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MILD SOLUTIONS ARE WEAK SOLUTIONS IN A CLASS OF (NON)LINEAR MEASURE-VALUED EVOLUTION EQUATIONS ON A BOUNDED DOMAIN

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Abstract. We study the connection between mild and weak solutions for a class of measure-valued evolution equations on the bounded domain $[0,1]$. Mass moves, driven by a velocity field that is either a function of the spatial variable only, $v = v(x)$, or depends on the solution μ itself: $v = v[\mu](x)$. The flow is stopped at the boundaries of $[0,1]$, while mass is gated away by a certain right-hand side. In previous works [16, 18], we showed the existence and uniqueness of appropriately defined mild solutions for $v = v(x)$ and $v = v[\mu](x)$, respectively. In the current paper we define weak solutions (by specifying the weak formulation and the space of test functions). The main result is that the aforementioned mild solutions are weak solutions, both when $v = v(x)$ and when $v = v[\mu](x)$.

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1 Introduction

Measure-valued evolution equations have been used in a large number of recent mathematical publications to model for instance animal aggregations [6, 9], structured populations [1, 7, 13, 21], pedestrian dynamics [12], and defects in metallic crystals [28]. The majority of works that study well-posedness of measure-valued equations and properties of their solutions treat these equations in the full space –see for instance also [2, 8, 10, 11, 27]– although many relevant problems involve *boundaries* and *bounded domains*. Examples of such problems –apart from the ones mentioned above– are intracellular transport processes, cf. [17, Section 1], and manufacturing chains [20]. Defining mathematically and physically 'correct' boundary conditions is a challenge, however. The present paper is a continuation of the author's work (in collaboration with Hille and Muntean) that focuses explicitly on bounded domains and boundary conditions.

Our first step is to consider a one-dimensional measure-valued transport equation restricted to the unit interval $[0,1]$ where mass moves according to a prescribed velocity field v and stops when reaching the boundary. A short-hand notation for this equation is:

$$\frac{\partial}{\partial t}\mu_t + \frac{\partial}{\partial x}(v\,\mu_t) = F_f(\mu_t). \tag{1.1}$$

Here, the perturbation map $F_f : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ is given by $F_f(\mu) := f \cdot \mu$.
In [16], we proved the well-posedness of this equation, in the sense of *mild solutions*, and the convergence of solutions corresponding to a sequence $(f_n)_{n\in\mathbb{N}}$ in the right-hand side. Some specific choices for $(f_n)_{n\in\mathbb{N}}$ represent for instance effects in a boundary layer that approximate, as $n \to \infty$, sink or source effects localized on the boundary (flux boundary conditions). The boundary layer corresponds to exactly those regions in $[0,1]$ where the functions f_n are nonzero.

Next, we want to consider (1.1) for velocity fields that are no longer *fixed* elements of $\mathrm{BL}([0,1])$. Instead of v, we write $v[\mu]$ for the velocity field that depends functionally on the measure μ. An example is

$$v[\mu](x) := \int_{[0,1]} \mathcal{K}(x-y)\,d\mu(y) = (\mathcal{K} * \mu)(x), \tag{1.2}$$

where the convolution encodes nonlocal interactions due to a kernel $\mathcal{K}$ in a population with distribution μ. This is a widely used choice of v, e.g. in interacting particle systems or biological aggregation models. The example (1.2) is a special case of the class of velocity fields (see Assumption 4.1) that are admissible in the framework of the current paper.

For such solution-dependent $v = v[\mu]$, the transport equation on $[0,1]$ becomes

$$\frac{\partial}{\partial t}\mu_t + \frac{\partial}{\partial x}(v[\mu_t]\,\mu_t) = F_f(\mu_t). \tag{1.3}$$

The well-posedness of (1.3) for $f \in \mathrm{BL}([0,1])$ was proved in [18] in the sense of mild solutions. The analysis turns out to build on the analysis for (1.1), hence it was useful to consider (1.1) before the more general (1.3). For $v = v[\mu]$, mild solutions are defined as the limit of a sequence $(\mu^k)_{k\in\mathbb{N}}$ of so-called *Euler approximations*. Such μ^k is constructed on each subinterval $(t_j^k, t_{j+1}^k]$ as a mild solution to (1.1) for velocity $v[\mu^k_{t_j^k}]$. Within a subinterval, $v[\mu^k_{t_j^k}]$ is a *fixed* element of $\mathrm{BL}([0,1])$ that is the same for all time $t \in (t_j^k, t_{j+1}^k]$. For further details on how μ^k is defined, see Section 4.

The works [5] and [21] treat comparable models, however posed on *infinite* domains. They consider weak solutions, and in fact, they construct those weak solutions that –roughly speaking– correspond to 'our' mild solutions.

The aim of the current paper is:

to investigate how and in which sense the mild solutions from [16, 18] correspond to weak solutions like the ones in e.g. [5, 21].

Note that the essence of weak solutions lies in the specific choice of the weak formulation and of the space of test functions that appear in the definition; cf. e.g. Definition 3.3 with weak formulation (3.2) and space of test functions (3.3).

Compared to e.g. [5, 21], our case is more complicated due to the bounded domain; the material flow is induced by the velocity v in the interior, but it is stopped once characteristics reach any of the boundary points. The *stopped flow* introduces subtleties when trying to find the appropriate definition of weak solutions. The domain in [21] is in fact $[0, \infty)$. Their velocity is required to point inward at $x = 0$, though, which is sufficient to make sure that no mass escapes the domain. For us, a demand on the sign of the velocity at $x = 0$ or $x = 1$ is too restrictive; cf. the remark we make about this in [18, Section 1].

In the current work, we overcome these difficulties and give the appropriate definition of weak solutions. We believe this indeed is the appropriate definition because of the following main result of this paper, that consists of two parts. Formulated in plain words in a pseudo-theorem, the first part of this result reads:

Theorem. *Mild solutions to* (1.1) *are weak solutions (in an appropriate sense).*

A more precise formulation follows in Theorem 3.5. Next, we use this property on each of the subintervals in an Euler approximation and show that in the limit as the mesh size goes to zero, we obtain a weak solution to (1.3). In other words:

Theorem. *Mild solutions to* (1.3) *are weak solutions.*

This result is stated in full detail in Theorem 4.8.

Our justification for speaking about *appropriate* definition, is exactly the fact that

we show the relation between these weak solutions and mild solutions in this paper (more about this in Section 4, directly after Definition 4.4). Mild solutions have a considerable advantage over weak solutions in the sense that it is directly clear how they should be interpreted, whereas defining weak solutions involves some seemingly arbitrary choices. Which choices to make is not directly evident from modelling considerations.
On the other hand, as was argued in [18, Section 1], the mild formulation in terms of the variation of constants formula (3.1) follows directly from a probabilistic interpretation. For more details, see [16, Section 6]. Moreover, the exact form of the variation of contsants formula is unambiguous, provided the system that is to be modelled.
Subsequently, mild solutions for $v = v[\mu]$ follow in a straight-forward manner, using the variation of constants formula as a building block; cf. (4.2) and Definition 4.5.

The structure of this paper is as follows. Section 2 provides preliminaries on the stopped flow on the interval $[0,1]$ induced by the velocity field $v : [0,1] \to \mathbb{R}$. In Section 3, we recall the results from [16] regarding the existence and uniqueness of mild solutions to (1.1). We introduce the concept of weak solutions, and show in Theorem 3.5 that the mild solutions from [16] are weak solutions. Section 4 briefly recalls the main ideas from [18]: the construction of Euler approximations using solutions to the variation of constants formula as building blocks. Theorem 4.5 repeats the result that Euler approximations converge as the mesh size goes to zero; this result is an alternative way of saying that mild solutions to (1.3) exist and are unique. We show in Section 4 that these mild solutions are weak solutions (in an appropriate sense). The paper is concluded by a section (Section 5) in which we discuss the wider context of our results and the open issues that are subject for follow-up work.

2 Preliminaries

This section contains the preliminaries that are needed for the arguments in this paper. These preliminaries were presented before in [16, 18]. We assume that the reader is familiar with elementary measure-theoretical concepts, such as finite Borel measures, the total variation norm $\|\cdot\|_{\mathrm{TV}}$, and the dual bounded Lipschitz norm $\|\cdot\|^*_{\mathrm{BL}}$. An overview of the basic concepts used in this paper can be found in Appendix A.

The rest of this section is devoted to properties of the flow induced on $[0,1]$ by some fixed $v \in \mathrm{BL}([0,1])$, a bounded Lipschitz velocity field. This flow is a fundamental mechanism in the model considered in this paper.

We assume that a single particle ('individual') is moving in the domain $[0,1]$ deterministically, described by the differential equation for its position $x(t)$ at time t:

$$\begin{cases} \dot{x}(t) = v(x(t)), \\ x(0) = x_0. \end{cases} \tag{2.1}$$

A solution to (2.1) is unique, it exists for time up to reaching the boundary 0 or 1 and depends continuously on initial conditions. Let $x(\,\cdot\,; x_0)$ be this solution and I_{x_0} be its *maximal* interval of existence. Define

$$\tau_\partial(x_0) := \sup I_{x_0} \in [0,\infty],$$

i.e. $\tau_\partial(x_0)$ is the time at which the solution starting at x_0 reaches the boundary (if it happens) when x_0 is an interior point. Note that $\tau_\partial(x_0) = 0$ when x_0 is a boundary point where v points outwards, while $\tau_\partial(x_0) > 0$ when x_0 is a boundary point where v vanishes or points inwards.

The *stopped flow* on $[0,1]$ associated to v is the family of maps $\Phi_t : [0,1] \to [0,1]$, $t \geqslant 0$, defined by

$$\Phi_t(x_0) := \begin{cases} x(t;x_0), & \text{if } t \in I_{x_0}, \\ x(\tau_\partial(x_0);x_0), & \text{otherwise.} \end{cases} \tag{2.2}$$

To lift the dynamics to the space of measures, we define $P_t : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ by means of the push-forward under Φ_t: for all $\mu \in \mathcal{M}([0,1])$,

$$P_t\mu := \Phi_t\#\mu = \mu \circ \Phi_t^{-1}; \tag{2.3}$$

see (A.2). Clearly, P_t maps positive measures to positive measures and P_t is mass preserving on positive measures. Since the family of maps $(\Phi_t)_{t\geqslant 0}$ forms a semigroup, so do the maps P_t in the space $\mathcal{M}([0,1])$. That is, $(P_t)_{t\geqslant 0}$ is a *Markov semigroup* on $\mathcal{M}[0,1]$ (cf. [24]). The basic estimate

$$\|P_t\mu\|_{\mathrm{TV}} \leqslant \|\mu\|_{\mathrm{TV}} \tag{2.4}$$

holds for $\mu \in \mathcal{M}([0,1])$. In [18, Section 2.2] a number of other properties (bounds and Lipschitz-like estimates) of $(P_t)_{t\geqslant 0}$ are given.

3 Mild and weak solutions for prescribed velocity

Mild solutions to (1.1) are defined in the following sense:

Definition 3.1 (See [16, Definition 2.4]). A *measure-valued mild solution* to the Cauchy-problem associated to (1.1) on $[0,T]$ with initial value $\nu \in \mathcal{M}([0,1])$ is a continuous map $\mu : [0,T] \to \mathcal{M}([0,1])_{\mathrm{BL}}$ that is $\|\cdot\|_{\mathrm{TV}}$-bounded and that satisfies the variation of constants formula

$$\mu_t = P_t\,\nu \;+\; \int_0^t P_{t-s}F_f(\mu_s)\,ds \qquad \text{for all } t \in [0,T]. \tag{3.1}$$

Amongst others, we showed in [16] that mild solutions in the sense of Definition 3.1 exist and are unique. We repeat those results in the following theorem.

Theorem 3.2 (Existence and uniqueness of mild solutions to (1.1)). *Let $f : [0,1] \to \mathbb{R}$ be a piecewise bounded Lipschitz function such that $v(x) \neq 0$ at any point x of discontinuity of f. Then for each $T \geqslant 0$ and $\mu_0 \in \mathcal{M}([0,1])$ there exists a unique continuous and locally $\|\cdot\|_{\mathrm{TV}}$-bounded solution to* (3.1).

Proof. See [16, Propositions 3.1 and 3.3] for details. □

In the rest of this section we will compare mild solutions provided by Theorem 3.2 to solutions in a different sense: weak solutions. Recall that $\langle \cdot, \cdot \rangle$ denotes the duality paring between finite Borel measures on $[0,1]$ and bounded measurable functions on $[0,1]$; see (A.1).

Definition 3.3 (Weak solution to (1.1))**.** Fix $T \geqslant 0$, let $v \in \mathrm{BL}([0,1])$ and let $f : [0,1] \to \mathbb{R}$ be piecewise bounded Lipschitz. Then $\mu : [0,T] \to \mathcal{M}([0,1])$ is a *weak solution* to (1.1) corresponding to initial condition $\nu_0 \in \mathcal{M}([0,1])$, if

$$\langle \mu_T, \psi(\cdot,T)\rangle - \langle \nu_0, \psi(\cdot,0)\rangle = \int_0^T \langle \mu_t, \partial_t\psi(\cdot,t) + \partial_x\psi(\cdot,t)\cdot v\rangle \, dt + \int_0^T \langle F_f(\mu_t), \psi(\cdot,t)\rangle \, dt \tag{3.2}$$

is satisfied for all

$$\psi \in \Lambda^T := \Big\{ \psi \in C^1([0,1]\times[0,T]) \,:\, \partial_x\psi(0,t) = \partial_x\psi(1,t) = 0 \text{ for all } 0 \leqslant t \leqslant T \Big\}. \tag{3.3}$$

Remark 3.4. Note that the boundary conditions imposed on the test functions are closely related to the behaviour of the stopped flow at the boundary points $x = 0$ and $x = 1$: no flux. This is a general phenomenon. See for instance [26, pp. 63–64 and 140], where several spaces of test functions are given, depending on which behaviour at the boundary is to be modelled. For the sake of completeness, we note that [26] treats Brownian motion (diffusion). The used operator A is the corresponding infinitesimal generator acting –as is common in probabilistic literature– *on the test functions*, not on the solution itself.

The main result of this section is the following theorem.

Theorem 3.5 (Mild solutions to (1.1) are weak solutions)**.** *Let $\mu : [0,T] \to \mathcal{M}([0,1])$ be the mild solution provided by Theorem 3.2, corresponding to initial value $\nu_0 \in \mathcal{M}([0,1])$. Then, μ is a weak solution of* (1.1).

For the proof of Theorem 3.5 we were inspired by the proof of [22, Proposition 3.7].

Proof. Let ψ be an arbitrary element from the set of test functions given in (3.3). Recall that I_y is the maximal interval of existence of a solution to (2.1) –i.e. restricted to $[0,1]$– with initial condition y. Recall moreover that $\tau_\partial(y) = \sup I_y$, i.e. $\tau_\partial(y)$ is the time at which the solution starting at y reaches the boundary (if it happens) when y is an interior point. Note that $\tau_\partial(y) = 0$ when y is a boundary point where v points outwards, while $\tau_\partial(y) > 0$ when y is a boundary point where v vanishes or points inwards.

Consider

$$\int_0^T \langle P_t\, \nu_0, \partial_x\psi(\cdot,t)\cdot v + \partial_t\psi(\cdot,t)\rangle \; dt$$

$$= \int_0^T \langle \nu_0, \partial_x\psi(\Phi_t(\cdot),t)\cdot v(\Phi_t(\cdot)) + \partial_t\psi(\Phi_t(\cdot),t)\rangle \; dt$$

$$= \int_{[0,1]} \int_0^T \Big(\partial_x\psi(\Phi_t(y),t)\cdot v(\Phi_t(y)) + \partial_t\psi(\Phi_t(y),t)\Big)\, dt\, d\nu_0(y)$$

$$= \int_{[0,1]} \int_0^{\tau_\partial(y)\wedge T} \frac{d}{dt}\psi(\Phi_t(y),t)\, dt\, d\nu_0(y)$$

$$+ \int_{[0,1]} \int_{\tau_\partial(y)\wedge T}^{T} \Big(\partial_x\psi(\Phi_t(y),t)\cdot v(\Phi_t(y)) + \partial_t\psi(\Phi_t(y),t)\Big)\, dt\, d\nu_0(y), \quad (3.4)$$

where the *truncation* is defined as $\tau_\partial(y)\wedge T := \min(\tau_\partial(y),T)$; this is a continuous function in y. Interchanging the order of integration is allowed by Fubini's theorem, because the integrand is bounded. The subdivision of the domain $[0,T]$ with respect to τ_∂ is necessary, since the semigroup Φ_t represents the *stopped* flow and therefore the identity $\frac{d}{dt}\Phi_t(y) = v(\Phi_t(y))$ is only valid if $t \in I_y$. Hence, only in the first integral on the right-hand side of (3.4), the chain rule

$$\frac{d}{dt}\psi(\Phi_t(y),t) = \partial_x\psi(\Phi_t(y),t)\cdot v(\Phi_t(y)) + \partial_t\psi(\Phi_t(y),t) \quad (3.5)$$

can be used. Note that at time $\tau_\partial(y)\wedge T$ this identity at least holds one-sidedly as $t \nearrow (\tau_\partial(y)\wedge T)$, which is sufficient for the first integral on the right-hand side to be correct.

Define, for $z \in \{0,1\}$, the sets

$$\Omega_z^T := \{y \in [0,1] \,:\, \tau_\partial(y) < T \text{ and } \Phi_{\tau_\partial(y)}(y) = z\}. \quad (3.6)$$

These are connected subsets of $[0,1]$.
If $y \in \Omega^T := [0,1] \setminus \big(\Omega_0^T \cup \Omega_1^T\big)$, then $\tau_\partial(y) \geqslant T$, and obviously

$$\int_{\tau_\partial(y)\wedge T}^{T} \Big(\partial_x\psi(\Phi_t(y),t)\cdot v(\Phi_t(y)) + \partial_t\psi(\Phi_t(y),t)\Big)\, dt \; = 0 \quad (3.7)$$

in the second integral on the right-hand side of (3.4), since the domain of integration is a nullset (in fact, a single point).
If $y \in \Omega_0^T \cup \Omega_1^T$, then $\tau_\partial(y) < T$ and $\Phi_t(y) \in \{0,1\}$ for all $(\tau_\partial(y)\wedge T) \leqslant t \leqslant T$. Therefore,

taking into account the test functions' boundary conditions $\partial_x\psi(0,t) = \partial_x\psi(1,t) = 0$, we have that

$$\partial_x\psi(\Phi_t(y),t) = 0 \tag{3.8}$$

for all $y \in \Omega_0^T \cup \Omega_1^T$ and $(\tau_\partial(y) \wedge T) \leqslant t \leqslant T$.

Due to (3.7) and (3.8), the second term on the right-hand side of (3.4) can be written as

$$\begin{aligned}
&\int\limits_{[0,1]} \int\limits_{\tau_\partial(y)\wedge T}^{T} \Big(\partial_x\psi(\Phi_t(y),t)\cdot v(\Phi_t(y)) + \partial_t\psi(\Phi_t(y),t)\Big)\,dt\,d\nu_0(y) \\
&\qquad = \int\limits_{\Omega_0^T} \int\limits_{\tau_\partial(y)}^{T} \partial_t\psi(0,t)\,dt\,d\nu_0(y) + \int\limits_{\Omega_1^T} \int\limits_{\tau_\partial(y)}^{T} \partial_t\psi(1,t)\,dt\,d\nu_0(y) \\
&\qquad = \int\limits_{\Omega_0^T} \Big(\psi(\underbrace{0}_{=\Phi_T(y)},T) - \psi\big(0,\tau_\partial(y)\big)\Big)\,d\nu_0(y) \\
&\qquad\quad + \int\limits_{\Omega_1^T} \Big(\psi(\underbrace{1}_{=\Phi_T(y)},T) - \psi\big(1,\tau_\partial(y)\big)\Big)\,d\nu_0(y).
\end{aligned} \tag{3.9}$$

The first term on the right-hand side of (3.4) we treat as follows:

$$\begin{aligned}
&\int\limits_{[0,1]} \int\limits_{0}^{\tau_\partial(y)\wedge T} \frac{d}{dt}\psi(\Phi_t(y),t)\,dt\,d\nu_0(y) \\
&\qquad = \int\limits_{\Omega^T} \int\limits_{0}^{T} \frac{d}{dt}\psi(\Phi_t(y),t)\,dt\,d\nu_0(y) \\
&\qquad\quad + \int\limits_{\Omega_0^T} \int\limits_{0}^{\tau_\partial(y)} \frac{d}{dt}\psi(\Phi_t(y),t)\,dt\,d\nu_0(y) + \int\limits_{\Omega_1^T} \int\limits_{0}^{\tau_\partial(y)} \frac{d}{dt}\psi(\Phi_t(y),t)\,dt\,d\nu_0(y) \\
&\qquad = \int\limits_{\Omega^T} \Big(\psi\big(\Phi_T(y),T\big) - \psi(y,0)\Big)\,d\nu_0(y) \\
&\qquad\quad + \int\limits_{\Omega_0^T} \Big(\psi(0,\tau_\partial(y)) - \psi(y,0)\Big)\,d\nu_0(y) + \int\limits_{\Omega_1^T} \Big(\psi(1,\tau_\partial(y)) - \psi(y,0)\Big)\,d\nu_0(y).
\end{aligned} \tag{3.10}$$

Note that, for *all* $y \in [0,1]$, the function $t \mapsto \psi(\Phi_t(y),t)$ is differentiable for all $t \geqslant 0$. If $t = \tau_\partial(y)$, then the differentiability follows from the boundary conditions on ψ.

Combining (3.4) with (3.9) and (3.10), we obtain that

$$\int_0^T \langle P_t\,\nu_0, \partial_x\psi(\cdot,t)\cdot v + \partial_t\psi(\cdot,t)\rangle\;dt = \int_{[0,1]} \Big(\psi\big(\Phi_T(y),T\big) - \psi(y,0)\Big)\,d\nu_0(y)$$
$$= \langle P_T\,\nu_0, \psi(\cdot,T)\rangle - \langle \nu_0, \psi(\cdot,0)\rangle\,. \tag{3.11}$$

Next, consider

$$\int_0^T \left\langle \int_0^t P_{t-s}F_f(\mu_s)\,ds, \partial_x\psi(\cdot,t)\cdot v + \partial_t\psi(\cdot,t)\right\rangle\,dt$$
$$= \int_0^T\int_0^t \langle F_f(\mu_s), \partial_x\psi(\Phi_{t-s}(\cdot),t)\cdot v(\Phi_{t-s}(\cdot)) + \partial_t\psi(\Phi_{t-s}(\cdot),t)\rangle\;ds\,dt$$
$$= \int_0^T \left\langle F_f(\mu_s), \int_s^T \Big(\partial_x\psi(\Phi_{t-s}(\cdot),t)\cdot v(\Phi_{t-s}(\cdot)) + \partial_t\psi(\Phi_{t-s}(\cdot),t)\Big)\,dt\right\rangle\,ds \tag{3.12}$$

We subdivide the domain of the spatial integration into Ω_0^{T-s}, Ω_1^{T-s} and $\Omega^{T-s} := [0,1]\setminus(\Omega_0^{T-s}\cup\Omega_1^{T-s})$, with the sets Ω_z^{T-s} defined analogous to (3.6).
For each $z\in\{0,1\}$, we have

$$\int_0^T\int_{\Omega_z^{T-s}}\int_s^T \Big(\partial_x\psi(\Phi_{t-s}(y),t)\cdot v(\Phi_{t-s}(y)) + \partial_t\psi(\Phi_{t-s}(y),t)\Big)\,dt\,dF_f(\mu_s)(y)\,ds$$
$$= \int_0^T\int_{\Omega_z^{T-s}}\int_s^{\tau_\partial(y)+s} \frac{d}{dt}\psi(\Phi_{t-s}(y),t)\,dt\,dF_f(\mu_s)(y)\,ds$$
$$+ \int_0^T\int_{\Omega_z^{T-s}}\int_{\tau_\partial(y)+s}^T \Big(\underbrace{\partial_x\psi(z,t)}_{=0}\cdot v(z) + \underbrace{\partial_t\psi(z,t)}_{=\frac{d}{dt}\psi(z,t)}\Big)\,dt\,dF_f(\mu_s)(y)\,ds$$
$$= \int_0^T\int_{\Omega_z^{T-s}} \Big(\psi\big(z,\tau_\partial(y)+s\big) - \psi(y,s)\Big)\,dF_f(\mu_s)(y)\,ds$$
$$+ \int_0^T\int_{\Omega_z^{T-s}} \Big(\psi(z,T) - \psi\big(z,\tau_\partial(y)+s\big)\Big)\,dF_f(\mu_s)(y)\,ds$$
$$= \int_0^T\int_{\Omega_z^{T-s}} \Big(\psi\big(\Phi_{T-s}(y),T\big) - \psi(y,s)\Big)\,dF_f(\mu_s)(y)\,ds. \tag{3.13}$$

Considering the spatial domain of integration Ω^{T-s}, we find

$$\int_0^T \int_{\Omega^{T-s}} \int_s^T \Big(\partial_x \psi(\Phi_{t-s}(y), t) \cdot v(\Phi_{t-s}(y)) + \partial_t \psi(\Phi_{t-s}(y), t) \Big) \, dt \, dF_f(\mu_s)(y) \, ds$$

$$= \int_0^T \int_{\Omega^{T-s}} \int_s^T \frac{d}{dt} \psi(\Phi_{t-s}(y), t) \, dt \, dF_f(\mu_s)(y) \, ds$$

$$= \int_0^T \int_{\Omega^{T-s}} \Big(\psi\big(\Phi_{T-s}(y), T\big) - \psi(y, s) \Big) \, dF_f(\mu_s)(y) \, ds. \tag{3.14}$$

Together, (3.12), (3.13) and (3.14) yield

$$\int_0^T \left\langle \int_0^t P_{t-s} F_f(\mu_s) \, ds, \partial_x \psi(\cdot, t) \cdot v + \partial_t \psi(\cdot, t) \right\rangle dt = \int_0^T \big\langle F_f(\mu_s), \psi\big(\Phi_{T-s}(\cdot), T\big) \big\rangle \, ds$$

$$- \int_0^T \langle F_f(\mu_s), \psi(\cdot, s) \rangle \, ds. \tag{3.15}$$

It follows from (3.11) and (3.15), and from the variation of constants formula (3.1) that

$$\int_0^T \langle \mu_t, \partial_x \psi(\cdot, t) \cdot v + \partial_t \psi(\cdot, t) \rangle \, dt = \langle P_T \, \nu_0, \psi(\cdot, T) \rangle - \langle \nu_0, \psi(\cdot, 0) \rangle$$

$$+ \int_0^T \big\langle F_f(\mu_s), \psi\big(\Phi_{T-s}(\cdot), T\big) \big\rangle \, ds - \int_0^T \langle F_f(\mu_s), \psi(\cdot, s) \rangle \, ds. \tag{3.16}$$

Note that

$$\int_0^T \big\langle F_f(\mu_s), \psi\big(\Phi_{T-s}(\cdot), T\big) \big\rangle \, ds = \int_0^T \langle P_{T-s} \, F_f(\mu_s), \psi(\cdot, T) \rangle \, ds$$

$$= \left\langle \int_0^T P_{T-s} \, F_f(\mu_s) \, ds, \psi(\cdot, T) \right\rangle,$$

and hence

$$\langle P_T \, \nu_0, \psi(\cdot, T) \rangle + \int_0^T \big\langle F_f(\mu_s), \psi\big(\Phi_{T-s}(\cdot), T\big) \big\rangle \, ds = \langle \mu_T, \psi(\cdot, T) \rangle \, .$$

Equation (3.16) can thus be written as

$$\int_0^T \langle \mu_t, \partial_x \psi(\cdot, t) \cdot v + \partial_t \psi(\cdot, t) \rangle \, dt = \langle \mu_T, \psi(\cdot, T) \rangle - \langle \nu_0, \psi(\cdot, 0) \rangle - \int_0^T \langle F_f(\mu_s), \psi(\cdot, s) \rangle \, ds,$$

which shows that μ is a weak solution of (1.1). □

4 Mild and weak solutions for measure-dependent velocity

In this section we summarize the results of [18] and compare the concept of mild solutions from [18] to weak solutions of (1.3). In [18], we generalized the assumptions on v from [16] in the following way to measure-dependent velocity fields:

Assumption 4.1 (Assumptions on the measure-dependent velocity field)**.** Assume that $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$ is a mapping such that:

(i) $v[\mu] \in \mathrm{BL}([0,1])$, for each $\mu \in \mathcal{M}([0,1])$.

Furthermore, assume that for any $R > 0$ there are constants K_R, L_R, M_R such that for all $\mu, \nu \in \mathcal{M}([0,1])$ satisfying $\|\mu\|_{\mathrm{TV}} \leqslant R$ and $\|\nu\|_{\mathrm{TV}} \leqslant R$, the following estimates hold:

(ii) $\|v[\mu]\|_\infty \leqslant K_R$,

(iii) $|\, v[\mu]\,|_{\mathrm{L}} \leqslant L_R$, and

(iv) $\|v[\mu] - v[\nu]\|_\infty \leqslant M_R\, \|\mu - \nu\|^*_{\mathrm{BL}}$.

In [18], we proved well-posedness of (1.3):

$$\frac{\partial}{\partial t}\mu_t + \frac{\partial}{\partial x}(v[\mu_t]\,\mu_t) = F_f(\mu_t)$$

on $[0,1]$, in the sense of mild solutions; cf. Definition 4.4 and Theorem 4.5. Like in [18], in this paper we restrict ourselves to f that is bounded Lipschitz on $[0,1]$. See Section 5 for further discussion on this assumption.

Mild solutions for measure-dependent $v = v[\mu]$ are defined using mild solutions for fixed $v \in \mathrm{BL}([0,1])$ as a building block via an Euler-like approach. We first summarize the required notation used in [18].

Let $v \in \mathrm{BL}([0,1])$ and $f \in \mathrm{BL}([0,1])$ be arbitrary. For all $t \geqslant 0$, we define $Q_t : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ to be the operator that maps the initial condition to the solution in the sense of Definition 3.1. Theorem 3.2 guarantees that this operator is well-defined and continuous for $\|\cdot\|^*_{\mathrm{BL}}$. Moreover, Q preserves positivity, due to [16, Corollary 3.4]. The set of operators $(Q_t)_{t\geqslant 0}$ constitutes a semigroup and has useful other properties, like certain Lipschitz estimates. The exact results and their proofs can be found in [18, Section 2.3].
In the sequel we will write e.g. Q^v and $Q^{v'}$ to distinguish between the semigroups Q on $\mathcal{M}([0,1])$ associated to $v \in \mathrm{BL}([0,1])$ and $v' \in \mathrm{BL}([0,1])$, respectively.

We now introduce the aforementioned forward-Euler-like approach to construct approximate solutions. Let $T > 0$ be given. Let $N \geqslant 1$ be fixed and define a set $\alpha \subset [0, T]$ as follows:

$$\alpha := \big\{ t_j \in [0, T] \ : \ 0 \leqslant j \leqslant N, \ t_0 = 0, \ t_N = T, \ t_j < t_{j+1} \big\}, \tag{4.1}$$

which we call a *partition* of the interval $[0, T]$. Here, N denotes the number of *subintervals* in α.

Let $\mu_0 \in \mathcal{M}([0, 1])$ be fixed. For a given partition $\alpha := \{t_0, \ldots, t_N\} \subset [0, T]$, define a measure-valued trajectory $\mu \in C([0, T]; \mathcal{M}([0, 1]))$ by

$$\begin{cases} \mu_t := Q^{v_j}_{t-t_j} \, \mu_{t_j}, & \text{if } t \in (t_j, t_{j+1}]; \\ v_j := v[\mu_{t_j}]; \\ \mu_{t=0} = \mu_0, \end{cases} \tag{4.2}$$

for all $j \in \{0, \ldots, N-1\}$. Here, $(Q^v_t)_{t \geqslant 0}$ denotes the semigroup introduced above. Note that by Assumption 4.1, Part (i), $v_j = v[\mu_{t_j}] \in \mathrm{BL}([0, 1])$ for each j.

We call this a forward-Euler-like approach, because it is the analogon of the forward Euler method for ODEs (cf. e.g. [4, Chapter 2]). See [18, Section 3] for further explanation.

The conditions in Parts (ii)–(iv) of Assumption 4.1 are only required to hold for measures in a TV-norm bounded set, in view of the following lemma:

Lemma 4.2. *Let $\mu_0 \in \mathcal{M}([0, 1])$ be given and let $v : \mathcal{M}([0, 1]) \times [0, 1] \to \mathbb{R}$ satisfy Assumption 4.1(i). For a given partition $\alpha := \{t_0, \ldots, t_N\} \subset [0, T]$, let $\mu \in C([0, T]; \mathcal{M}([0, 1]))$ be defined by* (4.2)*. Then for all $t \in [0, T]$*

$$\|\mu_t\|^*_{\mathrm{BL}} \ \leqslant \ \|\mu_t\|_{\mathrm{TV}} \ \leqslant \ \|\mu_0\|_{\mathrm{TV}} \exp(\|f\|_\infty T).$$

This bound is in particular independent of t, N and the distribution of points within α.

Proof. See the proof of [18, Lemma 3.4] for details. □

We construct sequences of Euler approximations, each following from a sequence of partitions $(\alpha_k)_{k \in \mathbb{N}}$ that satisfies the following assumption:

Assumption 4.3 (Assumptions on the sequence of partitions)**.** Let $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence of partitions of $[0, T]$ and let $(N_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ be the corresponding sequence such that each α_k is of the form

$$\alpha_k := \big\{ t^k_j \in [0, T] \ : \ 0 \leqslant j \leqslant N_k, \ t^k_0 = 0, \ t^k_{N_k} = T, \ t^k_j < t^k_{j+1} \big\}. \tag{4.3}$$

Define

$$M^{(k)} := \max_{j \in \{0, \ldots, N_k - 1\}} t^k_{j+1} - t^k_j \tag{4.4}$$

for all $k \in \mathbb{N}$. Assume that the sequence $(M^{(k)})_{k \in \mathbb{N}}$ is nonincreasing and $M^{(k)} \to 0$ as $k \to \infty$.

A mild solution is defined as follows:

Definition 4.4 (See [18, Definition 3.8])**.** Let the space of continuous maps from $[0,T]$ to $\mathcal{M}([0,1])$ be endowed with the metric defined for all $\mu,\nu \in C([0,T];\mathcal{M}([0,1]))$ by

$$\sup_{t\in[0,T]} \|\mu_t - \nu_t\|^*_{\mathrm{BL}}. \tag{4.5}$$

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of partitions satisfying Assumption 4.3. For each $k \in \mathbb{N}$, let $\mu^k \in C([0,T];\mathcal{M}([0,1]))$ be defined by (4.2) with partition α_k. Then, for any such sequence of partitions $(\alpha_k)_{k\in\mathbb{N}}$, any limit of a subsequence of $(\mu^k)_{k\in\mathbb{N}}$ is called a *(measure-valued) mild solution* of (1.3).

The name *mild solutions* is appropriate, first of all because they are constructed from piecewise mild solutions in the sense of Definition 3.1.
Moreover, as their name suggests, *weak* solutions in general constitute a weaker solution concept than mild solutions. That is, a mild solution (meaning: a solution of the variation of constants formula) is in general necessarily a weak solution. See e.g. [15, p. 4–5], [22, Proposition 3.7], Theorem 3.5 in this paper, and the way in which mild and weak solutions are connected on [25, pp. 258–259]. In all of these references, the equations treated are simpler than (1.3) that is considered in this section. Here, mild solutions are the ones constructed in [18], being defined as the limit of Euler approximations. They are not solutions of the variation of constants formula. However, they are still elements of the set of weak solutions, as we will show in Theorem 4.8. This implication is an extra justification for the name *mild* solutions.

In the rest of this paper we focus on *positive* measure-valued solutions, because these are the only physically relevant solutions in many applications.

Theorem 4.5 (Existence and uniqueness of mild solutions to (1.3))**.** *Let $\mu_0 \in \mathcal{M}^+([0,1])$ be given and let $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$ satisfy Assumption 4.1. Endow the space $C([0,T];\mathcal{M}([0,1]))$ with the metric defined by* (4.5)*. Then, there is a unique element of $C([0,T];\mathcal{M}^+([0,1]))$ with initial condition μ_0, that is a mild solution in the sense of Definition 4.4. That is, for each sequence of partitions $(\alpha_k)_{k\in\mathbb{N}}$ satisfying Assumption 4.3, the corresponding sequence $(\mu^k)_{k\in\mathbb{N}}$ defined by* (4.2) *is a sequence in $C([0,T];\mathcal{M}^+([0,1]))$ and has a unique limit as $k\to\infty$.*
Moreover, this limit is independent of the choice of $(\alpha_k)_{k\in\mathbb{N}}$.

Proof. See [18, Theorem 3.10] for details. □

Definition 4.6 (Weak solution to (1.3))**.** Fix $T \geqslant 0$, let $f \in \mathrm{BL}([0,1])$ and let $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$ satisfy Assumption 4.1. Then $\mu : [0,T] \to \mathcal{M}([0,1])$ is a *weak solution* to (1.3) corresponding to initial condition $\nu_0 \in \mathcal{M}([0,1])$, if

$$\langle \mu_T, \psi(\cdot,T)\rangle - \langle \nu_0, \psi(\cdot,0)\rangle = \int_0^T \langle \mu_t, \partial_t\psi(\cdot,t) + \partial_x\psi(\cdot,t)\cdot v[\mu_t]\rangle\, dt + \int_0^T \langle F_f(\mu_t), \psi(\cdot,t)\rangle\, dt \tag{4.6}$$

is satisfied for all $\psi \in \Lambda^T$, with Λ^T as defined in (3.3).

To show that mild solutions in the sense of Definition 4.4 are weak solutions in the sense of Definition 4.6, the following result is useful.

Lemma 4.7 (Cf. portmanteau theorem)**.** *Let* $(\nu^k)_{k\in\mathbb{N}} \subset \mathcal{M}([0,1])$ *and assume there is an* $R > 0$ *such that* $\nu^k([0,1]) \leqslant R$ *for all* $k \in \mathbb{N}$*. Let* $\nu \in \mathcal{M}([0,1])$*. The following are equivalent:*

(a) $\langle \nu^k, \phi \rangle \to \langle \nu, \phi \rangle$ *as* $k \to \infty$ *for all* $\phi \in \mathrm{BL}([0,1])$*;*

(b) $\langle \nu^k, \phi \rangle \to \langle \nu, \phi \rangle$ *as* $k \to \infty$ *for all* $\phi \in C_b([0,1])$*.*

Proof. Apply [23, Theorem 13.16] to the sequence $(\nu^k/R)_{k\in\mathbb{N}} \subset \mathcal{M}_{\leqslant 1}([0,1])$, and use the equivalence "(ii)⇔(iii)" therein, which implies the equivalence with (b) above. ☐

The main result of this paper is the following theorem.

Theorem 4.8 (Mild solutions to (1.3) are weak solutions)**.** *Let* $\mu : [0,T] \to \mathcal{M}^+([0,1])$ *be the mild solution provided by Theorem 4.5, corresponding to initial value* $\nu_0 \in \mathcal{M}^+([0,1])$*. Then,* μ *is a weak solution of* (1.3)*.*

The proof of Theorem 4.8 uses ideas from the proof of [22, Proposition 4.9].

Proof. Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of partitions satisfying Assumption 4.3, and for each $k \in \mathbb{N}$, let $\mu^k \in C([0,T]; \mathcal{M}([0,1]))$ be the corresponding sequence of Euler approximations defined by (4.2). Since μ is the unique mild solution, $\mu = \lim_{k\to\infty} \mu^k$ holds with convergence in the metric given by (4.5).

Fix a $\psi \in \Lambda^T$. For each $j \in \{0, \ldots, N_k - 1\}$ we apply Theorem 3.5 to subinterval $[t_j^k, t_{j+1}^k]$ in the Euler approximation. Note that the restriction $\psi|_{[t_j^k, t_{j+1}^k]}$ is an appropriate test function on the corresponding domain $[0,1] \times [t_j^k, t_{j+1}^k]$. The individual test functions on these subdomains are therefore all derived from the *same* ψ on $[0,1] \times [0,T]$. Moreover, the required regularity per subdomain and the spatial boundary conditions are simply inherited from ψ.
For each $j \in \{0, \ldots, N_k - 1\}$ we thus have

$$\left\langle \mu^k_{t^k_{j+1}}, \psi(\cdot, t^k_{j+1}) \right\rangle - \left\langle \mu^k_{t^k_j}, \psi(\cdot, t^k_j) \right\rangle = \int_{t^k_j}^{t^k_{j+1}} \left\langle \mu^k_t, \partial_t \psi(\cdot,t) + \partial_x \psi(\cdot,t) \cdot v[\mu^k_{t^k_j}] \right\rangle dt + \int_{t^k_j}^{t^k_{j+1}} \langle F_f(\mu^k_t), \psi(\cdot,t) \rangle \, dt, \tag{4.7}$$

while $\mu^k_{t^k_0} = \mu^k_0 = \nu_0$. Note that within the integrals we do not need to write $\psi|_{[t^k_j, t^k_{j+1}]}$, but we can simply use ψ. Summation over $j \in \{0, \ldots, N_k - 1\}$ yields

$$\begin{aligned}
\left\langle \mu^k_T, \psi(\cdot, T) \right\rangle - \langle \nu_0, \psi(\cdot, 0) \rangle &= \sum_{j=0}^{N_k-1} \int_{t^k_j}^{t^k_{j+1}} \left\langle \mu^k_t, \partial_t \psi(\cdot, t) + \partial_x \psi(\cdot, t) \cdot v[\mu^k_{t^k_j}] \right\rangle dt \\
&\quad + \int_0^T \left\langle F_f(\mu^k_t), \psi(\cdot, t) \right\rangle dt \\
&= \int_0^T \left\langle \mu^k_t, \partial_t \psi(\cdot, t) + \partial_x \psi(\cdot, t) \cdot \bar{v}^k(t, \cdot) \right\rangle dt \\
&\quad + \int_0^T \left\langle F_f(\mu^k_t), \psi(\cdot, t) \right\rangle dt. \qquad (4.8)
\end{aligned}$$

Here, $\bar{v}^k : [0, T] \to \mathrm{BL}([0, 1])$ is defined by $\bar{v}^k(t, \cdot) := v[\mu^k_{t^k_j}]$ whenever $t \in (t^k_j, t^k_{j+1}]$, while $\bar{v}^k(0, \cdot) = v[\nu_0]$.

Note that, for each $t \in [0, T]$ fixed, $\psi(\cdot, t) \in \mathrm{BL}([0, 1])$ due to the assumed regularity on the test functions. Since $\mu^k \to \mu$ with respect to the metric in (4.5) as $k \to \infty$, we have in particular that

$$\left\langle \mu^k_T, \psi(\cdot, T) \right\rangle \overset{k\to\infty}{\longrightarrow} \langle \mu_T, \psi(\cdot, T) \rangle . \qquad (4.9)$$

The second term on the right-hand side of (4.8) we treat as follows:

$$\begin{aligned}
\left| \int_0^T \left\langle F_f(\mu^k_t), \psi(\cdot, t) \right\rangle dt - \int_0^T \langle F_f(\mu_t), \psi(\cdot, t) \rangle \, dt \right| &\leqslant \int_0^T \left| \left\langle \mu^k_t - \mu_t, \psi(\cdot, t) \cdot f(\cdot) \right\rangle \right| dt \\
&\leqslant \int_0^T \|\psi(\cdot, t)\|_{\mathrm{BL}} \cdot \|f\|_{\mathrm{BL}} \cdot \|\mu^k_t - \mu_t\|^*_{\mathrm{BL}} \, dt, \qquad (4.10)
\end{aligned}$$

where we used that for each $t \in [0, T]$ the product $\psi(\cdot, t) \cdot f(\cdot)$ is bounded Lipschitz and $\|\psi(\cdot, t) \cdot f(\cdot)\|_{\mathrm{BL}} \leqslant \|\psi(\cdot, t)\|_{\mathrm{BL}} \cdot \|f\|_{\mathrm{BL}}$; cf. (A.4).

Since $\psi \in C^1([0, 1] \times [0, T])$, we know that

$$\sup_{[0,1]\times[0,T]} |\psi| < \infty, \quad \text{and} \quad \sup_{[0,1]\times[0,T]} |\partial_x \psi| < \infty,$$

and hence,

$$\sup_{t\in[0,T]} \|\psi(\cdot, t)\|_{\mathrm{BL}} < \infty.$$

Consequently, we have

$$\left| \int_0^T \left\langle F_f(\mu_t^k), \psi(\cdot,t) \right\rangle \, dt - \int_0^T \left\langle F_f(\mu_t), \psi(\cdot,t) \right\rangle \, dt \right| \leqslant \sup_{t\in[0,T]} \|\psi(\cdot,t)\|_{\mathrm{BL}} \cdot \|f\|_{\mathrm{BL}} \cdot \underbrace{\sup_{t\in[0,T]} \|\mu_t^k - \mu_t\|_{\mathrm{BL}}^*}_{\to 0 \text{ as } k\to\infty} \cdot T, \tag{4.11}$$

and thus

$$\int_0^T \left\langle F_f(\mu_t^k), \psi(\cdot,t) \right\rangle \, dt \stackrel{k\to\infty}{\longrightarrow} \int_0^T \left\langle F_f(\mu_t), \psi(\cdot,t) \right\rangle \, dt. \tag{4.12}$$

Finally, we consider the first term on the right-hand side of (4.8).
For each $t \in [0,T]$, it holds that $\|\mu_t^k - \mu_t\|_{\mathrm{BL}}^* \to 0$ as $k \to \infty$ since the convergence $\mu^k \to \mu$ is in the metric (4.5). Then in particular, $\left\langle \mu_t^k, \phi \right\rangle \to \left\langle \mu_t, \phi \right\rangle$ for all $\phi \in \mathrm{BL}([0,1])$. Since $\mu_t^k([0,1]) = \|\mu_t^k\|_{\mathrm{TV}}$, Lemma 4.2 provides a bound on $\mu_t^k([0,1])$ that is independent of k and t. We can therefore apply Lemma 4.7, and conclude that

$$\left\langle \mu_t^k, \phi \right\rangle \to \left\langle \mu_t, \phi \right\rangle, \quad \text{as } k \to \infty, \tag{4.13}$$

for all $\phi \in C_b([0,1])$.
For each $t \in [0,T]$, we have that $\partial_t\psi(\cdot,t), \partial_x\psi(\cdot,t) \in C_b([0,1])$ by the assumption that $\psi \in C_b^1([0,1] \times [0,T])$. Moreover, $v[\mu_t] \in \mathrm{BL}([0,1])$ and thus $\partial_x\psi(\cdot,t) \cdot v[\mu_t] \in C_b([0,1])$. Hence, (4.13) implies that

$$\left\langle \mu_t^k, \partial_t\psi(\cdot,t) \right\rangle \to \left\langle \mu_t, \partial_t\psi(\cdot,t) \right\rangle, \text{ and} \tag{4.14}$$

$$\left\langle \mu_t^k, \partial_x\psi(\cdot,t) \cdot v[\mu_t] \right\rangle \to \left\langle \mu_t, \partial_x\psi(\cdot,t) \cdot v[\mu_t] \right\rangle \tag{4.15}$$

as $k \to \infty$.

For each $t \in [0,T]$, the function $\partial_x\psi(\cdot,t) \cdot \left(\bar{v}^k(t,\cdot) - v[\mu_t]\right)$ is in $C_b([0,1])$ and therefore

$$\begin{aligned} \left| \left\langle \mu_t^k, \partial_x\psi(\cdot,t) \cdot \bar{v}^k(t,\cdot) \right\rangle - \left\langle \mu_t^k, \partial_x\psi(\cdot,t) \cdot v[\mu_t] \right\rangle \right| &= \left| \int_{[0,1]} \partial_x\psi(x,t) \cdot \left(\bar{v}^k(t,x) - v[\mu_t](x)\right) \, d\mu_t^k(x) \right| \\ &\leqslant \sup_{\tau\in[0,T]} \|\partial_x\psi(\cdot,\tau)\|_\infty \cdot \|\bar{v}^k(t,\cdot) - v[\mu_t]\|_\infty \cdot \|\mu_t^k\|_{\mathrm{TV}}. \end{aligned} \tag{4.16}$$

Due to Lemma 4.2, it holds for each $t \in [0,T]$ that $\|\mu_t^k\|_{\mathrm{TV}} \leqslant R := \|\nu_0\|_{\mathrm{TV}} \exp(\|f\|_\infty T)$. The measure μ_t is positive for all $t \in [0,T]$, and thus

$$\|\mu_t\|_{\mathrm{TV}} = \|\mu_t\|_{\mathrm{BL}}^* \leqslant \underbrace{\|\mu_t^k\|_{\mathrm{BL}}^*}_{\leqslant R} + \underbrace{\|\mu_t^k - \mu_t\|_{\mathrm{BL}}^*}_{\to 0}, \tag{4.17}$$

whence $\|\mu_t\|_{\mathrm{TV}} \leqslant R$. We can now use Assumption 4.1(iv) to estimate the term $\|\bar{v}^k(t,\cdot) - v[\mu_t]\|_\infty$.
Without loss of generality, assume that $t > 0$. Let $j \in \{0,\dots,N_k\}$ be such that $t_j^k < t \leqslant t_{j+1}^k$. Then

$$\begin{aligned}\|\bar{v}^k(t,\cdot) - v[\mu_t]\|_\infty = \|v[\mu^k_{t_j^k}] - v[\mu_t]\|_\infty &\leqslant \|v[\mu^k_{t_j^k}] - v[\mu_{t_j^k}]\|_\infty + \|v[\mu_{t_j^k}] - v[\mu_t]\|_\infty \\ &\leqslant M_R\,\|\mu^k_{t_j^k} - \mu_{t_j^k}\|^*_{\mathrm{BL}} + M_R\,\|\mu_{t_j^k} - \mu_t\|^*_{\mathrm{BL}} \\ &\leqslant M_R \sup_{\tau\in[0,T]} \|\mu^k_\tau - \mu_\tau\|^*_{\mathrm{BL}} + M_R\,\|\mu_{t_j^k} - \mu_t\|^*_{\mathrm{BL}}.\end{aligned} \tag{4.18}$$

The first term on the right-hand side goes to zero, because of the uniform convergence of μ^k to μ. The second term on the right-hand side goes to zero because $t \mapsto \mu_t$ is continuous and $t - t_j^k \leqslant M^{(k)} \to 0$ as $k \to \infty$. Consequently, the left-hand side of (4.18) must vanish as $k \to \infty$.
It follows from (4.16) and the fact that the left-hand side of (4.18) goes to zero, that

$$\left|\left\langle \mu_t^k, \partial_x\psi(\cdot,t)\cdot\bar{v}^k(t,\cdot)\right\rangle - \left\langle \mu_t^k, \partial_x\psi(\cdot,t)\cdot v[\mu_t]\right\rangle\right| \to 0, \text{ as } k\to\infty. \tag{4.19}$$

This results yields, together with (4.15), that for all $t \in [0,T]$

$$\left\langle \mu_t^k, \partial_x\psi(\cdot,t)\cdot\bar{v}^k(t,\cdot)\right\rangle \to \left\langle \mu_t, \partial_x\psi(\cdot,t)\cdot v[\mu_t]\right\rangle, \text{ as } k\to\infty, \tag{4.20}$$

due to the triangle inequality.

Combining (4.14) and (4.20), we obtain that for all $t \in [0,T]$

$$\left\langle \mu_t^k, \partial_t\psi(\cdot,t) + \partial_x\psi(\cdot,t)\cdot\bar{v}^k(t,\cdot)\right\rangle \to \left\langle \mu_t, \partial_t\psi(\cdot,t) + \partial_x\psi(\cdot,t)\cdot v[\mu_t]\right\rangle, \text{ as } k\to\infty. \tag{4.21}$$

Since for each $t \in [0,T]$ and $k \in \mathbb{N}$, the function $\partial_t\psi(\cdot,t) + \partial_x\phi(\cdot,t)\cdot\bar{v}^k(t,\cdot)$ is an element of $C_b([0,1])$, it holds that

$$\left|\left\langle \mu_t^k, \partial_t\psi(\cdot,t) + \partial_x\phi(\cdot,t)\cdot\bar{v}^k(t,\cdot)\right\rangle\right| \leqslant \left(\sup_{\tau\in[0,T]}\|\partial_t\psi(\cdot,\tau)\|_\infty + \sup_{\tau\in[0,T]}\|\partial_x\psi(\cdot,\tau)\|_\infty\cdot K_R\right)\cdot R, \tag{4.22}$$

with the same $R = \|\nu_0\|_{\mathrm{TV}}\exp(\|f\|_\infty T)$ as before. Here we used Lemma 4.2 and Assumption 4.1(ii).

By the dominated convergence theorem, cf. e.g. [3, Theorem 4.2], we now obtain from (4.21) and (4.22) in particular that

$$\int_0^T \left\langle \mu_t^k, \partial_t\psi(\cdot,t) + \partial_x\psi(\cdot,t)\cdot\bar{v}^k(t,\cdot)\right\rangle dt \to \int_0^T \left\langle \mu_t, \partial_t\psi(\cdot,t) + \partial_x\psi(\cdot,t)\cdot v[\mu_t]\right\rangle dt, \text{ as } k\to\infty. \tag{4.23}$$

By taking the limit $k \to \infty$ in (4.8), while taking (4.9), (4.12) and (4.23) into account, we obtain

$$\langle \mu_T, \psi(\cdot, T)\rangle - \langle \nu_0, \psi(\cdot, 0)\rangle = \int_0^T \langle \mu_t, \partial_t \psi(\cdot, t) + \partial_x \psi(\cdot, t) \cdot v[\mu_t]\rangle \, dt + \int_0^T \langle F_f(\mu_t), \psi(\cdot, t)\rangle \, dt.$$

Since $\psi \in \Lambda^T$ is chosen arbitrarily, this proves the statement of the theorem. □

5 Context of the results and open issues

In the Discussion section of [18], we explained that in fact we would like to consider mild solutions (with measure-dependent velocity) corresponding to a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{BL}([0,1])$, such that $f_n \to f$ pointwise, and f is *piecewise* bounded Lipschitz. Let $(\mu^{f_n})_{n\in\mathbb{N}}$ denote the corresponding sequence of measure-valued mild solutions, with measure-dependent velocity $v = v[\mu^{f_n}]$.
In [16] we specifically focussed on such sequence $(f_n)_{n\in\mathbb{N}}$ that describes a vanishing boundary layer in which mass is gated away from the domain. Assume there are regions around 0 and 1 in which mass decays, and that these regions shrink to zero width. That is, f_n is nonzero only in a region around $x = 0$ and $x = 1$, respectively. Moreover, this region shrinks to zero as $n \to \infty$ and $f_n \to f$, where f satisfies $f(x) = 0$ if $x \in (0,1)$ and $f(0) = f(1) = -1$.

We want to know whether the sequence $(\mu^{f_n})_{n\in\mathbb{N}}$ converges, and whether the limit coincides with the mild solution corresponding to f (if this mild solution exists). Mild solutions were obtained in [18] as the limit of Euler approximations. Let the approximating sequence corresponding to f_n be indexed by k, and let $\mu^{f_n,k}$ be one such Euler approximation. The question is now whether the limits $k \to \infty$ and $n \to \infty$ commute. The following scheme shows the four limit processes involved:

(A)

$$\begin{array}{ccccc} & \boxed{\mu^{f_n,k}} & \overset{n\to\infty}{\longrightarrow} & \boxed{\mu^{f,k}} & \\ \textbf{(C)} & \downarrow {\scriptstyle k\to\infty} & & {\scriptstyle k\to\infty} \downarrow & \textbf{(B)} \\ & \boxed{\mu^{f_n}} & \overset{n\to\infty}{\longrightarrow} & \boxed{\mu^{f}} & \end{array}$$

(D)

Note that each of the limits should be understood as convergence in the metric given by (4.5).

At this moment we are not yet able to prove that this scheme represents reality, but the results of this paper yield additional insight. Regarding the limit processes (A), (B), (C) and (D) in the scheme above, the following can be said:

(A) For fixed $k \in \mathbb{N}$, there is an obvious *candidate* for $\lim_{n\to\infty} \mu^{f_n,k}$, namely $\mu^{f,k}$, the Euler approximation corresponding to f. Note that, for fixed k, $\mu^{f,k}$ is well-defined by (4.2) and Theorem 3.2 (that is, [16, Propositions 3.1 and 3.3]). However, it is nontrivial to actually show that $\sup_{t\in[0,T]} \|\mu_t^{f_n,k} - \mu_t^{f,k}\|_{\mathrm{BL}}^* \to 0$ as $n \to \infty$.
The most straight-forward way to prove this, would be to look at the interval $(t_j^k, t_{j+1}^k]$, estimate $\|\mu_\tau^{f_n,k} - \mu_\tau^{f,k}\|_{\mathrm{BL}}^*$ for arbitrary $\tau \in (t_j^k, t_{j+1}^k]$, take the supremum over τ and finally the maximum over j. We will now point out what the problem is with this strategy.
Like in (4.2), let the semigroup Q denote the operator that maps initial data to the solution in the sense of Definition 3.1. From now on, we use $Q^{u,g}$ to denote the semigroup associated to velocity $u \in \mathrm{BL}([0,1])$ and right-hand side F_g, where g is piecewise bounded Lipschitz. For any $\tau \in (t_j^k, t_{j+1}^k]$, we have

$$\mu_\tau^{f_n,k} = Q^{u,f_n}_{\tau - t_j^k}\, \mu^{f_n,k}_{t_j^k},$$
$$\mu_\tau^{f,k} = Q^{\bar{u},f}_{\tau - t_j^k}\, \mu^{f,k}_{t_j^k},$$

with $u = v[\mu^{f_n,k}_{t_j^k}]$ and $\bar{u} = v[\mu^{f,k}_{t_j^k}]$. To estimate $\|\mu_\tau^{f_n,k} - \mu_\tau^{f,k}\|_{\mathrm{BL}}^*$ from above, one would use the triangle inequality and obtain three terms of the form

$$\|Q^{w,g}_{\tau - t_j^k}\, \nu - Q^{\bar{w},\bar{g}}_{\tau - t_j^k}\, \bar{\nu}\|_{\mathrm{BL}}^*, \tag{5.1}$$

for the appropriate choices of

$$w, \bar{w} \in \{v[\mu^{f_n,k}_{t_j^k}], v[\mu^{f,k}_{t_j^k}]\}, \qquad g, \bar{g} \in \{f_n, f\}, \qquad \nu, \bar{\nu} \in \{\mu^{f_n,k}_{t_j^k}, \mu^{f,k}_{t_j^k}\}.$$

Specifically, we would use the triangle inequality in such a way that in each term on the right-hand side two of these three pairs of variables are the same (e.g. $w = \bar{w}$, $g = \bar{g}$ and $\nu \neq \bar{\nu}$). An estimate for each term of the form (5.1) follows e.g. from [16, Proposition 3.5], [16, Proposition 4.2], [18, Corollary 2.9] or [18, Lemma 2.10]. There is freedom in how exactly we apply the triangle inequality, and thus in which specific terms of the form (5.1) appear on the right-hand side. However, each of these approaches results in an upper bound that depends on the Lipschitz constant of f_n (mostly via the bounded Lipschitz norm of f_n), which is unbounded as $n \to \infty$ if f is discontinuous.
For the moment, it is therefore still an open question whether $\mu^{f_n,k}$ converges to $\mu^{f,k}$ as $n \to \infty$.

(B) We emphasize that the results of Theorems 3.5 and 4.8 in this paper, do not hinge on the assumption that $f \in \mathrm{BL}([0,1])$. Theorem 3.5 is stated explicitly to hold for any f that is piecewise bounded Lipschitz. Theorem 4.8 is only restricted to $f \in \mathrm{BL}([0,1])$, because it builds on Theorem 4.5, which we managed to prove only for continuous f in [18, Theorem 3.10]. Let us *assume* that Theorem 4.5 does provide the convergence of Euler approximations *even* for f that is piecewise bounded Lipschitz. Note that Theorem 3.2 demands that the discontinuities of f and zeroes of v do not coincide; for the sake of the argument here, ignore this complication and assume that we *can* generalize Theorem 4.5 to piecewise bounded Lipschitz functions f. In that case the statement of Theorem 4.8 still holds: the resulting mild solution is a weak solution. The current proof of Theorem 4.8 requires a slight modification, though, since in (4.10) we used that $f \in \mathrm{BL}([0,1])$ to obtain immediately an estimate against $\|f\|_{\mathrm{BL}}$. The convergence in (4.12), can however be obtained for f piecewise bounded Lipschitz, using arguments very much like the ones leading to (4.23). These arguments involve the portmanteau theorem (in a slightly more general form than Lemma 4.7) and the dominated convergence theorem. See Appendix B for more details.

(C) Since n is fixed in this step and since $f_n \in \mathrm{BL}([0,1])$ for each n, this convergence result is covered by [18, Theorem 3.10]. In the current work, we show that the limit μ^{f_n} is a weak solution; see Theorem 4.8.

(D) We *assumed* above that μ^f can be obtained as the limit of the Euler approximations $\mu^{f,k}$, and we stress here that this is only an assumption. When trying to relate the mild solutions μ^{f_n} to the mild solution μ^f by letting n tend to infinity, one encounters the following problem: a mild solution is defined as the limit of a sequence of Euler approximations, but this does not provide a useful characterization of the limit itself. We suggested in [18] to use the weak formulation of the problem as an alternative characterization. In the current paper we show that μ^{f_n} is a weak solution. A next step would be to show that this solution converges in some sense (e.g. weakly) as $n \to \infty$.

Even if the addressed problems in steps (A)–(D) would be resolved, an additional argument is needed to conclude that the two limits $k \to \infty$ and $n \to \infty$ commute – and thus the scheme above is fully correct.
Under the aforementioned assumption that $\mu^{f,k} \to \mu^f$ as $k \to \infty$, the mild solution obtained via the route (A)-(B) is a weak solution, as argued above. On the other hand, our considerations regarding route (C)-(D) lead to an alternative μ^f, obtained as the (weak?) limit of the weak solutions μ^{f_n}. It might be possible to show (easily) that this $\lim_{n\to\infty} \mu^{f_n}$ is also a weak solution.

Finally, assume that we want to compare $\lim_{k\to\infty} \mu^{f,k}$, resulting from (A)-(B), to $\lim_{n\to\infty} \mu^{f_n}$, resulting from (C)-(D), based on the fact that they are both weak solutions. We did not prove in this paper that weak solutions are unique. To be able to identify $\lim_{k\to\infty} \mu^{f,k}$ with $\lim_{n\to\infty} \mu^{f_n}$, therefore an extra uniqueness criterion or selection criterion might be needed. The issue of uniqueness is nontrivial and it probably plays a role which specific weak formulation is used and which space of test functions is chosen. This topic is ‘work in progress’ and will (hopefully) be the subject of a follow-up paper.

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Until 2015, I was a member of the Centre for Analysis, Scientific computing and Applications (CASA), and the Institute for Complex Molecular Systems (ICMS) at Eindhoven University of Technology, The Netherlands, supported financially by the Netherlands Organisation for Scientific Research (NWO), Graduate Programme 2010. Some of the results presented here, were obtained during my time in Eindhoven.

A Basics of measure theory

We denote by $\mathcal{M}([0,1])$ the space of finite Borel measures on the interval $[0,1]$ and by $\mathcal{M}^+([0,1])$ the convex cone of positive measures included in it. For $x \in [0,1]$, δ_x denotes the Dirac measure at x. Let

$$\langle \mu, \phi \rangle := \int_{[0,1]} \phi \, d\mu \tag{A.1}$$

denote the natural pairing between measures $\mu \in \mathcal{M}([0,1])$ and bounded measurable functions ϕ. The *push-forward* or *image measure* of μ under Borel measurable $\Phi : [0,1] \to [0,1]$ is the measure $\Phi\#\mu$ defined on Borel sets $E \subset [0,1]$ by

$$(\Phi\#\mu)(E) := \mu\big(\Phi^{-1}(E)\big). \tag{A.2}$$

One easily verifies that $\langle \Phi\#\mu, \phi \rangle = \langle \mu, \phi \circ \Phi \rangle$.

The *total variation norm* $\|\cdot\|_{\mathrm{TV}}$ on $\mathcal{M}([0,1])$ is defined by

$$\|\mu\|_{\mathrm{TV}} := \sup\left\{ \langle \mu, \phi \rangle \ : \ \phi \in C_b([0,1]),\ \|\phi\|_\infty \leqslant 1 \right\},$$

where $C_b([0,1])$ is the Banach space of real-valued bounded continuous functions on $[0,1]$ equipped with the supremum norm $\|\cdot\|_\infty$. It follows immediately that for $\Phi : [0,1] \to [0,1]$ continuous, $\|\Phi\#\mu\|_{\mathrm{TV}} \leqslant \|\mu\|_{\mathrm{TV}}$.

In [16, 18] and in this paper, we mainly use a different norm on $\mathcal{M}([0,1])$ that we will introduce now. Let $\mathrm{BL}([0,1])$ be the vector space of real-valued bounded Lipschitz functions on $[0,1]$, equipped with the norm

$$\|\phi\|_{\mathrm{BL}} := \|\phi\|_\infty + |\phi|_{\mathrm{L}} \tag{A.3}$$

for which this space is a Banach space [19, 14]. Here,

$$|\phi|_{\mathrm{L}} := \sup\left\{ \frac{|\phi(x) - \phi(y)|}{|x-y|} \ : \ x, y \in [0,1],\ x \neq y \right\}$$

is the Lipschitz constant of an arbitrary $\phi \in \mathrm{BL}([0,1])$. With this norm $\mathrm{BL}([0,1])$ is a Banach algebra for pointwise product of functions:

$$\|\phi \cdot \psi\|_{\mathrm{BL}} \leq \|\phi\|_{\mathrm{BL}} \, \|\psi\|_{\mathrm{BL}}. \tag{A.4}$$

Let $\|\cdot\|_{\mathrm{BL}}^*$ be the dual norm of $\|\cdot\|_{\mathrm{BL}}$ on the dual space $\mathrm{BL}([0,1])^*$, i.e. for any $x^* \in \mathrm{BL}([0,1])^*$ its norm is given by

$$\|x^*\|_{\mathrm{BL}}^* := \sup\{|\langle x^*, \phi\rangle| \;:\; \phi \in \mathrm{BL}([0,1]),\ \|\phi\|_{\mathrm{BL}} \leqslant 1\}\,.$$

A linear *embedding* of $\mathcal{M}([0,1])$ into $\mathrm{BL}([0,1])^*$ is provided by the map $\mu \mapsto I_\mu$ with $I_\mu(\phi) := \langle \mu, \phi\rangle$; see [14, Lemma 6]. Thus $\|\cdot\|_{\mathrm{BL}}^*$ induces a norm on $\mathcal{M}([0,1])$, which is denoted by the same symbols. It is called the dual bounded Lipschitz norm or Dudley norm. Generally, $\|\mu\|_{\mathrm{BL}}^* \leqslant \|\mu\|_{\mathrm{TV}}$ for all $\mu \in \mathcal{M}([0,1])$. For positive measures the two norms coincide:

$$\|\mu\|_{\mathrm{BL}}^* = \mu([0,1]) = \|\mu\|_{\mathrm{TV}} \qquad \text{for all } \mu \in \mathcal{M}^+([0,1]). \tag{A.5}$$

In general, the space $\mathcal{M}([0,1])$ is not complete for $\|\cdot\|_{\mathrm{BL}}^*$. We denote by $\overline{\mathcal{M}}([0,1])_{\mathrm{BL}}$ its completion, viewed as closure of $\mathcal{M}([0,1])$ within $\mathrm{BL}([0,1])^*$. The space $\mathcal{M}^+([0,1])$ is complete for $\|\cdot\|_{\mathrm{BL}}^*$, hence closed in $\mathcal{M}([0,1])$ and $\overline{\mathcal{M}}([0,1])_{\mathrm{BL}}$.

The $\|\cdot\|_{\mathrm{BL}}^*$-norm is convenient also for integration. In [16, Appendix C] some technical results about integration of measure-valued maps were collected.

B Proof of convergence statement (4.12) for discontinuous f

The proof is based on the dominated convergence theorem and makes use of the portmanteau theorem.

For each $\nu \in \mathcal{M}([0,1])$,

$$\|F_f(\nu)\|_{\mathrm{TV}} \leqslant \|f\|_\infty \cdot \|\nu\|_{\mathrm{TV}} \tag{B.1}$$

holds, even for f piecewise bounded Lipschitz, hence not necessarily in $C_b([0,1])$. This inequality was previously used in the proof of [16, Proposition 3.1]; a proof can be found in [15, Lemma 4.3.1]. The proof makes use of an approximation of f by C_b-functions.
Note that each mild solution $(\nu_t)_{t\in[0,T]}$ corresponding to a fixed $v \in \mathrm{BL}([0,1])$ and with f piecewise bounded Lipschitz satisfies

$$\|\nu_t\|_{\mathrm{TV}} \leqslant \|\nu_0\|_{\mathrm{TV}} \exp(\|f\|_\infty t), \tag{B.2}$$

according to [16, Proposition 3.3]; the proof of that claim was left to the reader there. The argument is based on Gronwall's Lemma applied to the estimate

$$\|\nu_t\|_{\mathrm{TV}} \leqslant \|\nu_0\|_{\mathrm{TV}} + \int_0^t \|f\|_\infty \cdot \|\nu_s\|_{\mathrm{TV}}\, ds, \tag{B.3}$$

while the latter inequality follows from the variation of constants formula and the estimate $\|P_\tau \mu\|_{\mathrm{TV}} \leqslant \|\mu\|_{\mathrm{TV}}$ that holds for any $\mu \in \mathcal{M}([0,1])$, due to (2.4). Moreover, (B.1) is used to obtain (B.3).

The arguments in [18, Lemmas 3.4 and 2.8(i)] build on (B.2), and thus the result of Lemma 4.2 in this paper is still valid if f is piecewise bounded Lipschitz. That is, the uniform estimate

$$\|\mu_t^k\|_{\mathrm{TV}} \leqslant \|\nu_0\|_{\mathrm{TV}} \exp(\|f\|_\infty T) \tag{B.4}$$

holds for all k and t.

For each $t \in [0,T]$ fixed, note that $\psi(\cdot,t)$ is in $C_b([0,1])$ and thus,

$$\left|\left\langle F_f(\mu_t^k), \psi(\cdot,t)\right\rangle\right| \leqslant \sup_{\tau\in[0,T]} \|\psi(\cdot,\tau)\|_\infty \cdot \|F_f(\mu_t^k)\|_{\mathrm{TV}} \leqslant \sup_{\tau\in[0,T]} \|\psi(\cdot,\tau)\|_\infty \cdot \|f\|_\infty \cdot \|\mu_t^k\|_{\mathrm{TV}}, \tag{B.5}$$

for each $k \in \mathbb{N}$, where the last step is due to (B.2). Combining (B.5) with (B.4), we obtain that the function $t \mapsto \left\langle F_f(\mu_t^k), \psi(\cdot,t)\right\rangle$ is bounded uniformly in k and for each $t \in [0,T]$ by the (constant) function $t \mapsto \sup_{\tau\in[0,T]} \|\psi(\cdot,\tau)\|_\infty \cdot \|f\|_\infty \cdot \|\nu_0\|_{\mathrm{TV}} \exp(\|f\|_\infty T)$.

Next we prove that $t \mapsto \left\langle F_f(\mu_t^k), \psi(\cdot,t)\right\rangle$ converges pointwise. We emphasize that in the rest of this proof we work here under the (unproved) hypothesis that the Euler approximations μ^k converge to some unique mild solution μ even if f is piecewise bounded Lipschitz.
For each $t \in [0,T]$, the function $x \mapsto f(x)\cdot\psi(x,t)$ is bounded and measurable, because f is piecewise bounded Lipschitz, and $\psi(\cdot,t) \in C_b^1([0,1])$. Trivially, $\left\langle F_f(\mu_t^k), \psi(\cdot,t)\right\rangle = \left\langle \mu_t^k, f\cdot\psi(\cdot,t)\right\rangle$ holds.
By hypothesis $\mu^k \to \mu$ in the metric (4.5), hence we know in particular that $\left\langle \mu_t^k, \phi\right\rangle \to \langle \mu_t, \phi\rangle$ as $k \to \infty$ for all $\phi \in \mathrm{BL}([0,1])$. Instead of the simple version of the portmanteau theorem presented here in Lemma 4.7, we use [23, Theorem 13.16], which states that convergence against $\mathrm{BL}([0,1])$ is equivalent to convergence against bounded measurable functions that are discontinuous only on a nullset. One such function is $x \mapsto f(x)\cdot\psi(x,t)$, since f is assumed to have finitely many discontinuities. Hence, $\left\langle \mu_t^k, f\cdot\psi(\cdot,t)\right\rangle \to \langle \mu_t, f\cdot\psi(\cdot,t)\rangle$ as $k \to \infty$ for all $t \in [0,T]$, or equivalently

$$\left\langle F_f(\mu_t^k), \psi(\cdot,t)\right\rangle \to \langle F_f(\mu_t), \psi(\cdot,t)\rangle\,, \text{ as } k \to \infty. \tag{B.6}$$

Due to the uniform bound on $t \mapsto \left\langle F_f(\mu_t^k), \psi(\cdot,t)\right\rangle$ and the pointwise convergence (B.6), the dominated convergence theorem yields in particular that (4.12) holds even for f that is *piecewise* bounded Lipschitz:

$$\int_0^T \left\langle F_f(\mu_t^k), \psi(\cdot,t)\right\rangle dt \overset{k\to\infty}{\longrightarrow} \int_0^T \langle F_f(\mu_t), \psi(\cdot,t)\rangle\, dt. \tag{B.7}$$

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