infty} \theta_{\xi_n}(0,\cdot) = 0 \ (\in H) \ \text{ in } X'; \tag{5.37}$$

$$\overline{\chi}(0,\cdot) = \lim_{n\to\infty} \chi_{\xi_n}(0,\cdot) = 0 \ (\in H) \ \text{ in } X'. \tag{5.38}$$

Since the solutions to the Cauchy problem {(5.35)–(5.38)} are uniquely determined, we observe that the convergence (5.10) holds by putting $\theta = \overline{\theta}$ and $\chi = \overline{\chi}$. Thus, the proof of this lemma has been completed. □

Corollary 5.1. *Assume the same conditions as in Theorem 4.1. Let $\xi \in [-1,1] \setminus \{0\}$. Then, the Gâteaux derivative of the cost functional J^ε is continuous in the following sense:*

$$\begin{aligned} &D_{(\tilde{f},\tilde{h},\tilde{\ell})} J^\varepsilon(f + \xi z, h + \xi y_0, \ell + \xi y_L) \to D_{(\tilde{f},\tilde{h},\tilde{\ell})} J^\varepsilon(f,h,\ell) \\ &\text{for all } (f,h,\ell) \in \mathcal{U}_{ad}, \ (z, y_0, y_L) \in \mathcal{U}_{ad} \text{ and any direction } (\tilde{f}, \tilde{h}, \tilde{\ell}) \in \mathcal{U}_{ad} \end{aligned} \tag{5.39}$$

as $\xi \to 0$, where

$$D_{(\tilde{f},\tilde{h},\tilde{\ell})}J^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$$
$$:= \lim_{\lambda\to 0} \frac{J^\varepsilon(f+\xi z+\lambda\tilde{f}, h+\xi y_0+\lambda\tilde{h}, \ell+\xi y_L+\lambda\tilde{\ell}) - J^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)}{\lambda}$$

and

$$D_{(\tilde{f},\tilde{h},\tilde{\ell})}J^\varepsilon(f,h,\ell) := \lim_{\lambda\to 0} \frac{J^\varepsilon(f+\lambda\tilde{f}, h+\lambda\tilde{h}, \ell+\lambda\tilde{\ell}) - J^\varepsilon(f,h,\ell)}{\lambda}.$$

Proof. Note from (5.9) that

$$\begin{aligned} & D_{(\tilde{f},\tilde{h},\tilde{\ell})}J^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L) \\ = \; & c_0\int_0^T ((u_\xi - u_d)(t), \theta_\xi(t))dt + c_1\int_0^T ((w_\xi - w_d)(t), \chi_\xi(t))dt \\ & +m_0 a_0^2 \int_0^T ((f+\xi z)(t), \tilde{f}(t))dt + m_1 a_1^2 \int_0^T (h+\xi y_0)(t)\tilde{h}(t)dt \\ & +m_2 a_2^2 \int_0^T (\ell+\xi y_L)(t)\tilde{\ell}(t)dt \end{aligned} \tag{5.40}$$

for any $(f,h,\ell) \in \mathcal{U}_{ad}$, any $(z,y_0,y_L) \in \mathcal{U}_{ad}$ and any direction $(\tilde{f},\tilde{h},\tilde{\ell}) \in \mathcal{U}_{ad}$, where $(u_\xi, w_\xi)= \Lambda^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$ is the solution to $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f+\xi z, h+\xi y_0, \ell+\xi y_L)^\varepsilon$ on $[0,T]$, u_d and w_d are the given target profiles in $L^2(0,T;H)$, and $(\theta_\xi, \chi_\xi)(= D_{(\tilde{f},\tilde{h},\tilde{\ell})}\Lambda^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L))$ is the pair of functions obtained in **(I)** of Lemma 5.2.

Therefore, taking into account (5.10), (5.11) and (5.40), we easily see that the convergence (5.39) holds. □

Lemma 5.4. *Suppose the same conditions in Theorem 4.1. For any $\xi \in [-1,1] \setminus \{0\}$, $(f,h,\ell) \in \mathcal{U}_{ad}$, $(z,y_0,y_L) \in \mathcal{U}_{ad}$ and any direction $(\tilde{f},\tilde{h},\tilde{\ell}) \in \mathcal{U}_{ad}$, let $(p_\xi, q_\xi) = \Lambda_{ad}^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$. Then, (p_ξ, q_ξ) converges to $(p,q) = \Lambda_{ad}^\varepsilon(f,h,\ell)$ in $L^2(0,T;X) \times L^2(0,T;H)$ as $\xi \to 0$.*

Proof. For any $\xi \in [-1,1] \setminus \{0\}$, $(f,h,\ell) \in \mathcal{U}_{ad}$, $(z,y_0,y_L) \in \mathcal{U}_{ad}$ and any direction $(\tilde{f},\tilde{h},\tilde{\ell}) \in \mathcal{U}_{ad}$, let $(u_\xi, w_\xi) = \Lambda^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$ and $(u,w) := \Lambda^\varepsilon(f,h,\ell)$. Then, from the convergence result of solutions to $(\mathrm{P})^\varepsilon$ (cf. (5.2)), we have (cf. (5.11)):

$$(u_\xi, w_\xi) \to (u,w) \quad \text{in } L^2(0,T;H) \times C([0,T];H) \quad \text{as } \xi \to 0.$$

Also, note from **(II)** of Proposition 3.3 that $(p_\xi, q_\xi) = \Lambda_{ad}^\varepsilon(f+\xi z, h+\xi y_0, \ell+\xi y_L)$ satisfies the following adjoint system on $[0,T]$:

$$p_\xi \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X);$$
$$q_\xi \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H);$$

$$-p_\xi' - (p_\xi)_{xx} - q_\xi = c_0(u_\xi - u_d) \quad \text{in } Q; \tag{5.41}$$

$$
\begin{aligned}
&\int_t^T (-p'_\xi(\tau), z(\tau))d\tau + \int_t^T \langle -q'_\xi(\tau), z(\tau)\rangle d\tau \\
&\quad +\kappa \int_t^T ((a^\varepsilon)'((w_\xi)_x(\tau))(q_\xi)_x(\tau), z_x(\tau))\, d\tau \\
&\quad + \int_t^T ((K^\varepsilon)'(w_\xi(\tau))q_\xi(\tau), z(\tau))\, d\tau \\
&\quad + \int_t^T \big([3\nu(w_\xi(\tau))^2 - 1]q_\xi(\tau), z(\tau)\big)\, d\tau \\
&= c_1 \int_t^T (w_\xi(\tau) - w_d(\tau), z(\tau))\, d\tau \\
&\qquad \text{for all } t \in [0,T] \text{ and } z \in L^2(0,T;X);
\end{aligned}
\tag{5.42}
$$

$$
-(p_\xi)_x(t,0) + n_0 p_\xi(t,0) = (p_\xi)_x(t,L) + n_0 p_\xi(t,L) = 0, \quad t \in (0,T), \tag{5.43}
$$

$$
p_\xi(T,x) = q_\xi(T,x) = 0, \quad x \in (0,L). \tag{5.44}
$$

Multiplying (5.41) by $p_\xi(\tau)$, and applying the Schwarz inequality, we get:

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{d\tau}|p_\xi(\tau)|_H^2 + |(p_\xi)_x(\tau)|_H^2 + n_0|p_\xi(\tau,0)|^2 + n_0|p_\xi(\tau,L)|^2 \\
&\le |p_\xi(\tau)|_H^2 + \frac{1}{2}|q_\xi(\tau)|_H^2 + \frac{c_0^2}{2}|(u_\xi - u_d)(\tau)|_H^2 \quad \text{for a.e. } \tau \in (0,T).
\end{aligned}
$$

Then, applying the Gronwall lemma to the above inequality, we get the following estimate:

$$
\begin{aligned}
&\frac{1}{2}|p_\xi(t)|_H^2 + \int_t^T |(p_\xi)_x(\tau)|_H^2 d\tau + n_0 \int_t^T |p_\xi(\tau,0)|^2 d\tau + n_0 \int_t^T |p_\xi(\tau,L)|^2 d\tau \\
\le\ &\frac{1}{2}e^{2T}\left(|q_\xi|^2_{L^2(t,T;H)} + c_0^2|u_\xi - u_d|^2_{L^2(t,T;H)}\right) \quad \text{for all } t \in [0,T].
\end{aligned}
\tag{5.45}
$$

Next, multiplying (5.41) by $-p'_\xi(\tau)$ and applying the Schwarz inequality, we get:

$$
\begin{aligned}
&\frac{1}{2}|p'_\xi(\tau)|_H^2 - \frac{d}{d\tau}\left\{\frac{1}{2}|(p_\xi)_x(\tau)|_H^2 d\tau + \frac{n_0}{2}|p_\xi(\tau,0)|^2 + \frac{n_0}{2}|p_\xi(\tau,L)|^2\right\} \\
\le\ & |q_\xi(\tau)|_H^2 + c_0^2|(u_\xi - u_d)(\tau)|_H^2 \quad \text{for a.e. } \tau \in (0,T).
\end{aligned}
\tag{5.46}
$$

Therefore, integrating (5.46) in τ over $[t,T]$, we get:

$$
\begin{aligned}
&\frac{1}{2}\int_t^T |p'_\xi(\tau)|_H^2 d\tau + \frac{1}{2}|(p_\xi)_x(t)|_H^2 + \frac{n_0}{2}|p_\xi(t,0)|^2 + \frac{n_0}{2}|p_\xi(t,L)|^2 \\
\le\ & \int_t^T |q_\xi(\tau)|_H^2 d\tau + c_0^2|u_\xi - u_d|^2_{L^2(0,T;H)} \quad \text{for all } t \in [0,T].
\end{aligned}
\tag{5.47}
$$

Next, assigning $q_\xi(\tau)$ to z in (5.42) and applying the Schwarz inequality, we observe from (5.14) and (5.15) that:

$$
\begin{aligned}
&\frac{1}{2}|q_\xi(t)|_H^2 + \varepsilon\kappa \int_t^T |(q_\xi)_x(\tau)|_H^2 d\tau \\
\le\ & 2\int_t^T |q_\xi(\tau)|_H^2 d\tau + \frac{1}{2}\int_t^T |p'_\xi(\tau)|_H^2 d\tau + \frac{c_1^2}{2}\int_t^T |(w_\xi - w_d)(\tau)|_H^2 d\tau \\
&\qquad \text{for all } t \in [0,T].
\end{aligned}
\tag{5.48}
$$

Then, taking into account (5.11) and (5.47) and applying the Gronwall lemma to (5.48), we get the following estimate:

$$\begin{aligned} & |q_\xi(t)|_H^2 + \varepsilon\kappa \int_t^T |(q_\xi)_x(t)|_H^2 dt \\ \le\ & N_8 \left(c_0^2 |u-u_d|^2_{L^2(0,T;H)} + c_1^2 |w-w_d|^2_{L^2(0,T;H)} + 1 \right) \quad \text{for all } t \in [0,T], \end{aligned} \tag{5.49}$$

where $N_8 > 0$ is some constant independent of $\xi \in [-1,1] \setminus \{0\}$. Therefore it follows from (5.45), (5.47) with (5.49) that:

$$\begin{aligned} & |p_\xi(t)|_H^2 + \int_t^T |(p_\xi)_x(\tau)|_H^2 d\tau + n_0 \int_t^T |p_\xi(\tau,0)|^2 d\tau + n_0 \int_t^T |p_\xi(\tau,L)|^2 \tau \\ \le\ & N_9 \left(c_0^2 |u-u_d|^2_{L^2(0,T;H)} + c_1^2 |w-w_d|^2_{L^2(0,T;H)} + 1 \right) \quad \text{for all } t \in [0,T] \end{aligned} \tag{5.50}$$

and

$$\begin{aligned} & \int_t^T |p_\xi'(\tau)|_H^2 d\tau + |(p_\xi)_x(t)|_H^2 + n_0 |p_\xi(t,0)|^2 + n_0 |p_\xi(t,L)|^2 \\ \le\ & N_{10} \left(c_0^2 |u-u_d|^2_{L^2(0,T;H)} + c_1^2 |w-w_d|^2_{L^2(0,T;H)} + 1 \right) \quad \text{for all } t \in [0,T], \end{aligned} \tag{5.51}$$

where $N_9 > 0$ and $N_{10} > 0$ are positive constants independent of $\xi \in [-1,1] \setminus \{0\}$.

Also, we observe from (5.42) with (5.22) and (5.49)–(5.51) that

$$|q_\xi'|_{L^2(0,T;X')} \le N_{11} \left(c_0^2 |u-u_d|^2_{L^2(0,T;H)} + c_1^2 |w-w_d|^2_{L^2(0,T;H)} + 1 \right) \tag{5.52}$$

for some positive constant $N_{11} > 0$ independent of $\xi \in [-1,1] \setminus \{0\}$.

Therefore, by (5.33), (5.49)–(5.52), the uniqueness of solutions to the adjoint system {(3.5)–(3.8)} and the arguments similar to the proof of Lemma 5.3 (cf. (5.35)–(5.38)), namely, taking the limits in (5.41)–(5.44) as $\xi \to 0$, we observe that

$$\begin{gathered} (p_\xi, q_\xi) = \Lambda^\varepsilon_{ad}(f + \xi z, h + \xi y_0, \ell + \xi y_L) \to (p,q) = \Lambda^\varepsilon_{ad}(f,h,\ell) \\ \text{in } L^2(0,T;H) \times L^2(0,T;H) \text{ as } \xi \to 0. \end{gathered} \tag{5.53}$$

Note that $(p,q) = \Lambda^\varepsilon_{ad}(f,h,\ell)$ satisfies the following system:

$$-p' - p_{xx} - q = c_0(u - u_d) \quad \text{in } Q; \tag{5.54}$$

$$-p_x(t,0) + n_0 p(t,0) = p_x(t,L) + n_0 p(t,L) = 0, \quad t \in (0,T), \tag{5.55}$$

$$p(T,x) = 0, \quad x \in (0,L). \tag{5.56}$$

Subtract (5.54) from (5.41), and multiply it by $p_\xi - p$. Then, applying the Schwarz inequality, we get:

$$\begin{aligned} & -\frac{1}{2}\frac{d}{d\tau}|(p_\xi - p)(\tau)|_H^2 + |(p_\xi - p)_x(\tau)|_H^2 + n_0 |(p_\xi - p)(\tau,0)|^2 + n_0 |(p_\xi - p)(\tau,L)|^2 \\ \le\ & |(p_\xi - p)(\tau)|_H^2 + \frac{1}{2}|(q_\xi - q)(\tau)|_H^2 + \frac{c_0^2}{2}|(u_\xi - u)(\tau)|_H^2 \quad \text{for a.e. } \tau \in (0,T). \end{aligned}$$

Then, applying the Gronwall lemma to the above inequality, we get the following estimate:

$$\begin{aligned} & \frac{1}{2}|(p_\xi - p)(t)|_H^2 + \int_t^T |(p_\xi - p)_x(\tau)|_H^2 d\tau + n_0 \int_t^T |(p_\xi - p)(\tau,0)|^2 d\tau \\ & \quad + n_0 \int_t^T |(p_\xi - p)(\tau, L)|^2 d\tau \\ \leq & \ \frac{1}{2} e^{2T} \left(|q_\xi - q|^2_{L^2(t,T;H)} + c_0^2 |u_\xi - u|^2_{L^2(t,T;H)} \right) \qquad \text{for all } t \in [0,T], \end{aligned}$$

which implies from (5.53) and the convergence result of solutions to $(\mathrm{P})^\varepsilon$ (cf. (5.11)) that

$$p_\xi \to p \text{ in } L^2(0,T;X) \text{ as } \xi \to 0.$$

Thus, the proof of this lemma has been completed. □

The following function acts a key-role in the proof of Theorem 4.1.

Definition 5.1. We define the function $\gamma : [0,\infty) \to [0,\infty)$ by

$$\begin{aligned} \gamma(t) := \inf \{ & \|\xi(z, y_0, y_L)\|_{\mathcal{U}_{ad}} \ ; \\ & \|(\xi z + a_0 p_\xi - a_0 p, \xi y_0 + a_1 p_\xi - a_1 p, \xi y_L + a_2 p_\xi - a_2 p)\|_{\mathcal{U}_{ad}} \geq t \}, \end{aligned} \tag{5.57}$$

where $(p,q) = \Lambda^\varepsilon_{ad}(f,h,\ell)$ and $(p_\xi, q_\xi) = \Lambda^\varepsilon_{ad}(f + \xi z, h + \xi y_0, \ell + \xi y_L)$ for $\xi \in [-1,1] \setminus \{0\}$, $(f,h,\ell) \in \mathcal{U}_{ad}$ and $(z, y_0, y_L) \in \mathcal{U}_{ad}$. Clearly, $\gamma(\cdot)$ is a well-defined increasing function with $\gamma(0) = 0$ (cf. Lemma 5.4).

Lemma 5.5. *Assume the same conditions as in Theorem 4.1. Let* $\{(f_k, h_k, \ell_k); k = 1, 2, \cdots, n\}$ *be a sequence in* $\mathcal{U}_{ad}$ *defined by the numerical algorithm* (NA). *Let* $(p_n, q_n) = \Lambda^\varepsilon_{ad}(f_n, h_n, \ell_n)$, $\beta \in (0,1)$ *and* $\mu \in (0,1)$. *Put* $f_n^{p_n} := a_0(p_n + m_0 a_0 f_n)$, $h_n^{p_n} := a_1(p_n(\cdot,0) + m_1 a_1 h_n)$ *and* $\ell_n^{p_n} := a_2(p_n(\cdot,L) + m_2 a_2 \ell_n)$. *Assume that*

$$\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}} \neq 0. \tag{5.58}$$

Then, there is a minimal constant $j_n \in \mathbb{N} \cup \{0\}$ *such that*

$$\begin{aligned} & J^\varepsilon \left(f_n - \beta^{j_n} f_n^{p_n}, h_n - \beta^{j_n} h_n^{p_n}, \ell_n - \beta^{j_n} \ell_n^{p_n} \right) - J^\varepsilon(f_n, h_n, \ell_n) \\ \leq & \ -\mu \beta^{j_n} \|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|^2_{\mathcal{U}_{ad}}. \end{aligned} \tag{5.59}$$

Proof. By (5.58) and the definition of the Gâteaux derivative of $J^\varepsilon(\cdot)$, there is a constant $\delta_{\mu,n} > 0$ such that

$$\left| \frac{J^\varepsilon \left(f_n - \lambda f_n^{p_n}, h_n - \lambda h_n^{p_n}, \ell_n - \lambda \ell_n^{p_n} \right) - J^\varepsilon(f_n, h_n, \ell_n)}{\lambda} - D_{(-f_n^{p_n}, -h_n^{p_n}, -\ell_n^{p_n})} J^\varepsilon(f_n, h_n, \ell_n) \right|$$

$$< (1-\mu) \|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|^2_{\mathcal{U}_{ad}} \quad \text{for any } \lambda \in (-\delta_{\mu,n}, \delta_{\mu,n}) \setminus \{0\}. \tag{5.60}$$

Note from [31, Proof of Theorem 3.4] that

$$\begin{aligned} D_{(-f_n^{p_n}, -h_n^{p_n}, -\ell_n^{p_n})} J^\varepsilon(f_n, h_n, \ell_n) &= -|f_n^{p_n}|^2_{L^2(0,T;H)} - |h_n^{p_n}|^2_{L^2(0,T)} - |\ell_n^{p_n}|^2_{L^2(0,T)} \\ &= -\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|^2_{\mathcal{U}_{ad}}. \end{aligned}$$

Therefore, it follows from (5.60) that

$$\begin{aligned} & J^\varepsilon\left(f_n-\lambda f_n^{p_n}, h_n-\lambda h_n^{p_n}, \ell_n-\lambda \ell_n^{p_n}\right)-J^\varepsilon(f_n, h_n, \ell_n) \\ < \quad & -\lambda\mu\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}}^2 \quad \text{for any } \lambda \in (0, \delta_{\mu,n}). \end{aligned}$$

Therefore, we have only to take a minimal constant $j_n \in \mathbb{N} \cup \{0\}$ such that

$$0 < \beta^{j_n} < \delta_{\mu,n}.$$

Thus, the proof of this lemma has been completed. □

Lemma 5.6. *Assume the same conditions as in Theorem 4.1. Let* $\{(f_k, h_k, \ell_k); k = 1, 2, \cdots, n\}$ *be a sequence in* $\mathcal{U}_{ad}$ *defined by the numerical algorithm* (NA). *Let* $(p_n, q_n) = \Lambda_{ad}^\varepsilon(f_n, h_n, \ell_n)$, $\beta \in (0,1)$ *and* $\mu \in (0,1)$. *Put* $f_n^{p_n} := a_0(p_n + m_0 a_0 f_n)$, $h_n^{p_n} := a_1(p_n(\cdot, 0) + m_1 a_1 h_n)$ *and* $\ell_n^{p_n} := a_2(p_n(\cdot, L) + m_2 a_2 \ell_n)$. *Assume that*

$$\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}} \neq 0. \tag{5.61}$$

Also, let j_n *be the constant obtained in Lemma 5.5. Then, we have*

$$\begin{aligned} & \beta\gamma\left((1-\mu)\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}}\right) \\ \leq \quad & \beta^{j_n} \max\{m_0 a_0^2, m_1 a_1^2, m_2 a_2^2\}\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}}, \end{aligned} \tag{5.62}$$

where $\gamma(\cdot)$ *is the function defined by* (5.57) *in Definition 5.1.*

Proof. From the definition of j_n obtained in Lemma 5.5, we see that

$$\begin{aligned} & J^\varepsilon\left(f_n-\frac{\beta^{j_n}}{\beta} f_n^{p_n}, h_n-\frac{\beta^{j_n}}{\beta} h_n^{p_n}, \ell_n-\frac{\beta^{j_n}}{\beta} \ell_n^{p_n}\right)-J^\varepsilon(f_n, h_n, \ell_n) \\ > \quad & -\mu\frac{\beta^{j_n}}{\beta}\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}}^2. \end{aligned} \tag{5.63}$$

Here, by the mean-value theorem and the continuity of $D_{(-f_n^{p_n}, -h_n^{p_n}, -\ell_n^{p_n})} J^\varepsilon(f_n + \xi z, h_n + \xi h_0, \ell_n + \xi h_L)$ with respect to ξ, we observe that there is a constant $\theta \in (0,1)$ satisfying

$$\begin{aligned} & J^\varepsilon\left(f_n-\frac{\beta^{j_n}}{\beta} f_n^{p_n}, h_n-\frac{\beta^{j_n}}{\beta} h_n^{p_n}, \ell_n-\frac{\beta^{j_n}}{\beta} \ell_n^{p_n}\right)-J^\varepsilon(f_n, h_n, \ell_n) \\ = \quad & \int_0^{\frac{\beta^{j_n}}{\beta}} D_{(-f_n^{p_n}, -h_n^{p_n}, -\ell_n^{p_n})} J^\varepsilon\left(f_n-\xi f_n^{p_n}, h_n-\xi h_n^{p_n}, \ell_n-\xi \ell_n^{p_n}\right) d\xi \\ = \quad & \frac{\beta^{j_n}}{\beta} D_{(-f_n^{p_n}, -h_n^{p_n}, -\ell_n^{p_n})} J^\varepsilon\left(f_n-\theta\frac{\beta^{j_n}}{\beta} f_n^{p_n}, h_n-\theta\frac{\beta^{j_n}}{\beta} h_n^{p_n}, \ell_n-\theta\frac{\beta^{j_n}}{\beta} \ell_n^{p_n}\right) \\ = \quad & \frac{\beta^{j_n}}{\beta}\int_0^T\left(a_0 p_\theta(t)+m_0 a_0^2\left(f_n-\theta\frac{\beta^{j_n}}{\beta} f_n^{p_n}\right)(t), -f_n^{p_n}(t)\right) dt \\ & +\frac{\beta^{j_n}}{\beta}\int_0^T\left(a_1 p_\theta(t)+m_1 a_1^2\left(h_n-\theta\frac{\beta^{j_n}}{\beta} h_n^{p_n}\right)(t), -h_n^{p_n}(t)\right) dt \\ & +\frac{\beta^{j_n}}{\beta}\int_0^T\left(a_2 p_\theta(t)+m_2 a_2^2\left(\ell_n-\theta\frac{\beta^{j_n}}{\beta} \ell_n^{p_n}\right)(t), -\ell_n^{p_n}(t)\right) dt, \end{aligned} \tag{5.64}$$

where $(p_\theta, q_\theta) = \Lambda^\varepsilon_{ad}\left(f_n - \theta\frac{\beta^{j_n}}{\beta}f_n^{p_n}, h_n - \theta\frac{\beta^{j_n}}{\beta}h_n^{p_n}, \ell_n - \theta\frac{\beta^{j_n}}{\beta}\ell_n^{p_n}\right)$.

It follows from (5.63) and (5.64) that

$$\begin{aligned}
& (1-\mu)\,\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|^2_{\mathcal{U}_{ad}} \\
\le\ & \|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|^2_{\mathcal{U}_{ad}} \\
& + \int_0^T \left(a_0 p_\theta(t) + m_0 a_0^2\left(f_n - \theta\frac{\beta^{j_n}}{\beta}f_n^{p_n}\right)(t), -f_n^{p_n}(t)\right)dt \\
& + \int_0^T \left(a_1 p_\theta(t) + m_1 a_1^2\left(h_n - \theta\frac{\beta^{j_n}}{\beta}h_n^{p_n}\right)(t), -h_n^{p_n}(t)\right)dt \\
& + \int_0^T \left(a_2 p_\theta(t) + m_2 a_2^2\left(\ell_n - \theta\frac{\beta^{j_n}}{\beta}\ell_n^{p_n}\right)(t), -\ell_n^{p_n}(t)\right)dt \\
=\ & \int_0^T \left(a_0 p_\theta(t) + m_0 a_0^2\left(-\theta\frac{\beta^{j_n}}{\beta}f_n^{p_n}\right)(t) - a_0 p_n(t), -f_n^{p_n}(t)\right)dt \\
& + \int_0^T \left(a_1 p_\theta(t) + m_1 a_1^2\left(-\theta\frac{\beta^{j_n}}{\beta}h_n^{p_n}\right)(t) - a_1 h_n(t), -h_n^{p_n}(t)\right)dt \\
& + \int_0^T \left(a_2 p_\theta(t) + m_2 a_2^2\left(-\theta\frac{\beta^{j_n}}{\beta}\ell_n^{p_n}\right)(t) - a_2 \ell_n(t), -\ell_n^{p_n}(t)\right)dt \\
\le\ & \left|a_0 p_\theta(t) + m_0 a_0^2\left(-\theta\frac{\beta^{j_n}}{\beta}f_n^{p_n}\right) - a_0 p_n\right|_{L^2(0,T;H)} |f_n^{p_n}|_{L^2(0,T;H)} \\
& + \left|a_1 p_\theta(t) + m_1 a_1^2\left(-\theta\frac{\beta^{j_n}}{\beta}h_n^{p_n}\right) - a_1 h_n\right|_{L^2(0,T)} |h_n^{p_n}|_{L^2(0,T)} \\
& + \left|a_2 p_\theta(t) + m_2 a_2^2\left(-\theta\frac{\beta^{j_n}}{\beta}\ell_n^{p_n}\right) - a_2 \ell_n\right|_{L^2(0,T)} |\ell_n^{p_n}|_{L^2(0,T)}\,.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& (1-\mu)\,\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}} \\
\le & \left\|\left(a_0 p_\theta + m_0 a_0^2\left(-\theta\frac{\beta^{j_n}}{\beta}f_n^{p_n}\right) - a_0 p_n, a_1 p_\theta + m_1 a_1^2\left(-\theta\frac{\beta^{j_n}}{\beta}h_n^{p_n}\right) - a_1 h_n,\right.\right. \\
& \qquad\qquad \left.\left. a_2 p_\theta + m_2 a_2^2\left(-\theta\frac{\beta^{j_n}}{\beta}\ell_n^{p_n}\right) - a_2 \ell_n\right)\right\|_{\mathcal{U}_{ad}}.
\end{aligned} \tag{5.65}$$

By the definition of the function γ, we observe from (5.65) and $\theta \in (0,1)$ that

$$\begin{aligned}
& \gamma\left((1-\mu)\,\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}}\right) \\
\le\ & \theta\frac{\beta^{j_n}}{\beta}\left\|\left(m_0 a_0^2 f_n^{p_n}, m_1 a_1^2 h_n^{p_n}, m_2 a_2^2 \ell_n^{p_n}\right)\right\|_{\mathcal{U}_{ad}} \\
\le\ & \theta\frac{\beta^{j_n}}{\beta}\max\{m_0 a_0^2, m_1 a_1^2, m_2 a_2^2\}\,\|(f_n^{p_n}, h_n^{p_n}, \ell_n^{p_n})\|_{\mathcal{U}_{ad}}\,,
\end{aligned}$$

which implies that the inequality (5.62) holds. □

Now, we show the main Theorem 4.1 in this paper, which is concerned with the convergence for numerical algorithm (NA).

[**Proof of Theorem 4.1**] We show **(I)**. By (Step 5) in the numerical algorithm (NA)(cf. (5.59) or (5.68) below), we observe that $J^\varepsilon(f_n, h_n, \ell_n)$ is the non-increasing sequence with respect to n. Thus, from the non-negativity of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.8)), we infer that $\lim_{n\to\infty} J^\varepsilon(f_n, h_n, \ell_n)$ exists.

Now, we show **(II)** by contradiction. Therefore, we assume that (4.1) does not hold. Then, there are a constant $\delta > 0$ and a sequence $\{k\} \subset \mathbb{N}$ such that

$$|a_0(p_k + m_0 a_0 f_k)|_{L^2(0,T;H)} \ge \delta \quad \text{for any } k \in \mathbb{N}. \tag{5.66}$$

For simplicity, put $f_k^{p_k} := a_0(p_k + m_0 a_0 f_k)$, $h_k^{p_k} := a_1(p_k(\cdot,0) + m_1 a_1 h_k)$ and $\ell_k^{p_k} := a_2(p_k(\cdot,L) + m_2 a_2 \ell_k)$ for $k \in \mathbb{N}$. Then, note from (5.66) and the definition of norm $\|(\cdot,\cdot,\cdot)\|_{\mathcal{U}_{ad}}$ that

$$\|(f_k^{p_k}, h_k^{p_k}, \ell_k^{p_k})\|_{\mathcal{U}_{ad}} \ge |a_0(p_k + m_0 a_0 f_k)|_{L^2(0,T;H)} \ge \delta \quad \text{for any } k \in \mathbb{N}.$$

Therefore, from (5.66) and the definition of $\gamma(\cdot)$ in (5.57), it follows that

$$\gamma((1-\mu)\delta) \le \gamma\left((1-\mu)\,\|(f_k^{p_k}, h_k^{p_k}, \ell_k^{p_k})\|_{\mathcal{U}_{ad}}\right) \quad \text{for any } k \in \mathbb{N}, \tag{5.67}$$

where $\mu \in (0,1)$ is the stop parameter in **(Step 0)** of **(NA)**. Then, we observe from (5.59) and (5.62) that

$$\begin{aligned} & J^\varepsilon(f_{k+1}, h_{k+1}, \ell_{k+1}) - J^\varepsilon(f_k, h_k, \ell_k) \\ \le\ & -\mu\beta^{j_k}\,\|(f_k^{p_k}, h_k^{p_k}, \ell_k^{p_k})\|_{\mathcal{U}_{ad}}^2 \\ \le\ & -\frac{\mu\beta}{\max\{m_0 a_0^2, m_1 a_1^2, m_2 a_2^2\}}\gamma\left((1-\mu)\,\|(f_k^{p_k}, h_k^{p_k}, \ell_k^{p_k})\|_{\mathcal{U}_{ad}}\right)\|(f_k^{p_k}, h_k^{p_k}, \ell_k^{p_k})\|_{\mathcal{U}_{ad}} \\ \le\ & -\frac{\mu\beta}{\max\{m_0 a_0^2, m_1 a_1^2, m_2 a_2^2\}}\gamma((1-\mu)\delta)\,\delta \\ <\ & 0 \qquad\qquad \text{for any } k \in \mathbb{N}. \end{aligned} \tag{5.68}$$

By repeating the procedure (5.68), we get:

$$\begin{aligned} J^\varepsilon(f_{k+1}, h_{k+1}, \ell_{k+1}) &\le J^\varepsilon(f_k, h_k, \ell_k) - \frac{\mu\beta}{\max\{m_0 a_0^2, m_1 a_1^2, m_2 a_2^2\}}\gamma((1-\mu)\delta)\,\delta \\ &\le J^\varepsilon(f_{k-1}, h_{k-1}, \ell_{k-1}) - \frac{2\mu\beta}{\max\{m_0 a_0^2, m_1 a_1^2, m_2 a_2^2\}}\gamma((1-\mu)\delta)\,\delta \\ &\le \cdots \\ &\le J^\varepsilon(f_1, h_1, \ell_1) - \frac{k\mu\beta}{\max\{m_0 a_0^2, m_1 a_1^2, m_2 a_2^2\}}\gamma((1-\mu)\delta)\,\delta \\ &\qquad \text{for any } k \in \mathbb{N}. \end{aligned}$$

Thus, the above inequality implies that

$$J^\varepsilon(f_{k+1}, h_{k+1}, \ell_{k+1}) \to -\infty \qquad \text{as} \;\; k \to \infty. \tag{5.69}$$

This contradicts the non-negativity of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.8)). Hence, (4.1) holds. Similarly, we observe that (4.2) and (4.3) hold.

Finally, we show **(III)**. By **(I)** of Theorem 4.1 and the definition of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.8)), we observe that $\{f_n, h_n, \ell_n\}$ is bounded in $\mathcal{U}_{ad}$. Therefore, there are a subsequence $\{n_k\}$ of $\{n\}$ and a function $(f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**}) \in \mathcal{U}_{ad}$ such that $n_k \to \infty$,

$$f_{n_k} \to f^\varepsilon_{**} \quad \text{weakly in } L^2(0,T;H),$$

$$h_{n_k} \to h^\varepsilon_{**} \quad \text{weakly in } L^2(0,T),$$
$$\ell_{n_k} \to \ell^\varepsilon_{**} \quad \text{weakly in } L^2(0,T)$$

as $k \to \infty$. Then, from the convergence result of solutions to $(\mathrm{P})^\varepsilon$ in Lemma 5.1, we observe that (cf. (5.2))

$$\begin{aligned}\Lambda^\varepsilon(f_{n_k}, h_{n_k}, \ell_{n_k}) = (u_{n_k}, w_{n_k}) \to (u^\varepsilon_{**}, w^\varepsilon_{**}) = \Lambda^\varepsilon(f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**})\\ \text{in } L^2(0,T;H) \times C([0,T];H) \;\; \text{as } k \to \infty.\end{aligned} \tag{5.70}$$

Also, from (5.70) and the slight modification of the proof of Lemma 5.4, we infer that there are a function $(p^\varepsilon_{**}, q^\varepsilon_{**}) \in L^2(0,T;X) \times L^2(0,T;H)$ and a subsequence of $\{n_k\}$ (which we also denote $\{n_k\}$ for simplicity) satisfying

$$\begin{aligned}\Lambda^\varepsilon_{ad}(f_{n_k}, h_{n_k}, \ell_{n_k}) = (p_{n_k}, q_{n_k}) \to (p^\varepsilon_{**}, q^\varepsilon_{**}) = \Lambda^\varepsilon_{ad}(f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**})\\ \text{in } L^2(0,T;X) \times L^2(0,T;H) \;\; \text{as } k \to \infty.\end{aligned} \tag{5.71}$$

Therefore we infer (4.1)–(4.3), (5.70) and (5.71) that the assertion **(III)** of Theorem 4.1 holds. Thus, the proof of Theorem 4.1 has been completed. □

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