Advances in Mathematical
Sciences and Applications
Vol. 25, No. 1 (2016), pp. 133–179

GAKKOTOSHO
TOKYO JAPAN

RELATIVISTIC EQUATIONS FOR THE GENERALIZED CHAPMAN-ENSKOG HIERARCHY

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Abstract. In their book Chapman and Cowling have motivated a hierarchy of differential equations which they derived from the Boltzmann equation. I consider now a general version of this hierarchy and considering it up to a certain order N, which is called N-moments system. The system for $N = 1$ is the general Navier-Stokes system. In this paper I derive for arbitrary N a relativistic version for these partial differential equations. This is interesting because a covariant version of a time direction is used, which also leads to a representation of the matrix giving the hyperbolic geometry. This time direction is needed to derive a reduction of the differential equations, that is, to show that the N-moment system contains the $(N-1)$-moments system as a part.

Communicated by Editors; Received May 5, 2016.
Mathematics Subject Classification: 35Q75, 35Q35.
Physics and Astronomy Classification Scheme: 04.20.Cv.
Keywords: Relativity, Partial differential equations, Chapman-Enskog.

1 Introduction

In the book [3] Chapman & Cowling have presented a theory based on Boltzmann's equation of monatomic gases. Some of the results one finds in section 2. The Chapman-Enskog system is a hierarchy of conservation laws for the mass, the momentum, the second moments, the third moments and so on. The hierarchy of functions is denoted by $F_{i_1 \cdots i_{N+1}}$ and the conservation laws are as one can see in [9, Chap. 2 (3.15)] or in this paper in equation (10.8)

$$\sum_{j \geq 0} \partial_{y_j} F_{\alpha j} - \sum_{\beta \in \{0,\ldots,3\}^{N+1}} C_\alpha^\beta F_\beta = \underline{\mathbf{f}}_\alpha \qquad \text{for } \alpha \in \{0,\ldots,3\}^N \tag{1.1}$$

where $\beta = i_1 \cdots i_{N+1}$. We point out to section [8, Chap. 13] where also the entropy principle is presented. It is important to mention, that in this paper, except section 2, the higher moments $F_{i_1 \cdots i_{N+1}}$ are defined as independent variables and are denoted by $T_{i_1 \cdots i_{N+1}}$ so that, for example, the pressure p and the internal energy ε are independent from each other and not as in (2.5), only a constitutive relation between them based on Gibb's relation is assumed which would be a conseqence of the entropy principle. Therefore we have the general form

$$T_{k0} = \varrho \underline{v}_k \,, \quad T_k = \varrho \underline{v}_k + \mathbf{J}_k \,,$$
$$T_{kl0} = \varrho \underline{v}_k \underline{v}_l + \varepsilon_{kl} \,, \quad T_{kl} = \varrho \underline{v}_k \underline{v}_l + \Pi_{kl} \,,$$

which includes Navier-Stokes equations. Also the higher equations with terms for $N \geq 2$ are in the same way a generalization of the Navier-Stokes equation. Besides this the $(N+1)$-tensor $\left(T_{i_1 \cdots i_{N+1}}\right)_{i_1,\ldots,i_{N+1}}$ satifies (1.1). The possible symmetry of $T_{i_1 \cdots i_{N+1}}$ only refers to the first N indices.

The higher moments $T_{i_1 \cdots i_M}$, here $M = N+1$, satisfy an identity for different observers which is crucial for the entire theory. This identity says that $T_{i_1 \cdots i_M}$ are contravariant M-tensors (see definition (1.5))

$$T_{i_1 \cdots i_M} \circ Y = \sum_{\bar{i}_1,\ldots,\bar{i}_M \geq 0} Y_{i_1}{}'_{\bar{i}_1} \cdots Y_{i_M}{}'_{\bar{i}_M} T^*_{\bar{i}_1 \cdots \bar{i}_M} \tag{1.2}$$

as in (2.8) and (10.3). This means that one has an observer transformation between two observers $y = Y(y^*)$, where, to have a common description, $y \in \mathbb{R}^4$ are the time and space coordinates of one observer and $y^* \in \mathbb{R}^4$ the coordinates of a second observer, and finally Y denotes the observer transformation. Here $y = (t,x) \in \mathbb{R}^4$ in the classical sense. Now, the moments T are the quantities of one observer and the same moments T^* the quantities for the other observer. Similar is the notion for other quantities like the forces $\underline{\mathbf{f}}_\alpha$ which are denoted by $\underline{\mathbf{f}}^*_\alpha$ for the other observer. It is important that the differential equations of the system are the same for all observers, that is, the system (1.1) for T_β, C_α^β, $\underline{\mathbf{f}}_\alpha$ is the same for T^*_β, $C_\alpha^{*\beta}$, $\underline{\mathbf{f}}^*_\alpha$, see section 5 and section 10.

The difference between the classical formulation and the relativistic version lies in the Group of transformations, which in the relativistic case is based on Lorentz matrices (see section 12). It is also based on a matrix G which occurs in the conservation laws and is transformed by the rule $\mathrm{G}\circ Y = \mathrm{D}Y\mathrm{G}^* \left(\mathrm{D}Y\right)^{\mathrm{T}}$, that is, G is a contravariant tensor. There are special cases

$$\mathrm{G} = \mathrm{G}_\mathrm{c} = \begin{bmatrix} -\dfrac{1}{\mathrm{c}^2} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} \quad \text{and} \quad \mathrm{G} = \mathrm{G}_\infty = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{bmatrix},$$

where Id is the Identity in space and where G_∞ is the limit of G_c as $\mathrm{c} \to \infty$, hence G_∞ is the matrix in the classical limit. Therefore the matrix G_c in the relativistic case is invertible whereas G_∞ is not. This limit is justified although the speed of light in vacuum

$$\mathrm{c} = 2.99792458 \cdot 10^8 \frac{m}{s}$$

is a finite number. The inverse matrix G^{-1}, if it exists, has the transformation rule $\mathrm{G}^{*-1} = \left(\mathrm{D}Y\right)^{\mathrm{T}} \mathrm{G}^{-1}\circ Y\mathrm{D}Y$, that is, G^{-1} is a covariant tensor. And G^{-1} is the commonly used matrix in relativistic physics.

It is an essential step that in section 3 we introduce a covariant vector $\boldsymbol{e}$ with the identity

$$\boldsymbol{e}\bullet\mathrm{G}\boldsymbol{e} = -\frac{1}{\mathrm{c}^2}\,. \tag{1.3}$$

This vector is the "time direction" and plays an important role. With $\boldsymbol{e} = e_0'$ it is part of a dual basis $\{e_0', e_1', e_2', e_3'\}$ of $\{e_0, e_1, e_2, e_3\}$, and the matrix G has the non-unique representation 3.5(1)

$$\mathrm{G} = -\frac{1}{\mathrm{c}^2} e_0 {e_0}^{\mathrm{T}} + \sum_{i=1}^{n} e_i {e_i}^{\mathrm{T}}\ . \tag{1.4}$$

Therefore $\boldsymbol{e}$ is strongly connected to the matrix G and because $\boldsymbol{e}^* = \mathrm{D}Y^{\mathrm{T}}\boldsymbol{e}\circ Y$ it can occur in basis physical laws. One fundamental example is the definition of a 4-velocity $\underline{v}$ with $\boldsymbol{e}\bullet\underline{v} = 1$ in (5.10). This definition differs by a scalar multiplication from the known definition (see 5.4). Our definition of a 4-velocity plays an important role in the constitutive law 9.1 for fluids.

There is an essential application of the time vector $\boldsymbol{e}$, it is the reduction principle. It is obseved in 2.1 that in the classical limit the system of N^{th}-order moments contain the system for $(N-1)^{\mathrm{th}}$-order moments. This classical reduction principle is generalized in this paper to the relativistic case. In section 6 we treat the case $N = 1$, that is, the 4-momentum equation. Here the simplest case of a reduction occurs, choosing $\zeta = \eta\boldsymbol{e}$ as test function we are able to show that the mass equation is part of the 4-momentum equation, see 6.5. This prove in the general version is then presented to the equation of N^{th}-moments in 10.1. We use as test function $\zeta_{\alpha_1\cdots\alpha_N} := \boldsymbol{e}_{\alpha_1}\eta_{\alpha_2\cdots\alpha_N}$ and obtain this way a system of $(N-1)^{\mathrm{th}}$-moments. Therefore the relativistic version of the N^{th}-order moments is a generalization of the classical case.

One main tool in this paper is the principle of relativity. Its application to basic differential equations is presented in section 11. The quantities of these conservation laws have to satisfy certain transformation rules (11.3). This is applied to scalar conservation laws in (5.4), to the 4-momentum equation in (6.5) and to N^{th}-moment equations in (10.3) and (10.4). We mention that for Maxwell's equation in vacuum in [4, II Elektrodynamischer Teil] this method has been applied to Lorentz transformations.

In section 7 we require that the mass-momentum equation of section 6 should also describe the law of particles. Thus one has to consider the differential equation $\underline{\mathrm{div}}\, T = \underline{r}$ for distributions, here a one-dimensional curve Γ in spacetime $\mathbb{R}^4$. So $\Gamma := \{\underline{\xi}(s)\,;\ s \in \mathbb{R}\}$ is the evolving point, by which we mean that s is chosen so that $\boldsymbol{e}(\underline{\xi}(s))\bullet\partial_s\underline{\xi}(s) > 0$. It is shown in a rigorous way that the differential equation is equivalent to known ordinary differential equations.

Notation: The definition of a contravariant m-tensor $T = (T_{k_1\cdots k_m})_{k_1,\ldots,k_m}$ is

$$T_{k_1\cdots k_m}\circ Y = \sum_{\bar{k}_1,\ldots,\bar{k}_m \geq 0} Y_{k_1{}'\bar{k}_1} \cdots Y_{k_m{}'\bar{k}_m} T^*_{\bar{k}_1\cdots\bar{k}_m}\,, \tag{1.5}$$

and the definition of a covariant m-tensor $T = (T_{k_1\cdots k_m})_{k_1,\ldots,k_m}$

$$T^*_{\bar{k}_1\cdots\bar{k}_m} = \sum_{k_1,\ldots,k_m \geq 0} Y_{k_1{}'\bar{k}_1} \cdots Y_{k_m{}'\bar{k}_m} T_{k_1\cdots k_m}\circ Y\,. \tag{1.6}$$

In this connection a 4-matrix is a 2-tensor and a 4-vector a 1-tensor. Besides this we call a scalar quantity u "objective" if $u\circ Y = u^*$ is true. We denote with an underscore terms in spacetime which are usually meant in space only, so $\underline{\mathrm{div}}q = \sum_{i\geq 0}\partial_{y_i}q_i$ which in coordinates $y = (t, x)$ means $\underline{\mathrm{div}}q = \partial_t q_0 + \sum_{i\geq 1}\partial_{x_i}q_i$. Also $\underline{v}(y) \in \mathbb{R}^4$ is the spacetime version of the velocity. We mention that in section 2 the letter "c" denotes a variable whereas in the rest of the paper "c" is the speed of light.

2 Chapman-Enskog method

The classical Boltzmann equation is a differential equation for the probability $(t, x, c) \mapsto f(t, x, c)$. For a single species this equation reads

$$\partial_t f + \sum_{i=1}^{3} c_i\partial_{x_i} f + \sum_{i=1}^{3} \mathbf{g}_i\partial_{c_i} f = \mathbf{r}_B \tag{2.1}$$

with the additional equation $\sum_{i=1}^{n}\partial_{c_i}\mathbf{g}_i = 0$ for the external acceleration $\mathbf{g}$. The quantity f is the density of atoms at (t, x) with velocity c, and the acceleration $\mathbf{g}$ is a function of (t, x, c). Moreover, $\mathbf{r}_B$ is the collision product, which is also a function of (t, x, c). The Boltzmann equation is explained in many papers including the collision product, see for example [8, 5.2.1] and the literature cited there.

According to the probability f the ***higher moments*** are defined for $k_1, \ldots, k_M \in \{0, \ldots, 3\}$ by

$$F_{k_1\cdots k_M}(t, x) := \int_{\mathbb{R}^3} m\underline{c}_{k_1}\cdots\underline{c}_{k_M} f(t, x, c)\,\mathrm{d}c\,, \tag{2.2}$$

where m is the particle mass and $\underline{c}$ the extended velocity

$$\underline{c} := \begin{bmatrix} 1 \\ c \end{bmatrix},$$

that is $\underline{c} = (c_0, c) = (c_0, c_1, \ldots, c_3)$, $c_0 := 1$. We remark that $F_{k_1\cdots k_M}$ can be the same function for different indices, for example $F_{k0} = F_k$ and $F_{0kl} = F_{kl}$. If f is a solution of

Boltzmann's equation and decays fast enough for $|c| \to \infty$ then the higher moments satisfy the following system of partial differential equations in (t, x): For $i_1, \ldots, i_N \in \{0, \ldots, 3\}$ there holds

$$\partial_t F_{i_1 \cdots i_N} + \sum_{i=1}^{3} \partial_{x_i} F_{i_1 \cdots i_N i} = R_{i_1 \cdots i_N} \,, \tag{2.3}$$

$$\begin{aligned} R_{i_1 \cdots i_N}(t,x) &:= \int_{\mathbb{R}^3} m \underline{c}_{i_1} \cdots \underline{c}_{i_N} \mathbf{r}_B(t,x,c) \,\mathrm{d}c \\ &+ \sum_i \int_{\mathbb{R}^3} m \mathbf{g}_i(t,x,c) \partial_{c_i} (\underline{c}_{i_1} \cdots \underline{c}_{i_N}) f(t,x,c) \,\mathrm{d}c \end{aligned} \tag{2.4}$$

This has been proved in many books, see for example [8, 5.2.2] and [9, Chap.2 3.1 3.2 (3.15)]. The equations for $N \leq 2$ are the classical equations, where ϱ is assumed to be positive and for $N = 2$ one uses only the trace:

$$\begin{aligned} &\varrho := \int_{\mathbb{R}^3} m f \,\mathrm{d}c = F_0 \,, \quad \varrho v := \int_{\mathbb{R}^3} m f c \,\mathrm{d}c = (F_i)_{i=1,\ldots,n} \,, \\ &\Pi := \int_{\mathbb{R}^3} m f (c - v)(c - v)^{\mathrm{T}} \,\mathrm{d}c \,, \quad \mathbf{f} := \int_{\mathbb{R}^3} m f \mathbf{g} \,\mathrm{d}c \,, \\ &e = \varepsilon + \frac{\varrho}{2}|v|^2 = \frac{1}{2}\sum_{i=1}^{3} F_{ii} = \int_{\mathbb{R}^3} \frac{m}{2}|c|^2 f \,\mathrm{d}c \,, \\ &\varepsilon := \int_{\mathbb{R}^3} \frac{m}{2}|c - v|^2 f \,\mathrm{d}c = \frac{1}{2}\sum_{k=1}^{3} \Pi_{kk} \,, \\ &q := \int_{\mathbb{R}^3} m f |c - v|^2 (c - v) \,\mathrm{d}c \,, \quad g = \int_{\mathbb{R}^3} m \mathbf{g} \bullet (c - v) f \,\mathrm{d}c \,, \end{aligned} \tag{2.5}$$

and these quantities satisfy the following classical equations

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho v) = 0 \,, \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) = \mathbf{f} \,, \\ &\partial_t e + \operatorname{div}(e v + \Pi^{\mathrm{T}} v + q) = v \bullet \mathbf{f} + g \,. \end{aligned} \tag{2.6}$$

Here the special properties of the collision term are used, that is, $\mathbf{r}_B$ does not give any contribution to the mass, momentum, and energy. It is clear that the mass equation, which means $N = 0$, is part of the mass-momentum equation, which are the moments with $N = 1$. Similarly one can consider the moments for $N \leq 3$ with a trace for $N = 3$. One obtains Grad's 13-moment theory, see [5] and [9, Chap.2 (3.16) 3.4]. Then the equations with $N = 1$ are contained in the larger system with $N = 2$.

2.1 Observation. The system of moments (2.3) for $i_1, \ldots, i_N \in \{0, \ldots, 3\}$ contains the system for $(N-1)$-order moments.

We will not discuss this in detail but rather focus on the following fact which makes this observation obvious. By denoting the time and space coordinates $y = (y_0, y_1, y_2, y_3) = (t, x_1, x_2, x_3)$ the differential equation (2.3) reads

$$\sum_{k=0}^{3} \partial_{y_k} F_{i_1 \cdots i_N k} = R_{i_1 \cdots i_N} \tag{2.7}$$

for $i_1,\dots,i_N \in \{0,\dots,3\}$. This is the entirety of differential equations for moments less or equal N. For these moments the following transformation rule holds where the indices run from 0 to 3 and where $M = N+1$.

2.2 Transformation rule. For $k_1,\dots,k_M \in \{0,\dots,3\}$

$$F_{k_1\cdots k_M}\circ Y = \sum_{\bar k_1,\dots,\bar k_M=0}^{3} Y_{k_1\,'\bar k_1}\cdots Y_{k_M\,'\bar k_M} F^*_{\bar k_1\cdots\bar k_M}\,. \tag{2.8}$$

Proof. For f we have the transformation rule $f(t,x,c) = f^*(t^*,x^*,c^*)$ if

$$\begin{bmatrix} t \\ x \\ c \end{bmatrix} = \begin{bmatrix} T(t^*) \\ X(t^*,x^*) \\ \dot X(t^*,x^*) + Q(t^*)c^* \end{bmatrix},$$

where $T(t^*) = t^* + \mathrm{a}$ and $X(t^*,x^*) = Q(t^*)x^* + \mathrm{b}(t^*)$. Hence with

$$\begin{bmatrix} t \\ x \end{bmatrix} = Y\left(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\right) = \begin{bmatrix} T(t^*) \\ X(t^*,x^*) \end{bmatrix}, \quad \mathrm{D}Y = \begin{bmatrix} 1 & 0 \\ \dot X & Q \end{bmatrix},$$

we obtain the following rule for $F_{k_1,\dots,k_N}$

$$\begin{aligned} F_{k_1\cdots k_N}(t,x) &= \int_{\mathbb{R}^3} m\underline{c}_{k_1}\cdots\underline{c}_{k_N} f(t,x,c)\,\mathrm{d}c \\ &= \int_{\mathbb{R}^3} m\underline{c}_{k_1}\cdots\underline{c}_{k_N} f^*(t^*,x^*,c^*)\,\mathrm{d}c^* \\ &= \int_{\mathbb{R}^3} m\Big(\sum_{\bar k_1=0}^{3} Y_{k_1\,'\bar k_1}\underline{c}^*_{\bar k_1}\Big)\cdots\Big(\sum_{\bar k_N=0}^{3} Y_{k_N\,'\bar k_N}\underline{c}^*_{\bar k_N}\Big) f^*\,\mathrm{d}c^* \\ &= \sum_{\bar k_1,\dots,\bar k_N=0}^{3} Y_{k_1\,'\bar k_1}\cdots Y_{k_N\,'\bar k_N}\int_{\mathbb{R}^3} m\underline{c}^*_{\bar k_1}\cdots\underline{c}^*_{\bar k_N} f^*\,\mathrm{d}c^* \\ &= \sum_{\bar k_1,\dots,\bar k_N=0}^{3} (Y_{k_1\,'\bar k_1}\cdots Y_{k_N\,'\bar k_N})(t^*,x^*) F^*_{\bar k_1\cdots\bar k_N}(t^*,x^*)\,, \end{aligned}$$

where

$$\underline{c} = \begin{bmatrix} 1 \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ \dot X + Qc^* \end{bmatrix} = \mathrm{D}Y\,\underline{c}^*\,,$$

and of course $m = m^*$. □

This transformation rule can look quite complex if written in single terms. On the other hand, the general description can easily be remembered and is motivated by the following representation

$$F_{k_1\cdots k_M} = \varrho\underline{v}_{k_1}\cdots\underline{v}_{k_M} + \underline{\Pi}_{k_1\cdots k_M}\,. \tag{2.9}$$

We remark that the transformation rule (2.8) works also for arbitrary transformations Y, hence its a rule which we will postulate also in the relativistic case, see (10.3).

Furthermore, the transformation rule (2.8) is important in connection with the general rule (11.3). We define the physical properties of the quantities in (2.7) by saying that in the weak formulation

$$\int_{\mathbb{R}^4} \Big(\sum_{k=0}^{3} \partial_{y_k} \zeta_{i_1 \cdots i_N} \cdot F_{i_1 \cdots i_N k} + \zeta_{i_1 \cdots i_N} R_{i_1 \cdots i_N} \Big) \, \mathrm{dL}^4 = 0 \tag{2.10}$$

the test functions $\zeta_{i_1 \cdots i_N} \in C_0^\infty(\mathbb{R}^4)$ for $i_1, \ldots, i_N \in \{0, 1, \ldots, 3\}$ satisfy the following transformation rule

$$\zeta^*_{\bar{i}_1 \cdots \bar{i}_N} = \sum_{i_1, \ldots, i_N = 0}^{3} Y_{i_1\,\prime\bar{i}_1} \cdots Y_{i_N\,\prime\bar{i}_N} \zeta_{i_1 \cdots i_N} \circ Y$$

for $\bar{i}_1, \ldots, \bar{i}_N \in \{0, 1, \ldots, 3\}$ This means that $\zeta^* = Z^{\mathrm{T}} \zeta \circ Y$ where

$$Z_{(i_1, \ldots, i_N)(\bar{i}_1, \ldots, \bar{i}_N)} = Y_{i_1\,\prime\bar{i}_1} \cdots Y_{i_N\,\prime\bar{i}_N} \, .$$

By 11.1 this is satisfied if

$$F_{i_1 \cdots i_N k} \circ Y = \sum_{\bar{i}_1, \ldots, \bar{i}_N, \bar{k} = 0}^{n} Z_{(i_1, \ldots, i_N)(\bar{i}_1, \ldots, \bar{i}_N)} Y_{k\,\prime\bar{k}} F^*_{\bar{i}_1 \cdots \bar{i}_N \bar{k}} \tag{2.11}$$

and

$$\begin{aligned} R_{i_1 \cdots i_N} \circ Y = & \sum_{\bar{i}_1, \ldots, \bar{i}_N, \bar{k} = 0}^{n} \big(Z_{(i_1, \ldots, i_N)(\bar{i}_1, \ldots, \bar{i}_N)} \big)_{\prime\bar{k}} F^*_{\bar{i}_1 \cdots \bar{i}_N \bar{k}} \\ & + \sum_{\bar{i}_1, \ldots, \bar{i}_N}^{n} Z_{(i_1, \ldots, i_N)(\bar{i}_1, \ldots, \bar{i}_N)} R^*_{\bar{i}_1 \cdots \bar{i}_N} \, . \end{aligned} \tag{2.12}$$

Equation (2.11) is equivalent to (2.8) for $M = N + 1$ which was proved in 2.2. The proof of (2.12) you will find in [2, Chap V]. This presentation will serve us in section 10 as guide.

3 Time and space

The points in spacetime are denoted by $y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ and in spacetime a symmetric 4×4-matrix $y \mapsto \mathrm{G}(y)$ is given. This matrix describes the hyperbolic geometry, that is,

$$\begin{gathered} \text{G has a negative eigenvalue: } \lambda_0 < 0, \\ \text{the remaining eigenvalues of G are positive: } \lambda_i > 0 \text{ for } i \geq 1. \end{gathered} \tag{3.1}$$

If follows from the theory of symmetric matrices:

3.1 Theorem. Let λ_k, $k \geq 0$, be the eigenvalues of G as in (3.1), then there exists a orthonormal basis $\{e_0^\perp, \ldots, e_n^\perp\}$ with

$$\mathrm{G} = \sum_{k \geq 0} \lambda_k e_k^\perp \, (e_k^\perp)^{\mathrm{T}} = \sum_{k \geq 0} \lambda_k e_k^\perp \otimes e_k^\perp \, .$$

The eigenfunctions $e_k^\perp$ depend on y.

Proof. It is $\mathrm{G}e_k^\perp = \lambda_k e_k^\perp$ and $e_k^\perp \bullet e_l^\perp = \delta_{kl}$. □

We assume that to every point $y \in \mathbb{R}^4$ there exists a (dual) direction $\boldsymbol{e}(y) \neq 0$, which we call "direction of time", such that*

$$\boldsymbol{e} \bullet \mathrm{G} \boldsymbol{e} = -\frac{1}{\mathrm{c}^2}\,. \tag{3.2}$$

The "speed of light" $\mathrm{c} > 0$ occurs here in (3.2) for the first time, and is the same for all observers. This follows from the fact that (3.2) is objective as shown in 4.1. The "space" which is orthogonal to $\boldsymbol{e}$ we denote by

$$\boldsymbol{W}(y) := \{\boldsymbol{e}(y)\}^\perp = \{z \in \mathbb{R}^4\,;\ \boldsymbol{e}(y) \bullet z = 0\}\,. \tag{3.3}$$

This space $\boldsymbol{W}(y) \subset \mathbb{R}^4$ is 3-dimensional, orthogonal to $\{\boldsymbol{e}(y)\}$, and $\boldsymbol{e}(y)$ is a point in $\mathbb{R}^4$.

It turns out that $\boldsymbol{e}$ has to depend on y (this is related to the theorem on second derivatives of observer transformations), hence $\boldsymbol{e}$ in general is not constant. Now $(\boldsymbol{e}, \boldsymbol{W})$ describe the coordinates of the world for a single observer. In the next section we see what this description says for another observer. In the special case $\mathrm{G} = \mathrm{G_c}$ we have the situation of Lorentz observers, and this case serves as an example. It is here and hereinafter always $n = 3$.

3.2 Standard example. It is $\mathrm{G} = \mathrm{G_c}$, where

$$\mathrm{G_c} = -\frac{1}{\mathrm{c}^2}\underline{\mathrm{e}}_0\underline{\mathrm{e}}_0^{\mathrm{T}} + \sum_{i\geq 1}\underline{\mathrm{e}}_i\underline{\mathrm{e}}_i^{\mathrm{T}} = \begin{bmatrix} -\frac{1}{\mathrm{c}^2} & 0 \\ 0 & \mathrm{Id} \end{bmatrix}. \tag{3.4}$$

We then have as an example

$$\boldsymbol{e} := \underline{\mathrm{e}}_0 \quad \text{and} \quad \boldsymbol{W} := \operatorname{span}\{\underline{\mathrm{e}}_1, \dots, \underline{\mathrm{e}}_n\}\,.$$

Here $\{\underline{\mathrm{e}}_0, \dots, \underline{\mathrm{e}}_n\}$ is the standard orthonormal basis of $\mathbb{R} \times \mathbb{R}^n$. The classical limit one obtains for $\mathrm{c} \to \infty$ if the quantities have a certain limit.

About $\boldsymbol{e}$ the following statement is true.

3.3 Lemma. Let $e_0' := \boldsymbol{e}$ and let $\{e_1, \dots, e_n\}$ be a basis of $\boldsymbol{W} := \{\boldsymbol{e}(y)\}^\perp$, then there exists for $(\nu_1, \dots, \nu_n)$ one and only one e_0 and e_i' for $i = 1, \dots, n$ with $e_i' + \nu_i e_0' \in \boldsymbol{W}$, so that $\{e_0, \dots, e_n\}$ and $\{e_0', \dots, e_n'\}$ are dual basis, i.e.

$$e_k' \bullet e_l = \delta_{k,l} \text{ for } k, l = 0, \dots, n\,.$$

Hint: The free parameters $(\nu_1, \dots, \nu_n)$ correspond to a "velocity" of an observer transformation, see 3.9.

Proof. Using (3.3) it follows that $\{e_0', e_1, \dots, e_n\}$ is a basis of spacetime. Let with certain coefficients

$$e_0 := \mu e_0' + \sum_{j=1}^n \nu_j e_j$$

*We denote by "•" the inner product in spacetime.

be a vector. The property $e'_0 \bullet e_0 = 1$ of a dual matrix gives

$$1 = e'_0 \bullet e_0 = \mu |e'_0|^2 \quad \text{hence} \quad \mu = \frac{1}{|e'_0|^2} \,.$$

Next let with certain coefficients

$$e'_i := b_i e'_0 + \sum_{k=1}^{n} a_{ik} e_k \text{ for } i = 1, \ldots, n.$$

Then we obtain from the property of a dual basis for $i, j = 1, \ldots, n$

$$\delta_{i,j} = e'_i \bullet e_j = \sum_{k=1}^{n} a_{ik} e_k \bullet e_j = (AE)_{ij}$$
$$\text{where } A := (a_{ik})_{i,k=1,\ldots,n} \text{ and } E := (e_k \bullet e_j)_{k,j=1,\ldots,n} \,,$$

hence $\mathrm{Id} = AE$, that is $A = E^{-1}$. And for $i = 1, \ldots, n$

$$\begin{aligned} 0 = e'_i \bullet e_0 &= \Big(b_i e'_0 + \sum_{k=1}^{n} a_{ik} e_k \Big) \bullet \Big(\mu e'_0 + \sum_{j=1}^{n} \nu_j e_j \Big) \\ &= b_i \mu |e'_0|^2 + \sum_{j,k=1}^{n} a_{ik} \nu_j e_k \bullet e_j = b_i + \sum_{j=1}^{n} \delta_{i,j} \nu_j = b_i + \nu_i \,, \end{aligned}$$

hence $b_i = -\nu_i$. □

Two every basis there exists one and only one dual basis, this is a simple consequence of Functional Analysis. Now let $\{e_0, \ldots, e_n\}$ and $\{e'_0, \ldots, e'_n\}$ be any dual basis. If one defines $\boldsymbol{e} := e'_0$ and $W := \operatorname{span}\{e_1, \ldots, e_n\}$ then it follows (3.3), however, the property (3.2) is still to be satisfied. For given $\boldsymbol{e}$ with (3.2) one can define a dual basis also as follows.

3.4 Theorem. Let $\boldsymbol{e}$ with (3.2) and let $\boldsymbol{W} := \{\boldsymbol{e}\}^{\perp}$. We assume that

$$(z, w) \in \boldsymbol{W} \times \boldsymbol{W} \quad \longmapsto \quad z \bullet \mathrm{G}^{-1} w \in \mathbb{R}$$

is a scalar product, i.e. $w \bullet \mathrm{G}^{-1} w > 0$ for $w \in \boldsymbol{W} \setminus \{0\}$.

(1) Choose $e'_0 := \boldsymbol{e}$ and define $e_0 := -\mathrm{c}^2 \mathrm{G} e'_0$.

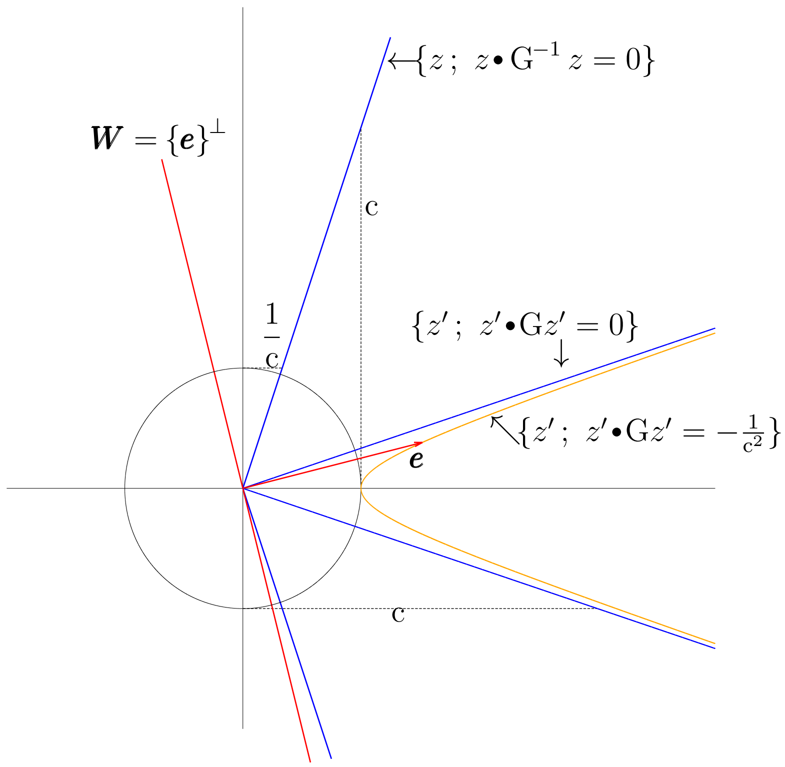
(2) Choose a basis $\{e_1, \ldots, e_n\}$ of $\boldsymbol{W}$ which is G^{-1}-orthogonal, i.e.

$$e_j \bullet \mathrm{G}^{-1} e_i = \delta_{i,j} \text{ für } i, j = 1, \ldots, n.$$

(3) Define $e'_i := \mathrm{G}^{-1} e_i$ for $i = 1, \ldots, n$.

Then $\{e_0, \ldots, e_n\}$ and $\{e'_0, \ldots, e'_n\}$ are dual basis and

$$\mathrm{G} = -\frac{1}{\mathrm{c}^2} e_0 {e_0}^{\mathrm{T}} + \sum_{i=1}^{n} e_i {e_i}^{\mathrm{T}} \,. \tag{3.5}$$



Proof of the duality. It is $e_0'\bullet e_0 = -\mathrm{c}^2 e_0'\bullet \mathrm{G}e_0' = 1$. Since $e_i \in \boldsymbol{W} = \{e_0'\}^\perp$ it is $e_0'\bullet e_i = 0$ for $i = 1, \ldots, n$. And $e_i'\bullet e_j = e_j\bullet \mathrm{G}^{-1} e_i = \delta_{i,j}$ for $i, j = 1, \ldots, n$ by construction. Since G^{-1} is symmetric it follows

$$e_i'\bullet e_0 = e_0\bullet e_i' = e_0\bullet(\mathrm{G}^{-1} e_i) = (\mathrm{G}^{-1} e_0)\bullet e_i = -\mathrm{c}^2 e_0'\bullet e_i = 0$$

for $i = 1, \ldots, n$. □

Proof of the representation of G. Define

$$\widetilde{\mathrm{G}} := -\frac{1}{\mathrm{c}^2} e_0\, e_0{}^{\mathrm{T}} + \sum_{i=1}^{n} e_i\, e_i{}^{\mathrm{T}} \;.$$

The dual basis implies $\widetilde{\mathrm{G}}e_0' = -\frac{1}{\mathrm{c}^2} e_0 = \mathrm{G}e_0'$ and $\widetilde{\mathrm{G}}e_i' = e_i = \mathrm{G}e_i'$ for $i = 1, \ldots, n$. Hence $\widetilde{\mathrm{G}} = \mathrm{G}$. □

If e_0' as in 3.4(1) and $\{e_1, \ldots, e_n\}$ as in 3.4(2) are chosen then we can represent e_0 as in the proof of 3.3

$$e_0 = \mu e_0' + \sum_{j=1}^{n} \nu_j e_j\,, \quad \mu = \frac{1}{|e_0'|^2}\,.$$

If we define $e_i' := \mathrm{G}^{-1} e_i$ for $i = 1, \ldots, n$ as in 3.4(3) then

$$\begin{aligned} 0 = e_j'\bullet e_0 = e_0\bullet \mathrm{G}^{-1} e_j &= \frac{1}{|e_0'|^2} e_0'\bullet \mathrm{G}^{-1} e_j + \sum_i \nu_i e_i\bullet \mathrm{G}^{-1} e_j \\ &= \frac{1}{|e_0'|^2} e_0'\bullet \mathrm{G}^{-1} e_j + \nu_j \quad \text{hence} \quad \nu_j = -\frac{e_0'\bullet \mathrm{G}^{-1} e_j}{|e_0'|^2} = -\frac{e_0'\bullet e_j'}{|e_0'|^2}. \end{aligned}$$

Consequently, the freedom in the choice of $(\nu_1, \dots, \nu_n)$ is given by the choice of the basis $\{e_1, \dots, e_n\}$ in 3.4(1). In 3.4 the following properties of the matrix G are addressed, where 3.5(3) is an essential property, whereas 3.5(1) is important for practical use.

3.5 Properties. For a symmetric matrix G consider the following properties:

(1) It is $\{e_0, \dots, e_n\}$ a basis and

$$\mathrm{G} = -\frac{1}{\mathrm{c}^2} e_0 \, {e_0}^{\mathrm{T}} + \sum_{i=1}^{n} e_i \, {e_i}^{\mathrm{T}} \; .$$

(2) It is $\{e_0, \dots, e_n\}$ a basis and $\{e'_0, \dots, e'_n\}$ the corresponding dual basis and

$$\mathrm{G} e'_0 = -\frac{1}{\mathrm{c}^2} e_0 \, , \quad \mathrm{G} e'_i = e_i \text{ für } i \geq 1 \, .$$

(3) It is $\boldsymbol{W} \subset \mathbb{R}^4$ a subspace of codimension 1 and

$$(z, w) \mapsto z \bullet \mathrm{G}^{-1} w \text{ on } \boldsymbol{W} \text{ a scalar produkt.}$$

The connection between these properties is the content of the following lemmata. The property 3.5(1) implies immediately, if $\{e'_0, \dots, e'_n\}$ is the dual basis,

$$\mathrm{G}^{-1} = -\mathrm{c}^2 e'_0 \, {e'_0}^{\mathrm{T}} + \sum_{i=1}^{n} e'_i \, {e'_i}^{\mathrm{T}} \; . \tag{3.6}$$

Let $\boldsymbol{e} = e'_0$. The property 3.5(2) implies immediately (3.2), since $\mathrm{G} e'_0 = -\frac{1}{\mathrm{c}^2} e_0$ implies

$$e'_0 \bullet \mathrm{G} e'_0 = -\frac{1}{\mathrm{c}^2} e'_0 \bullet e_0 = -\frac{1}{\mathrm{c}^2} \, .$$

3.6 Lemma. 3.5(1) and 3.5(2) are equivalent.
Remark: That 3.5(2) implies 3.5(1) has been proved in 3.4.

Proof 3.5(1)⇒3.5(2). With $\lambda_0 = -\frac{1}{\mathrm{c}^2}$ and $\lambda_i = 1$ for $i \geq 1$ it follows from 3.5(1), if we define $e'_k := \lambda_k \, \mathrm{G}^{-1} \, e_k$, that

$$\lambda_l e_l = \mathrm{G} e'_l = \Big(\sum_{k \geq 0} \lambda_k e_k \, {e_k}^{\mathrm{T}} \Big) e'_l = \sum_{k \geq 0} \lambda_k (e_k \bullet e'_l) e_k \, .$$

Hence it holds for all $k, l \geq 0$

$$\lambda_k e_k \bullet e'_l = \lambda_l \delta_{k,l} = \lambda_k \delta_{k,l} \, .$$

Since all $\lambda_k \neq 0$ we conclude $e_k \bullet e'_l = \delta_{k,l}$. This says that $\{e'_l ; \; l \geq 0\}$ is the dual basis. □

3.7 Lemma. Let 3.5(1) be true where $\{e'_0, \dots, e'_n\}$ is the dual basis of $\{e_0, \dots, e_n\}$. If $\boldsymbol{e} = e'_0$ and $\boldsymbol{W} = \{e'_0\}^{\perp}$ then the property 3.5(3) is true.

Proof. For z' and z we have the representation

$$z' = \sum_{k\geq 0} z'_k e'_k\,, \quad z'_k = z'\bullet e_k\,,$$
$$z = \sum_{k\geq 0} z_k e_k\,, \quad z_k = z\bullet e'_k$$

and we have $\boldsymbol{e} = e'_0$ and $\boldsymbol{W} = \{e'_0\}^{\perp} = \operatorname{span}\{e_1,\dots,e_n\}$. From 3.5(1) it follows

$$\mathrm{G}z' = -\frac{1}{\mathrm{c}^2}z'_0e_0 + \sum_{i\geq 1} z'_i e_i\,,$$
$$z'\bullet\mathrm{G}z' = -\frac{1}{\mathrm{c}^2}|z'_0|^2 + \sum_{i\geq 1}|z'_i|^2\,.$$

If $z' = \mathrm{G}^{-1}\,z$, that means $\mathrm{G}z' = z$, then

$$-\frac{1}{\mathrm{c}^2}z'_0e_0 + \sum_{i\geq 1} z'_i e_i = z = \sum_{k\geq 0} z_k e_k\,,$$

and therefore

$$z_0 = -\frac{1}{\mathrm{c}^2}z'_0\,, \quad z_i = z'_i \text{ for } i\geq 1.$$

Now let $z\in\boldsymbol{W}$, that is $z_0 = z\bullet e'_0 = 0$, and then also $z'_0 = 0$. This implies

$$z\bullet\,\mathrm{G}^{-1}\,z = (\mathrm{G}z')\bullet z' = z'\bullet\mathrm{G}z' = \sum_{i\geq 1}|z'_i|^2 \geq 0\,.$$

And this is strict positive if $z\neq 0$ which is equivalent to $z'\neq 0$. □

The following lemma shows that for all occurring matrices G the property 3.5(3) is satisfied, and therefore also the construction in 3.4.

3.8 Lemma. Let G be a matrix as in (3.1) and $\boldsymbol{e}$ with (3.2) and $\boldsymbol{W} = \{\boldsymbol{e}\}^{\perp}$. Then the property 3.5(3) is true.

Proof. Let $\lambda_0 < 0$ and $\lambda_i > 0$ for $i\geq 1$. For vectors z' one has the identity

$$\mathrm{G}z' = \sum_{k\geq 0}\lambda_k z'_k e^{\perp}_k\,, \quad z'_k := z'\bullet e^{\perp}_k$$

and therefore

$$z'\bullet\mathrm{G}z' = \sum_{k\geq 0}\lambda_k|z'_k|^2\,. \tag{3.7}$$

And it follows that for vectors z one has the identity

$$\mathrm{G}^{-1}\,z = \sum_{k\geq 0}\lambda_k^{-1}z_k e^{\perp}_k\,, \quad z_k := z\bullet e^{\perp}_k$$

hence

$$z\bullet\,\mathrm{G}^{-1}\,z = \sum_{k\geq 0}\frac{|z_k|^2}{\lambda_k}\,. \tag{3.8}$$

Since $\boldsymbol{e}\bullet \mathrm{G}\boldsymbol{e} = -\frac{1}{\mathrm{c}^2}$ by (3.2) and hence

$$\bar{\boldsymbol{e}}\bullet \mathrm{G}\bar{\boldsymbol{e}} = -|\lambda_0| = \lambda_0\,, \quad \bar{\boldsymbol{e}} := \mathrm{c}\sqrt{|\lambda_0|}\,\boldsymbol{e}\,.$$

The first identity (3.7) gives (with $z' = \bar{\boldsymbol{e}}$)

$$-|\lambda_0| = \bar{\boldsymbol{e}}\bullet \mathrm{G}\bar{\boldsymbol{e}} = -|\lambda_0|\cdot|\bar{\boldsymbol{e}}_0|^2 + \sum_{i\geq 1}\lambda_i|\bar{\boldsymbol{e}}_i|^2\,, \quad \bar{\boldsymbol{e}}_k := \bar{\boldsymbol{e}}\bullet e_k^{\perp}\,,$$

that is

$$|\bar{\boldsymbol{e}}_0|^2 = 1 + \frac{1}{|\lambda_0|}\sum_{i\geq 1}\lambda_i|\bar{\boldsymbol{e}}_i|^2\,. \tag{3.9}$$

Therefore $\bar{\boldsymbol{e}}_0 \neq 0$ and for $z \in \boldsymbol{W}\setminus\{0\} = \{\bar{\boldsymbol{e}}\}^{\perp}\setminus\{0\}$, that is,

$$0 = z\bullet\bar{\boldsymbol{e}} = z_0\bar{\boldsymbol{e}}_0 + \sum_{i\geq 1} z_i\bar{\boldsymbol{e}}_i \quad \text{hence} \quad -z_0 = \sum_{i\geq 1} z_i\frac{\bar{\boldsymbol{e}}_i}{\bar{\boldsymbol{e}}_0}\,,$$

we get

$$|z_0|^2 \leq \Big(\sum_{i\geq 1}|z_i|\frac{|\bar{\boldsymbol{e}}_i|}{|\bar{\boldsymbol{e}}_0|}\Big)^2 \leq \sum_{i\geq 1}\frac{|z_i|^2}{\lambda_i}\cdot\sum_{i\geq 1}\lambda_i\frac{|\bar{\boldsymbol{e}}_i|^2}{|\bar{\boldsymbol{e}}_0|^2}\,,$$

where from (3.9)

$$\sum_{i\geq 1}\lambda_i\frac{|\bar{\boldsymbol{e}}_i|^2}{|\bar{\boldsymbol{e}}_0|^2} < |\lambda_0|\,.$$

Then from the second identity (3.8)

$$z\bullet\,\mathrm{G}^{-1}\,z = -\frac{|z_0|^2}{|\lambda_0|} + \sum_{i\geq 1}\frac{|z_i|^2}{\lambda_i} \geq \sum_{i\geq 1}\frac{|z_i|^2}{\lambda_i}\cdot\Big(1 - \frac{1}{|\lambda_0|}\sum_{i\geq 1}\lambda_i\frac{|\bar{\boldsymbol{e}}_i|^2}{|\bar{\boldsymbol{e}}_0|^2}\Big) > 0$$

which had to be shown. □

That for given $\boldsymbol{e}$ the dual basis has free parameters $(\nu_1,\ldots,\nu_n)$ we have seen in the proof of 3.3. Therefore the representation 3.5(1) for G is not unique. We see this if we have a look at the standard matrix $\mathrm{G_c}$ in (3.4):

3.9 Theorem. Let $\mathrm{G} = \mathrm{G_c}$ and let $\{e_0,\ldots,e_n\}$ a (on y depending) basis. Then

$$\mathrm{G_c} = -\frac{1}{\mathrm{c}^2}e_0\,{e_0}^{\mathrm{T}} + \sum_{i=1}^{n} e_i\,{e_i}^{\mathrm{T}}$$

if and only if modulo the sign of each basis vector there exists a Lorentz matrix $\mathbf{L}_c(V,Q)$ (here V and Q depend on y) with

$$e_0 = \begin{bmatrix}\gamma\\ \gamma V\end{bmatrix}, \quad e_0' = -\frac{1}{\mathrm{c}^2}\,{\mathrm{G_c}}^{-1}\,e_0 = \begin{bmatrix}\gamma\\ -\dfrac{\gamma}{\mathrm{c}^2}V\end{bmatrix}, \qquad \text{and for } i\geq 1:$$

$$e_i = \begin{bmatrix}\dfrac{\gamma}{\mathrm{c}^2}V\bullet Q\mathbf{e}_i\\ Q\mathbf{e}_i + \dfrac{\gamma^2 V\bullet Q\mathbf{e}_i}{\mathrm{c}^2(\gamma+1)}V\end{bmatrix}, \quad e_i' = {\mathrm{G_c}}^{-1}\,e_i = \begin{bmatrix}-\gamma V\bullet Q\mathbf{e}_i\\ Q\mathbf{e}_i + \dfrac{\gamma^2 V\bullet Q\mathbf{e}_i}{\mathrm{c}^2(\gamma+1)}V\end{bmatrix}.$$

And of course $e_k'\bullet e_l = \delta_{kl}$ for $k,l\geq 0$.

Proof. Let us write the vectors e_k in components $e_k = (M_{1k}, \dots, M_{nk})$, so that the matrix $M = (M_{ij})_{ij}$ satisfies with the canonical basis vectors $\underline{\mathbf{e}}_i$ the equation $M_{ik} = e_k \bullet \underline{\mathbf{e}}_i$. Then $e_k = M\underline{\mathbf{e}}_k$ and with $\lambda_0 := -\frac{1}{c2}$ and $\lambda_i := 1$ for $i \geq 1$

$$\begin{aligned}\mathrm{G_c} &= \sum_{k\geq 0} \lambda_k e_k {e_k}^{\mathrm{T}} = \sum_{k\geq 0} \lambda_k M\underline{\mathbf{e}}_k (M\underline{\mathbf{e}}_k)^{\mathrm{T}} \\ &= M\Big(\sum_{k\geq 0} \lambda_k \underline{\mathbf{e}}_k {\underline{\mathbf{e}}_k}^{\mathrm{T}}\Big) M^{\mathrm{T}} = M \mathrm{G_c} M^{\mathrm{T}} \ .\end{aligned}$$

This says that the Matrix M keeps $\mathrm{G_c}$ unchanged, and therefore (see 12.1 under the assumption $M_{00} \geq 0$ and $\det M \geq 0$) implies that M is a Lorentz-Matrix. The rows of this matrix are $e_k = M\underline{\mathbf{e}}_k$ for $k \geq 0$. □

As generalization we give in 4.4 representations of G for different $\boldsymbol{e}$-vectors.

3.10 Remark (Classical physics). In the classical limit $\mathrm{c} \to \infty$ the basis $\{e_0, \dots, e_n\}$ and $\{e'_0, \dots, e'_n\}$ in 3.9 converge to

$$e_0 = \begin{bmatrix} 1 \\ V \end{bmatrix}, \quad e'_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and for } i \geq 1 :$$
$$e_i = \begin{bmatrix} 0 \\ Q\mathbf{e}_i \end{bmatrix}, \quad e'_i = \begin{bmatrix} -V \bullet Q\mathbf{e}_i \\ Q\mathbf{e}_i \end{bmatrix},$$

and the matrix is

$$\mathrm{G}_\infty = \sum_{i\geq 1} e_i {e_i}^{\mathrm{T}} \ .$$

4 Change of observer

Since the vector $\boldsymbol{e}$ will occur in the differential equations we have to guarantee the rule by which this quantity will change between observers. If $\boldsymbol{e}$ is this vector for one observer and $\boldsymbol{e}^*$ is this vector for another observer, the transformation rule is

$$\boldsymbol{e}^* = (\underline{\mathrm{D}}Y)^{\mathrm{T}} \, \boldsymbol{e} \circ Y \,, \tag{4.1}$$

where Y is the observer transformation. This means that $\boldsymbol{e}$ is a covariant vector.

4.1 Consistence. The transformations rule (4.1) is consistent with the assumption (3.2) and implies

$$\boldsymbol{W} \circ Y = \mathrm{D}Y \, \boldsymbol{W}^* \,.$$

Proof. It is

$$\begin{aligned}(\boldsymbol{e} \bullet \mathrm{G}\boldsymbol{e}) \circ Y &= (\boldsymbol{e} \circ Y) \bullet (\underline{\mathrm{D}}Y \mathrm{G}^* \underline{\mathrm{D}}Y^{\mathrm{T}} \boldsymbol{e} \circ Y) \\ &= (\underline{\mathrm{D}}Y^{\mathrm{T}} \boldsymbol{e} \circ Y) \bullet \mathrm{G}^* \underline{\mathrm{D}}Y^{\mathrm{T}} \boldsymbol{e} \circ Y = \boldsymbol{e}^* \bullet \mathrm{G}^* \boldsymbol{e}^* \,,\end{aligned}$$

hence the condition $\boldsymbol{e} \bullet \mathrm{G}\boldsymbol{e} = -\frac{1}{\mathrm{c}^2}$ is objective. And for w^* with $w := \mathrm{D}Y w^*$ it holds

$$(\boldsymbol{e} \bullet w) \circ Y = (\boldsymbol{e} \circ Y) \bullet \mathrm{D}Y w^* = (\mathrm{D}Y^{\mathrm{T}} \boldsymbol{e} \circ Y) \bullet w^* = \boldsymbol{e}^* \bullet w^* \,,$$

hence the condition $\boldsymbol{e} \bullet w = 0$, which defines $\boldsymbol{W}$, is objective. □

The transformation rule for $\boldsymbol{e}$ can be generalized to the basis elements, where again $n = 3$.

4.2 Consistence with basis (Definition). Let $\{e_0, \dots, e_n\}$ be a basis with dual basis $\{e'_0, \dots, e'_n\}$. Then the transformation rules for $k \geq 0$ are

$$\begin{aligned} e_k \circ Y &= \underline{\mathrm{D}}Y e^*_k \,, \\ e'^*_k &= (\underline{\mathrm{D}}Y)^{\mathrm{T}}\, e'_k \circ Y \,. \end{aligned} \tag{4.2}$$

Setting $\boldsymbol{e} = e'_0$ this is in accordance with (4.1). The definition shows: The basis elements e_k are contravariant vectors and the dual elements e'_k covariant vectors.
Assertion: The transformation rules are compatible with the definition of a dual basis.

Proof of the assertion. If the transformation rules are true for $\{e_0, \dots, e_n\}$ and if the dual basis $\{e'_0, \dots, e'_n\}$ for one observer is given, then it holds

$$\delta_{k,l} = (e'_k \bullet e_l) \circ Y = (e'_k \circ Y) \bullet (\underline{\mathrm{D}}Y e^*_l) = (\underline{\mathrm{D}}Y^{\mathrm{T}}\, e'_k \circ Y) \bullet e^*_l \,.$$

Hence $e'^*_k := \underline{\mathrm{D}}Y^{\mathrm{T}}\, e'_k \circ Y$ is the dual basis of $\{e^*_0, \dots, e^*_n\}$. The other way around, if the transformation rules are true for $\{e'_0, \dots, e'_n\}$ and if the dual basis $\{e_0, \dots, e_n\}$ for one observer is given, then it holds

$$\delta_{k,l} = (e'_k \bullet e_l) \circ Y = ((\underline{\mathrm{D}}Y)^{-\mathrm{T}}\, e'^*_k) \bullet e_l \circ Y = e'^*_k \bullet ((\underline{\mathrm{D}}Y)^{-1}\, e_l \circ Y) \,.$$

Hence $e^*_l := \underline{\mathrm{D}}Y^{-1}\, e_l \circ Y$ is the dual basis of $\{e'^*_0, \dots, e'^*_n\}$. □

It follows from the transformation rules (4.2) that $e'_k \bullet e_l$ are objective scalars.

4.3 Lemma. Let for the other observer

$$\mathrm{G}^* := -\frac{1}{\mathrm{c}^2} e^*_0\, e^{*\mathrm{T}}_0 + \sum_{i=1}^{n} e^*_i\, e^{*\mathrm{T}}_i$$

and let $y^* \mapsto y = Y(y^*)$ be the observer transformation. Then it holds for the local observer

$$\mathrm{G} = -\frac{1}{\mathrm{c}^2} e_0\, {e_0}^{\mathrm{T}} + \sum_{i=1}^{n} e_i\, {e_i}^{\mathrm{T}} \,,$$

if the basis is transformed according to (4.2).

Proof. Let $\lambda_0 = -\frac{1}{\mathrm{c}^2}$ and $\lambda_i = 1$ for $i \geq 1$. Then

$$\begin{aligned} \mathrm{G} \circ Y &= \underline{\mathrm{D}}Y \mathrm{G}^* (\underline{\mathrm{D}}Y)^{\mathrm{T}} = \sum_{k \geq 0} \lambda_k \underline{\mathrm{D}}Y e^*_k\, e^{*\mathrm{T}}_k\, (\underline{\mathrm{D}}Y)^{\mathrm{T}} \\ &= \sum_{k \geq 0} \lambda_k (\underline{\mathrm{D}}Y e^*_k)\, (\underline{\mathrm{D}}Y e^*_k)^{\mathrm{T}} = \Big(\sum_{k \geq 0} \lambda_k e_k\, {e_k}^{\mathrm{T}} \Big) \circ Y \,, \end{aligned}$$

since $e_k \circ Y = \underline{\mathrm{D}}Y e^*_k$ by (4.2). □

We remark that all observer transformations are allowed which convert the hyperbolic geometry again in a geometry of the same type. Since they are connected with the standard situation it follows from the group property that the general matrix G can be expressed by $\mathrm{G_c}$ and general transformations. Therefore we have the following theorem, where we assume that an observer has the matrix G and there exists an observer transformation Y to the standard observer.

4.4 Theorem. Let $\{e_0, \dots, e_n\}$ be an on y depending basis and $\{e'_0, \dots, e'_n\}$ the corresponding dual basis with $e'_0 \bullet \mathrm{G} e'_0 = -\frac{1}{\mathrm{c}^2}$. Then there exists modulo the sign of the basis elements a Lorentz matrix $\mathbf{L}_c(V,Q)$, which depends on y, with

$$e_k \circ Y = \mathrm{D}Y \mathbf{L}_c(V,Q) \underline{\mathbf{e}}_k \,, \quad \big(\mathrm{D}Y \mathbf{L}_c(V,Q)\big)^{\mathrm{T}} e'_k \circ Y = \underline{\mathbf{e}}_k \,.$$

Here $\underline{\mathbf{e}}_k$ are as in 3.2.

Proof. There exists an observer transformation $y = Y(y^*)$ such that y^* are the coordinates of the standard observer. Let e^*_k and e'^*_k be the corresponding basis vectors, that is,

$$e_k \circ Y = \underline{\mathrm{D}} Y e^*_k \,, \quad e'^*_k = (\underline{\mathrm{D}} Y)^{\mathrm{T}} e'_k \circ Y \,.$$

Then $\{e^*_0, \dots, e^*_n\}$ and $\{e'^*_0, \dots, e'^*_n\}$ are dual basis of the standard observer which satisfy $e'^*_0 \bullet \mathrm{G_c} e'^*_0 = -\frac{1}{\mathrm{c}^2}$, hence by 3.9 modulo the sign of the elements there exists a Lorentz matrix $\mathbf{L}_c(V,Q)$ with

$$e^*_k = \mathbf{L}_c(V,Q) \underline{\mathbf{e}}_k \,, \quad \mathbf{L}_c(V,Q)^{\mathrm{T}} e'^*_k = \underline{\mathbf{e}}_k \,.$$

This implies the assertion. □

5 Scalar conservation laws

We introduce here the simplest example of conservations laws, which is a single equation

$$\underline{\mathrm{div}}\, q = \mathbf{r} \tag{5.1}$$

in a domain $\mathcal{U} \subset \mathbb{R}^4$, where $q = (q_k)_{k=0,\dots,3}$ is a 4-vector and $\mathbf{r}$ an objective scalar. The weak version of this law reads

$$\int_{\mathcal{U}} \big(\underline{\nabla} \eta \bullet q + \eta \mathbf{r}\big) \,\mathrm{dL}^4 = 0 \tag{5.2}$$

for test functions $\eta \in \mathcal{C}^\infty_0(\mathcal{U};\mathbb{R})$. The equation (5.1) is called ***objective scalar equation*** if its weak form (5.2) for observer transformations $y = Y(y^*)$ obeys the transformation rule

$$\eta^* = \eta \circ Y \tag{5.3}$$

for scalar valued test functions $\eta \in \mathcal{C}^\infty_0(\mathcal{U};\mathbb{R})$. That means that for η^* as in (5.3) and with $\mathcal{U} = Y(\mathcal{U}^*)$

$$\int_{\mathcal{U}} \big(\underline{\nabla} \eta \bullet q + \eta \mathbf{r}\big) \,\mathrm{dL}^4 = \int_{\mathcal{U}^*} \big(\underline{\nabla} \eta^* \bullet q^* + \eta^* \mathbf{r}^*\big) \,\mathrm{dL}^4 \,.$$

This has been proved by a general theorem (see 11.1) and it is true if the quantities q and $\mathbf{r}$ satisfy the transformation rules

$$q\circ Y = \mathrm{D}Y q^* , \quad \mathbf{r}\circ Y = \mathbf{r}^* , \tag{5.4}$$

which written in components are

$$q_k\circ Y = \sum_{l=0}^{n} Y_{k}{}'{}_{l} q_l^* , \quad \mathbf{r}\circ Y = \mathbf{r}^* .$$

Quantities with this property, that is, q is an contravariant vector and $\mathbf{r}$ an objective scalar, are well-known. Let us now present some special classes of scalar equations.

Distributional form
A more general version is the distributional version

$$\underline{\mathrm{div}}\, q = \mathbf{r} \text{ in } \mathscr{D}'(\mathcal{U};\mathbb{R}) , \tag{5.5}$$

where $q \in \mathscr{D}'(\mathcal{U};\mathbb{R}^n)$ und $\mathbf{r} \in \mathscr{D}'(\mathcal{U};\mathbb{R})$ are distributions. This definition means for test functions $\eta \in \mathscr{D}(\mathcal{U};\mathbb{R}^n) = \mathcal{C}_0^\infty(\mathcal{U};\mathbb{R})$

$$\langle\, \underline{\nabla}\eta\, ,\, q\,\rangle_{\mathscr{D}(\mathcal{U})} + \langle\, \eta\, ,\, \mathbf{r}\,\rangle_{\mathscr{D}(\mathcal{U})} = 0 , \tag{5.6}$$

and for this the transformation rule (5.3) applies. We add this distributional version because we will use it for mass points in section 7. Similarly the distributional setting for other differential equations is defined.

Mass equation
The standard example of a scalar equation is the mass equation. In this case the flux q has the form

$$q = \varrho\underline{v} + \mathbf{J} . \tag{5.7}$$

We declare the ***mass equation*** as an objective scalar equation

$$\underline{\mathrm{div}}(\varrho\underline{v} + \mathbf{J}) = \mathbf{r} , \tag{5.8}$$

where the type of ϱ and $\underline{v}$ are defined in 5.1 and 5.2 below so that $\varrho\underline{v}$ is a contravariant vector. Also $\mathbf{J}$ is assumed to be a contravariant vector and $\mathbf{r}$ an objective scalar. Thus the condition (5.4) is satisfied.

5.1 Mass density (Definition). A quantity $\varrho \geq 0$ is called ***mass density***, if it is an objective scalar, that is, if for different observers $\varrho\circ Y = \varrho^*$. A mass density is a density over space and time (this is essential).

This means that if ϱ is continuous in space and time then

$$\varrho(y) = \lim_{U(y)\to\{y\}} \frac{1}{\mathrm{L}^4(U(y))} \int_{U(y)} \varrho \,\mathrm{dL}^4 , \tag{5.9}$$

where $U(y)$ is a neighbourhood of the spacetime point y. And for another observer with an observer transformation $y = Y(y^*)$ the ***mass in U(y)*** is

$$\int_{U(y)} \varrho \,\mathrm{dL}^4 = \int_{U^*(y^*)} \varrho^* \,\mathrm{dL}^4 \quad \text{if } U(y) = Y(U^*(y^*)) \ ,$$

where $\mathrm{L}^4(U(y)) = \mathrm{L}^4(U^*(y^*))$ because we assume that $\det \underline{\mathrm{D}}Y = 1$. (If ϱ is only integrable (5.9) holds only almost everywhere in space and time.)

5.2 Velocity (Definition). A quantity $\underline{v}$ is called a ***4-velocity***, if it is an objective 4-vector, that is, if for different observers $\underline{v}\circ Y = \underline{\mathrm{D}}Y\underline{v}^*$, the rule for a contravariant vector, and if with the (dual) time vector $\boldsymbol{e}$ in section 3 (see (3.3))

$$\boldsymbol{e}\bullet\underline{v} = 1\,. \tag{5.10}$$

This definition is objective, because $\boldsymbol{e}$ is a covariant vector.

Proof of objectivity. It is

$$(\boldsymbol{e}\bullet\underline{v})\circ Y = \boldsymbol{e}\circ Y\bullet(\underline{\mathrm{D}}Y\underline{v}^*) = (\underline{\mathrm{D}}Y^{\mathrm{T}}\boldsymbol{e}\circ Y)\bullet\underline{v}^* = \boldsymbol{e}^*\bullet\underline{v}^*\,,$$

since $\boldsymbol{e}$ is a covariant and $\underline{v}$ a contravariant vector. □

Here a remark on $\mathbf{J}$ and ϱ.

5.3 Lemma. Let $q = \varrho\underline{v} + \mathbf{J}$. An often used condition is $\boldsymbol{e}\bullet\mathbf{J} = 0$. This condition is objective and implies that $\varrho = \boldsymbol{e}\bullet q$. That is, if the condition on $\mathbf{J}$ is satisfied the mass density ϱ is the $\boldsymbol{e}$-component of the 4-flux in the mass equation.

Proof. Since $(\boldsymbol{e}\bullet\mathbf{J})\circ Y = (\boldsymbol{e}\circ Y)\bullet \mathrm{D}Y\mathbf{J}^* = (\mathrm{D}Y^{\mathrm{T}}\boldsymbol{e}\circ Y)\bullet\mathbf{J}^* = \boldsymbol{e}^*\bullet\mathbf{J}^*$, the condition on $\boldsymbol{e}\bullet\mathbf{J}$ is objective. And $\boldsymbol{e}\bullet q = \varrho\boldsymbol{e}\bullet\underline{v} + \boldsymbol{e}\mathbf{J} = \varrho$. □

The word "4-velocity" in connection to literature is described in 5.4, where we use the definition

$$||\underline{w}|| := \sqrt{\sum_{i\geq 1} |e_i'\bullet\underline{w}|^2} \quad \text{for } \underline{w}\in\mathbb{R}^4\,. \tag{5.11}$$

5.4 Remark on velocity. In literature a "4-velocity" is a contravariant vector $\underline{u}$ satisfying

$$\underline{u}\bullet \mathrm{G}^{-1}\,\underline{u} = -\mathrm{c}^2\,. \tag{5.12}$$

Reference: Usually one has a different coordinate system, but it says that $\big(\frac{1}{\mathrm{c}}\underline{u}\big)\bullet\widetilde{\mathrm{G}}\big(\frac{1}{\mathrm{c}}\underline{u}\big) = 1$ with $\widetilde{\mathrm{G}} := -\mathrm{G}^{-1}$ (see for example [7, I §7]).

(1) Equation (5.12) is objective, that is, is the same for all observers.

(2) If $\underline{u}$ satisfies (5.12) then $\boldsymbol{e}\bullet\underline{u} \neq 0$ and (if $\boldsymbol{e}\bullet\underline{u} > 0$)

$$\underline{v} := \frac{\underline{u}}{\boldsymbol{e}\bullet\underline{u}}$$

defines a 4-velocity as in 5.2. It is $||\underline{v}|| < \mathrm{c}$.

(3) Let $\underline{v}$ be a 4-velocity as in 5.2 with $||\underline{v}|| < \mathrm{c}$ then

$$\underline{u} := \gamma_{\underline{v}}\underline{v}\,, \quad \gamma_{\underline{v}} := \frac{1}{\sqrt{1-\frac{||\underline{v}||^2}{\mathrm{c}^2}}}\,,$$

satisfies equation (5.12).

(4) The classical formulas follow for $\mathrm{G} = \mathrm{G_c}$ and $\boldsymbol{e} = \underline{\mathbf{e}}_0$.

Proof (4). In the special case $\mathrm{G} = \mathrm{G_c}$ and $\boldsymbol{e} = \underline{\mathbf{e}}_0$ (and hence $e'_k = e_k = \underline{\mathbf{e}}_k$ for $k \geq 0$ in the standard case) it follows that $\underline{v} = (1, v)$ and $||\underline{v}|| = |v|$. □

Proof (1). That $\underline{u}$ is contravariant means $\underline{u}\circ Y = \mathrm{D}Y\underline{u}^*$, hence

$$\begin{aligned}&(\underline{u}\bullet \mathrm{G}^{-1}\,\underline{u})\circ Y = (\underline{\mathrm{D}}Y\underline{u}^*)\bullet \mathrm{G}^{-1}\,\underline{\mathrm{D}}Y\underline{u}^* \\ &= \underline{u}^*\bullet\big(\underline{\mathrm{D}}Y^{\mathrm{T}}\,\mathrm{G}^{-1}\,\underline{\mathrm{D}}Y\big)\underline{u}^* = \underline{u}^*\bullet(\mathrm{G}^*)^{-1}\,\underline{u}^*\,,\end{aligned}$$

since G^{-1} is a covariant tensor. □

Proof (2). It is $\boldsymbol{e} = e'_0$ and

$$\mathrm{G}^{-1} = -\mathrm{c}^2 e'_0\, e'^{\mathrm{T}}_0 + \sum_{i\geq 1} e'_i\, e'^{\mathrm{T}}_i\,.$$

Hence

$$-\mathrm{c}^2 = \underline{u}\bullet \mathrm{G}^{-1}\,\underline{u} = -\mathrm{c}^2|e'_0\bullet\underline{u}|^2 + \sum_{i\geq 1}|e'_i\bullet\underline{u}|^2$$

or

$$|e'_0\bullet\underline{u}|^2 = 1 + \frac{1}{\mathrm{c}^2}\sum_{i\geq 1}|e'_i\bullet\underline{u}|^2 = 1 + \frac{||\underline{u}||^2}{\mathrm{c}^2}$$

hence $|e'_0\bullet\underline{u}| \geq 1 > 0$. If $e'_0\bullet\underline{u} > 0$ this means

$$e'_0\bullet\underline{u} = \sqrt{1+\frac{||\underline{u}||^2}{\mathrm{c}^2}}\,.$$

Then, with $\underline{u}_k := e'_k\bullet\underline{u}$ for $k \geq 0$ we can write $\underline{u} = \sum_{k\geq 0}\underline{u}_k e_k$ and we obtain

$$\underline{v} := \frac{1}{e'_0\bullet\underline{u}}\underline{u} = \frac{\underline{u}}{\underline{u}_0} = e_0 + \sum_{i\geq 1}\frac{\underline{u}_i}{\underline{u}_0}e_i\,.$$

From this it follows immediately that $e'_0\bullet\underline{v} = 1$ and

$$||\underline{v}||^2 = \sum_{i\geq 1}\left|\frac{\underline{u}_i}{\underline{u}_0}\right|^2 = \frac{||\underline{u}||^2}{1+\frac{||\underline{u}||^2}{\mathrm{c}^2}} < \mathrm{c}^2\,.$$

□

Proof (3). Let $\underline{u} = \mu \underline{v}$. It should be

$$-\mathrm{c}^2 = \underline{u} \bullet \mathrm{G}^{-1}\, \underline{u} = \mu^2(-\mathrm{c}^2 |\underline{v}_0|^2 + \sum_{i \geq 1} |\underline{v}_i|^2)$$

where $\underline{v}_k := e'_k \bullet \underline{v}$. Since $\underline{v}_0 = 1$ this means $||\underline{v}|| < \mathrm{c}$ and

$$\mu^2 = \left(1 - \frac{||\underline{v}||^2}{\mathrm{c}^2}\right)^{-1}.$$

This proves the assertion. □

Diffusion

A mass equation (5.8) with a nonzero 4-flux $\mathbf{J}$ is called ***diffusion***. Usually $\mathbf{J}$ depends on the gradient of an objective scalar μ, for example, μ is the mass density or the chemical potential.

5.5 Theorem. Let μ be an objective scalar. Then the following is true.

(1) $\underline{\mathbf{J}} := -\mathrm{G}\underline{\nabla}\mu$ is a contravariant vector.

(2) $\mathbf{J} = -\mathrm{G}^{\mathrm{sp}}\underline{\nabla}\mu$ is a contravariant vector.

(3) $\mathbf{J}$ satisfies the objective property $\boldsymbol{e} \bullet \mathbf{J} = 0$.

(4) $\underline{\mathbf{J}}$ satisfies the objective property $\boldsymbol{e} \bullet \underline{\mathbf{J}} = -(\mathrm{G}\boldsymbol{e}) \bullet \underline{\nabla}\mu = \frac{1}{\mathrm{c}^2}\partial_{e_0}\mu$.

The diffusion equation becomes in the case 5.5(1)

$$\underline{\mathrm{div}}(\varrho \underline{v} - \mathrm{G}\underline{\nabla}\mu) = \mathbf{r}\,. \tag{5.13}$$

Before we prove this let us remark the following. In general we have the identities (see (3.5))

$$\mathrm{G} = -\frac{1}{\mathrm{c}^2} e_0\, e_0{}^{\mathrm{T}} + \sum_{i=1}^{n} e_i\, e_i{}^{\mathrm{T}}\ , \quad \boldsymbol{e} = e'_0\,.$$

By default G and $\boldsymbol{e}$ belong to those quantities appearing in the description of physical processes. Now consider the splitting

$$\mathrm{G} = \mathrm{G}^{\mathrm{ti}} + \mathrm{G}^{\mathrm{sp}}\,, \quad \mathrm{G}^{\mathrm{ti}} := -\frac{1}{\mathrm{c}^2} e_0\, e_0{}^{\mathrm{T}} = (\mathrm{G}e'_0)\, e_0{}^{\mathrm{T}}\ ,$$

that is, we get the identity

$$\mathrm{G}^{\mathrm{sp}} = \mathrm{G} - \mathrm{G}^{\mathrm{ti}} = \mathrm{G}\big(\underline{\mathrm{Id}} - e'_0\, e_0{}^{\mathrm{T}}\,\big)$$

and this contains also e_0. On the other hand $\boldsymbol{e} \bullet \underline{\mathbf{J}} = -(\mathrm{G}\boldsymbol{e}) \bullet \underline{\nabla}\mu$ only contains $\mathrm{G}\boldsymbol{e} = -\frac{1}{\mathrm{c}^2} e_0$, which in the classical limit $\mathrm{c} \to \infty$ goes to zero. But concerning $\mathbf{J}$ we also have $\mathrm{G}^{\mathrm{sp}} \to \mathrm{G}_\infty$ in the classical limit. So one has to clarify what in nature the diffusion does.

Proof (1). Since μ is an objective scalar, that is $\mu\circ Y = \mu^*$, we compute

$$\partial_{\bar{i}}\mu^* = \partial_{\bar{i}}(\mu\circ Y) = \sum_i (\partial_i\mu)\circ Y\, Y_{i\,'\bar{i}}$$

for $\bar{i} = 0,\dots,4$, hence

$$\underline{\nabla}_{y^*}\mu^* = \underline{\mathrm{D}}Y^{\mathrm{T}}\,\underline{\nabla}_y\mu\,,$$

that is $\underline{\nabla}\mu$ is a covariant vector. Then we obtain

$$\begin{aligned}-\underline{\mathbf{J}}\circ Y &= (\mathrm{G}\circ Y)(\underline{\nabla}\mu\circ Y) = \underline{\mathrm{D}}Y\,\mathrm{G}^*\,\underline{\mathrm{D}}Y^{\mathrm{T}}\,\underline{\nabla}\mu\circ Y\\ &= \underline{\mathrm{D}}Y\,\mathrm{G}^*\underline{\nabla}\mu^* = -\underline{\mathrm{D}}Y\,\underline{\mathbf{J}}^*\,,\end{aligned}$$

that is, $\underline{\mathbf{J}}$ is an objective 4-vector or a contravariant vector. □

Proof (3). We compute

$$\begin{aligned}\underline{\mathbf{J}} - (e_0'\bullet\underline{\mathbf{J}})\,e_0 &= -\mathrm{G}\underline{\nabla}\mu + (e_0'\bullet\mathrm{G}\underline{\nabla}\mu)\,e_0 = -\mathrm{G}\underline{\nabla}\mu + \big((\mathrm{G}e_0')\bullet\underline{\nabla}\mu\big)\,e_0\\ &= -\mathrm{G}\underline{\nabla}\mu - \frac{1}{\mathrm{c}^2}(e_0\bullet\underline{\nabla}\mu)\,e_0 = -\Big(\mathrm{G} + \frac{1}{\mathrm{c}^2}e_0\,{e_0}^{\mathrm{T}}\Big)\underline{\nabla}\mu = -\mathrm{G}^{\mathrm{sp}}\underline{\nabla}\mu = \mathbf{J}\,,\end{aligned}$$

hence $\boldsymbol{e}\bullet\mathbf{J} = e_0'\bullet(\underline{\mathbf{J}} - (e_0'\bullet\underline{\mathbf{J}})\,e_0) = 0$. □

We shall later in 9.2 have a reason for the following example.

5.6 Example. Let G be symmetric. Define the 4-flux $\mathbf{J}$ by

$$\mathbf{J}_i = \sum_k w_k G_{ki} \qquad \text{for } i \geq 0\,,$$

where w is a covariant vector and G a contravariant tensor.

(1) Then $\mathbf{J}$ is a contravariant vector.

(2) If $\boldsymbol{e}\bullet(Gw) = 0$ then $\boldsymbol{e}\bullet\mathbf{J} = 0$.

(3) If $G = \mathrm{G}^{\mathrm{sp}}$ then $\boldsymbol{e}\bullet\mathbf{J} = 0$.

Proof (1). Since G is a contravariant tensor, that is $G\circ Y = \underline{D}Y\,G^*\,\underline{D}Y^{\mathrm{T}}$, we conclude

$$\begin{aligned}\mathbf{J}_i\circ Y &= \sum_k (w_k G_{ki})\circ Y = \sum_{k\bar{k}\bar{i}} w_k\circ Y\, Y_{k\,'\bar{k}} Y_{i\,'\bar{i}} G^*_{\bar{k}\bar{i}}\\ &= \sum_{\bar{i}} Y_{i\,'\bar{i}} \sum_{k\bar{k}} Y_{k\,'\bar{k}} w_k\circ Y\, G^*_{\bar{k}\bar{i}} = \sum_{\bar{i}} Y_{i\,'\bar{i}} \sum_{\bar{k}} w^*_{\bar{k}} G^*_{\bar{k}\bar{i}} = \sum_{\bar{i}} Y_{i\,'\bar{i}} \mathbf{J}^*_{\bar{i}}\,,\end{aligned}$$

da $w^*_{\bar{k}} = \sum_k Y_{k\,'\bar{k}} w_k\circ Y$. □

Proof (2). $\boldsymbol{e}\bullet\mathbf{J} = \boldsymbol{e}\bullet(G^{\mathrm{T}}\,w) = \boldsymbol{e}\bullet(Gw) = 0$. □

Proof (3). $\boldsymbol{e}\bullet\mathbf{J} = \mathbf{J}^{\mathrm{T}}\,\boldsymbol{e} = (G^{\mathrm{T}}\,w)^{\mathrm{T}}\,\boldsymbol{e} = w^{\mathrm{T}}\,G\boldsymbol{e} = w^{\mathrm{T}}\,\mathrm{G}^{\mathrm{sp}}\boldsymbol{e} = 0$. □

6 Momentum equation

We introduce here the form of a relativistic momentum equation, that is in a domain $\mathcal{U} \subset \mathbb{R}^4$ we look at the differential equation

$$\underline{\operatorname{div}}\, T = \underline{r} \tag{6.1}$$

for a 4-tensor $T = (T_{ij})_{i,j=0,\dots,3}$ and a 4-field $\underline{r} = (\underline{r}_i)_{i=0,\dots,3}$ which contain the 4-forces. Written in coordinates this is

$$\sum_{j=0}^{3} \partial_{y_j} T_{ij} = \underline{r}_i \quad \text{for } i = 0, \dots, 3\,. \tag{6.2}$$

It is essential to introduce the weak formulation of this law and it reads for test functions $\zeta \in C_0^\infty(\mathcal{U};\mathbb{R}^4)$

$$\int_{\mathcal{U}} \Big(\sum_{i,j=0}^{3} \partial_{y_j}\zeta_i \cdot T_{ij} + \sum_{i=0}^{3} \zeta_i\, \underline{r}_i \Big)\, \mathrm{dL}^4 = 0\,. \tag{6.3}$$

The system (6.1) is called ***4-momentum equation*** if under observer transformations Y the test functions in (6.3) transform via

$$\zeta^* = (\underline{D}Y)^{\mathrm{T}}\, \zeta \circ Y\,. \tag{6.4}$$

Here we as always assume that $\det \underline{D}Y = 1$. This means with $\mathcal{U} = Y(\mathcal{U}^*)$

$$\int_{\mathcal{U}} \Big(\sum_{i,j=0}^{3} \partial_{y_j}\zeta_i \cdot T_{ij} + \sum_{i=0}^{3} \zeta_i \underline{r}_i \Big)\, \mathrm{dL}^4 = \int_{\mathcal{U}^*} \Big(\sum_{\bar{i},\bar{j}=0}^{3} \partial_{y^*_{\bar{j}}}\zeta^*_{\bar{i}} \cdot T^*_{\bar{i}\bar{j}} + \sum_{\bar{i}=0}^{3} \zeta^*_{\bar{i}}\, \underline{r}^*_{\bar{i}} \Big)\, \mathrm{dL}^4\,.$$

Here the test function ζ is a covariant vector by (6.4). This definition is satisfied (see 11.1) if the quantities T and $\underline{r}$ satisfy the transformation rules

$$\begin{aligned} T_{ij} \circ Y &= \sum_{\bar{i},\bar{j}=0}^{3} Y_{i\,\prime\bar{i}} Y_{j\,\prime\bar{j}} T^*_{\bar{i}\bar{j}}\,, \\ \underline{r}_i \circ Y &= \sum_{\bar{i},\bar{j}=0}^{3} Y_{i\,\prime\bar{i}\bar{j}} T^*_{\bar{i}\bar{j}} + \sum_{\bar{i}=0}^{3} Y_{i\,\prime\bar{i}}\, \underline{r}^*_{\bar{i}} \end{aligned} \tag{6.5}$$

for $i, j = 0, \dots, 3$. We mention that concerning the transformation rule (6.5) it is useful to make the following definition. Since $\underline{r} \circ Y$ depends linearly on T^* we can write $\underline{r}$ as in (6.7) below. After doing so the 4-momentum equation (6.2) reads

$$\sum_{j=0}^{3} \partial_{y_j} T_{ij} - \sum_{p,q=0}^{3} C_i^{pq} T_{pq} = \underline{\mathbf{f}}_i \quad \text{for } i = 0, \dots, 3\,, \tag{6.6}$$

where now $\underline{\mathbf{f}}$ is the real ***4-force*** besides the fictitious forces containing forces which we call ***Coriolis forces*** (containing acceleration). The Coriolis coefficients C_i^{pq} are defined in 6.1 and this definition takes care of the terms with second derivatives $Y_{\prime\bar{i}\bar{j}}$ in the transformation rule (6.5).

6.1 Coriolis Coefficients. With $C_i := (C_i^{pq})_{p,q\geq 0}$ let us write

$$\underline{r}_i = \underline{\mathbf{f}}_i + \sum_{p,q\geq 0} C_i^{pq} T_{pq} \quad \text{for } i \geq 0, \tag{6.7}$$

where $C_i^{pq} = C_i^{qp}$. Then the rule (6.5) for $\underline{r}$ is equivalent to $\underline{\mathbf{f}}\circ Y = \underline{\mathrm{D}}Y\underline{\mathbf{f}}^*$, that is, $\underline{\mathbf{f}}$ is a contravariant vector, and

$$\sum_{p,q\geq 0} Y_{p'\bar{p}} Y_{q'\bar{q}} C_i^{pq}\circ Y = \sum_{\bar{i}\geq 0} Y_{i'\bar{i}} C_{\bar{i}}^{*\bar{p}\bar{q}} + Y_{i'\bar{p}\bar{q}} \tag{6.8}$$

for all i and $(\bar{p},\bar{q})$. The matrix version of this transformation rule (6.8) is

$$\underline{\mathrm{D}}Y^{\mathrm{T}}\, C_i\circ Y\, \underline{\mathrm{D}}Y = \sum_{\bar{i}\geq 0} Y_{i'\bar{i}} C_{\bar{i}}^* + \underline{\mathrm{D}}^2 Y_i\,. \tag{6.9}$$

Proof. Using the above transformation rule for $\underline{\mathbf{f}}$ the second equation of (6.5) becomes

$$\sum_{p,q} C_i^{pq}\circ Y\, T_{pq}\circ Y = \sum_{\bar{p},\bar{q}} Y_{i'\bar{p}\bar{q}} T^*_{\bar{p}\bar{q}} + \sum_{\bar{i}} Y_{i'\bar{i}} \sum_{\bar{p},\bar{q}} C_{\bar{i}}^{*\bar{p}\bar{q}} T^*_{\bar{p}\bar{q}}\,.$$

Inserting on the left side for $T\circ Y$ the first equation of (6.5) gives

$$\sum_{\bar{p},\bar{q}} \sum_{p,q} C_i^{pq}\circ Y\, Y_{p'\bar{p}} Y_{q'\bar{q}} T^*_{\bar{p}\bar{q}} = \sum_{\bar{p},\bar{q}} Y_{i'\bar{p}\bar{q}} T^*_{\bar{p}\bar{q}} + \sum_{\bar{i}} Y_{i'\bar{i}} \sum_{\bar{p},\bar{q}} C_{\bar{i}}^{*\bar{p}\bar{q}} T^*_{\bar{p}\bar{q}}\,.$$

Now compare the coefficients of $T^*_{\bar{p}\bar{q}}$. □

6.2 Remark (Classical physics). In the classical limit $\mathrm{c}\to\infty$ the derivatives $Y_{k'x_ix_j}$ are zero, only the second derivatives

$$\left.\begin{aligned} &Y_{k'tt} \\ &Y_{k'tx_j} = Y_{k'x_jt} \end{aligned}\right\} \text{ for } j,k\geq 1 \text{ can be nonzero.}$$

Or, if we write $Y_k = X_k$ for $k\geq 1$, where $X(t^*,x^*) = Q(t^*)x^* + \mathrm{b}(t^*)$, the nonzero terms are those depending on $\ddot{X}$ and $\dot{Q}$. Therefore the Coriolis coefficient C is of the form

$$C_0 = 0\,, \quad C_k = \begin{bmatrix} a_k & {d_k}^{\mathrm{T}} \\ d_k & 0 \end{bmatrix} \text{ for } k\geq 1$$

with

$$\begin{aligned} Q^{\mathrm{T}}\, d_k &= \textstyle\sum_{\bar{k}} Q_{k\bar{k}} d^*_{\bar{k}} + \left(\dot{Q}_{kj}\right)_j\,, \\ a_k + 2\dot{X}\bullet d_k &= \textstyle\sum_{\bar{k}} Q_{k\bar{k}} a^*_{\bar{k}} + \ddot{X}_k\,. \end{aligned} \tag{6.10}$$

This gives for the Coriolis terms the well known fictitious forces

$$\sum_{p,q\geq 0} C_k^{pq} T_{pq} = a_k T_{00} + \sum_{p\geq 0} d_{kp}(T_{k0} + T_{0k}) = a_k\varrho + \sum_{p\geq 0} d_{kp}(2\varrho v_k + \mathbf{J}_k)\,.$$

In the classical limit this has been computed with different notation in [1, 9 Force].

Proof. If Y is a Newton transformation, then the inhomogeneous term in the transformation rule for $\underline{r}$ is for $k \geq 1$

$$\sum_{\bar{i},\bar{j}\geq 0} Y_{k\,'\bar{i}\bar{j}} T^*_{\bar{i}\bar{j}} = \ddot{X}_k T^*_{00} + \sum_{\bar{j}\geq 1} \dot{Q}_{kj} (T^*_{0\bar{j}} + T^*_{\bar{j}0}) \,.$$

Here the first term is the k^{th}-component of the acceleration and the second term the k^{th}-component of the Coriolis force. The transformation rule for C is $C_0 = 0$ and for $k \geq 1$

$$\begin{bmatrix} 1 & \dot{X}^{\mathrm{T}} \\ 0 & Q^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} a_k & {d_k}^{\mathrm{T}} \\ d_k & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}$$
$$= \sum_{\bar{k}} Q_{k\bar{k}} \begin{bmatrix} a^*_{\bar{k}} & d^{*\mathrm{T}}_{\bar{k}} \\ d^*_{\bar{k}} & 0 \end{bmatrix} + \begin{bmatrix} \ddot{X}_k & \left(\dot{Q}_{kj}\right)_j^{\mathrm{T}} \\ \left(\dot{Q}_{kj}\right)_j & 0 \end{bmatrix},$$

which is equivalent to (6.10). These are the transformation rules for $(a_k)_k$ and $(d_k)_k$. □

The transformation rule (6.9) applies to $C = (C_k^{pq})_{kpq}$ and to the negative Christoffel symbols $\Gamma = \left(\Gamma^k_{pq}\right)_{kpq}$, see 6.3. Thus $B_k^{pq} := C_k^{pq} + \Gamma^k_{pq}$ satisfy the rule

$$\underline{\mathrm{D}}Y^{\mathrm{T}}\, B_k \circ Y\, \underline{\mathrm{D}}Y = \sum_{\bar{k}\geq 0} Y_{k\,'\bar{k}} B^*_{\bar{k}} \,.$$

Therefore if the B_k vanish for one observer, it is always true that $C_k^{pq} = -\Gamma^k_{pq}$.

6.3 Christoffel symbols. They are

$$\Gamma^k_{ij} := \frac{1}{2} \sum_l g^{kl} \big(g_{jl\,'i} + g_{il\,'j} - g_{ij\,'l} \big)$$

for $i, j, k \geq 0$ and $-\Gamma$ satisfies the transformation rule (6.8) (formulated for the Coriolis coefficients). Here we use

$$g^{kl} = -\mathrm{G}_{kl} \quad \text{and} \quad g_{kl} = -{\mathrm{G}^{-1}}_{kl} \,. \tag{6.11}$$

Proof. This is well known, see [7, §86]. We have the following transformation rules

$$g^{kl} \circ Y = \sum_{nm} Y_{k\,'n} Y_{l\,'m}\, g^{*nm} \text{ for } y = Y(y^*) \,,$$
$$g_{ij} = \sum_{pq} Y^*_{p\,'i} Y^*_{q\,'j}\, g^*_{pq} \circ Y^* \text{ for } y^* = Y^*(y) \,.$$

Now inserting this, we obtain

$$
\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}\sum_l g^{kl}\big(g_{jl\,'i} + g_{il\,'j} - g_{ij\,'l}\big)\\
&= \frac{1}{2}\sum_{lnmpq}(g^{*nm}Y_{k\,'n}Y_{l\,'m})\circ Y^*\Big(\big(g^*_{pq}\circ Y^* Y^*_{p\,'j}Y^*_{q\,'l}\big)_{\,'i}\\
&\qquad +\big(g^*_{pq}\circ Y^* Y^*_{p\,'i}Y^*_{q\,'l}\big)_{\,'j} - \big(g^*_{pq}\circ Y^* Y^*_{p\,'i}Y^*_{q\,'j}\big)_{\,'l}\Big)\\
&= \frac{1}{2}\sum_{lnmpq}(g^{*nm}Y_{k\,'n}Y_{l\,'m})\circ Y^*\Big(\big(g^*_{pq}\circ Y^*\big)_{\,'i}Y^*_{p\,'j}Y^*_{q\,'l}\\
&\qquad +\big(g^*_{pq}\circ Y^*\big)_{\,'j}Y^*_{p\,'i}Y^*_{q\,'l} - \big(g^*_{pq}\circ Y^*\big)_{\,'l}Y^*_{p\,'i}Y^*_{q\,'j}\Big)\\
&\quad +\frac{1}{2}\sum_{lnmpq}(g^{*nm}Y_{k\,'n}Y_{l\,'m})\circ Y^*\Big(g^*_{pq}\circ Y^*\big(Y^*_{p\,'j}Y^*_{q\,'l}\big)_{\,'i}\\
&\qquad +g^*_{pq}\circ Y^*\big(Y^*_{p\,'i}Y^*_{q\,'l}\big)_{\,'j} - g^*_{pq}\circ Y^*\big(Y^*_{p\,'i}Y^*_{q\,'j}\big)_{\,'l}\Big)\,.
\end{aligned}
$$

The first sum is

$$
\begin{aligned}
&\frac{1}{2}\sum_{lnmpq}(g^{*nm}Y_{k\,'n}Y_{l\,'m})\circ Y^*\Big(\big(g^*_{pq}\circ Y^*\big)_{\,'i}Y^*_{p\,'j}Y^*_{q\,'l}\\
&\qquad\qquad +\big(g^*_{pq}\circ Y^*\big)_{\,'j}Y^*_{p\,'i}Y^*_{q\,'l} - \big(g^*_{pq}\circ Y^*\big)_{\,'l}Y^*_{p\,'i}Y^*_{q\,'j}\Big)\\
&= \frac{1}{2}\sum_{nmpq}(g^{*nm}Y_{k\,'n})\circ Y^*\Big(\big(g^*_{pq}\circ Y^*\big)_{\,'i}Y^*_{p\,'j}\underbrace{\sum_l Y^*_{q\,'l}Y_{l\,'m}\circ Y^*}_{=\delta_{qm}}\\
&+\big(g^*_{pq}\circ Y^*\big)_{\,'j}Y^*_{p\,'i}\underbrace{\sum_l Y^*_{q\,'l}Y_{l\,'m}\circ Y^*}_{=\delta_{qm}} - \sum_l\big(g^*_{pq}\circ Y^*\big)_{\,'l}Y_{l\,'m}\circ Y^*\,Y^*_{p\,'i}Y^*_{q\,'j}\Big)\\
&= \frac{1}{2}\sum_{nmp}(g^{*nm}Y_{k\,'n})\circ Y^*\Big(\big(g^*_{pm}\circ Y^*\big)_{\,'i}Y^*_{p\,'j} + \big(g^*_{pm}\circ Y^*\big)_{\,'j}Y^*_{p\,'i}\\
&\qquad\qquad - \sum_{qr}g^*_{pq\,'r}\circ Y^*\underbrace{\sum_l Y^*_{r\,'l}Y_{l\,'m}\circ Y^*}_{=\delta_{rm}}Y^*_{p\,'i}Y^*_{q\,'j}\Big)\\
&= \frac{1}{2}\sum_{nm}(g^{*nm}Y_{k\,'n})\circ Y^*\Big(\underbrace{\sum_{pq}g^*_{pm\,'q}Y^*_{q\,'i}Y^*_{p\,'j}}_{=\sum_{pq}g^*_{qm\,'p}Y^*_{p\,'i}Y^*_{q\,'j}} + \sum_{pq}g^*_{pm\,'q}Y^*_{q\,'j}Y^*_{p\,'i}\\
&\qquad\qquad - \sum_{pq}g^*_{pq\,'m}\circ Y^*\,Y^*_{p\,'i}Y^*_{q\,'j}\Big)\\
&= \frac{1}{2}\sum_{nmpq}(g^{*nm}Y_{k\,'n})\circ Y^*\big(g^*_{qm\,'p} + g^*_{pm\,'q} - g^*_{pq\,'m}\big)\circ Y^*\,Y^*_{p\,'i}Y^*_{q\,'j}\,.
\end{aligned}
$$

The second sum is

$$
\begin{aligned}
&\frac{1}{2}\sum_{lnmpq}(g^{*nm}Y_{k\,'n}Y_{l\,'m})\circ Y^*\Big(g^*_{pq}\circ Y^*\big(Y^*_{p\,'j}Y^*_{q\,'l}\big)_{\,'i}\\
&\qquad +g^*_{pq}\circ Y^*\big(Y^*_{p\,'i}Y^*_{q\,'l}\big)_{\,'j} - g^*_{pq}\circ Y^*\big(Y^*_{p\,'i}Y^*_{q\,'j}\big)_{\,'l}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{lnmpq} (g^{*nm} g^*_{pq} Y_{k\,'n} Y_{l\,'m})\circ Y^* \Big(Y^*_{p\,'ij} Y^*_{q\,'l} + Y^*_{p\,'j} Y^*_{q\,'il} \\
&\qquad\qquad + Y^*_{p\,'ji} Y^*_{q\,'l} + Y^*_{p\,'i} Y^*_{q\,'jl} - Y^*_{p\,'il} Y^*_{q\,'j} - Y^*_{p\,'i} Y^*_{q\,'jl} \Big) \\
&= \sum_{lnmpq} (g^{*nm} g^*_{pq} Y_{k\,'n} Y_{l\,'m})\circ Y^*\, Y^*_{p\,'ij} Y^*_{q\,'l} \\
&= \sum_{nmpq} (g^{*nm} g^*_{pq} Y_{k\,'n})\circ Y^*\, Y^*_{p\,'ij} \sum_{l} Y^*_{q\,'l} Y_{l\,'m}\circ Y^* \\
&= \sum_{nmp} (g^{*nm} g^*_{pm} Y_{k\,'n})\circ Y^*\, Y^*_{p\,'ij} = \sum_{n} Y_{k\,'n}\circ Y^*\, Y^*_{n\,'ij}\,.
\end{aligned}
$$

Together we have shown that

$$
\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}\sum_{l} g^{kl}\big(g_{jl\,'i} + g_{il\,'j} - g_{ij\,'l}\big) \\
&= \frac{1}{2}\sum_{nmpq} (g^{*nm} Y_{k\,'n})\circ Y^* \big(g^*_{qm\,'p} + g^*_{pm\,'q} - g^*_{pq\,'m}\big)\circ Y^*\, Y^*_{p\,'i} Y^*_{q\,'j} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \sum_{n} Y_{k\,'n}\circ Y^*\, Y^*_{n\,'ij} \\
&= \sum_{n} Y_{k\,'n}\circ Y^* \Big(\frac{1}{2}\sum_{mpq} \big(g^{*nm}(g^*_{qm\,'p} + g^*_{pm\,'q} - g^*_{pq\,'m})\big)\circ Y^*\, Y^*_{p\,'i} Y^*_{q\,'j} + Y^*_{n\,'ij} \Big) \\
&= \sum_{n} Y_{k\,'n}\circ Y^* \Big(\sum_{pq} \Gamma^{*n}_{pq}\circ Y^*\, Y^*_{p\,'i} Y^*_{q\,'j} + Y^*_{n\,'ij} \Big)\,,
\end{aligned}
$$

that is

$$
\Gamma^k_{ij} = \sum_{n} Y_{k\,'n}\circ Y^* \Big(\sum_{pq} \Gamma^{*n}_{pq}\circ Y^*\, Y^*_{p\,'i} Y^*_{q\,'j} + Y^*_{n\,'ij} \Big)\,. \tag{6.12}
$$

The equation (6.12) is equivalent to

$$
\sum_{k} Y^*_{n\,'k} \Gamma^k_{ij} = \sum_{pq} \Gamma^{*n}_{pq}\circ Y^*\, Y^*_{p\,'i} Y^*_{q\,'j} + Y^*_{n\,'ij}
$$

and this is equivalent to

$$
\sum_{ij} (Y_{i\,'p} Y_{j\,'q})\circ Y^*\, \Gamma^k_{ij} = \sum_{n} Y_{k\,'n}\circ Y^*\, \Gamma^{*n}_{pq}\circ Y^* + \sum_{nij} (Y_{k\,'n} Y_{i\,'p} Y_{j\,'q})\circ Y^*\, Y^*_{n\,'ij}\,.
$$

Since

$$
-Y_{k\,'pq} = \sum_{nij} (Y_{k\,'n} Y_{i\,'p} Y_{j\,'q})\, Y^*_{n\,'ij}\circ Y
$$

it is equivalent to

$$
\sum_{ij} Y_{i\,'p} Y_{j\,'q}\, \Gamma^k_{ij}\circ Y = \sum_{n} Y_{k\,'n}\, \Gamma^{*n}_{pq} - Y_{k\,'pq}
$$

which is (6.8) for $-\Gamma$. □

Therefore it is of interest to take the Christoffel symbols for a relativistic sequence and show that in the classical limit they converge to the formulas for the Coriolis coefficients in 6.2. The following lemma holds.

6.4 Identity for Γ. The Christoffel symbols satisfy (we use the definition (6.11))

$$
g_{ik\,'l} = \sum_{m} \big(g_{mk}\Gamma^m_{il} + g_{im}\Gamma^m_{kl} \big)\,.
$$

This is well known, see [7, §86].

Contains mass equation
It is an essential step to prove that the 4-momentum system (6.2) contains as part the mass equation. To explain this let us assume for the moment that the 4-tensor T has the form

$$T_{ij} = \varrho \underline{v}_i \underline{v}_j + \underline{\Pi}_{ij} \quad \text{for } i,j \geq 0, \tag{6.13}$$

where $\underline{v}$ is a 4-velocity as in 5.2 and ϱ is an objective scalar, the mass density. If we multiply this with the component $\boldsymbol{e}_i$ of the vector $\boldsymbol{e}$ and sum over i we get using that $\boldsymbol{e} \bullet \underline{v} = 1$

$$q_j := \textstyle\sum_i \boldsymbol{e}_i T_{ij} = \varrho \underline{v}_j + \sum_i \boldsymbol{e}_i \underline{\Pi}_{ij} \quad \text{for } j \geq 0, \tag{6.14}$$

where q has the form (5.7), that is, q is identical with a mass flux. We convert this to an argument with test functions by looking at the weak version (6.3). By (6.4) the test function ζ must be a covariant vector. With an objective scalar test function η we can set $\zeta = \eta \boldsymbol{e}$. Since $\boldsymbol{e}$ is a covariant vector this is a possible test function. Thus we obtain a weak equation written in η which is the "$\boldsymbol{e}$-component" of the relativistic momentum equation (6.2) and is identical to the mass conservation (5.2). Doing so for arbitrary T we get the

6.5 Mass conservation (Reduction). A part of the 4-momentum equation (6.1) reads

$$\underline{\operatorname{div}}\, q = \mathbf{r} \tag{6.15}$$

with

$$q_j := \sum_i \boldsymbol{e}_i T_{ij}\,, \quad \mathbf{r} := \sum_i \boldsymbol{e}_i \underline{r}_i + \sum_{i,j} \boldsymbol{e}_{i'j} T_{ij}\,.$$

If T is as in (6.13) then

$$\begin{aligned} q_j &= \varrho \underline{v}_j + \mathbf{J}_j\,, \quad \mathbf{J}_j := \sum_i \boldsymbol{e}_i \underline{\Pi}_{ij}\,, \\ \mathbf{r} &:= \sum_i \boldsymbol{e}_i \big(\underline{r}_i - \varrho(\underline{v} \bullet \underline{\nabla}) \underline{v}_i\big) + \sum_{i,j} \boldsymbol{e}_{i'j} \underline{\Pi}_{ij} \\ &= \boldsymbol{e} \bullet \underline{\mathbf{f}} + \sum_{p,q} (\boldsymbol{e} \bullet C^{pq} + \boldsymbol{e}_{p'q}) T_{pq}\,. \end{aligned}$$

Remark: We mention that the condition $\sum_{i,j} \boldsymbol{e}_i \boldsymbol{e}_j \underline{\Pi}_{i,j} = 0$ implies $\boldsymbol{e} \bullet \mathbf{J} = 0$.

Proof. We take

$$\zeta := \eta\, \boldsymbol{e} \tag{6.16}$$

as test function. If η is an objective scalar then

$$\zeta^* = \eta^* \boldsymbol{e}^* = \eta^*\, \mathrm{D}Y^{\mathrm{T}}\, \boldsymbol{e} \circ Y = \mathrm{D}Y^{\mathrm{T}}\, \zeta \circ Y\,,$$

hence ζ is an allowed test function and it follows from (6.3)

$$\begin{aligned} 0 &= \int_{\mathcal{U}} \big(\sum_{i,j} \partial_{y_j} \zeta_i T_{ij} + \sum_i \zeta_i \underline{r}_i \big) \, \mathrm{dL}^4 \\ &= \int_{\mathcal{U}} \big(\sum_{i,j} \partial_{y_j} (\eta \boldsymbol{e}_i) T_{ij} + \eta \sum_i \boldsymbol{e}_i \underline{r}_i \big) \, \mathrm{dL}^4 \\ &= \int_{\mathcal{U}} \Big(\sum_j \partial_{y_j} \eta \underbrace{\sum_i \boldsymbol{e}_i T_{ij}}_{= q_j} + \eta \big(\underbrace{\sum_{i,j} (\partial_{y_j} \boldsymbol{e}_i) T_{ij} + \sum_i \boldsymbol{e}_i \underline{r}_i}_{= \mathbf{r}} \big) \Big) \, \mathrm{dL}^4 \,. \end{aligned}$$

If T is as in (6.13) then

$$q_j = \sum_i \boldsymbol{e}_i (\varrho \underline{v}_i \underline{v}_j + \underline{\Pi}_{ij}) = \varrho (\boldsymbol{e} \bullet \underline{v}) \underline{v}_j + \sum_i \boldsymbol{e}_i \Pi_{ij}$$

with $\boldsymbol{e} \bullet \underline{v} = 1$, and one term of $\mathbf{r} = \sum_i \boldsymbol{e}_i \underline{r}_i + \sum_{i,j} \boldsymbol{e}_{i}{}'_j T_{ij}$ is

$$\sum_{i,j} \boldsymbol{e}_{i}{}'_j \varrho \underline{v}_i \underline{v}_j = \sum_{i,j} (\boldsymbol{e}_i \underline{v}_i)'_j \varrho \underline{v}_j - \sum_{i,j} \boldsymbol{e}_i \underline{v}_{i}{}'_j \varrho \underline{v}_j = -\varrho \sum_i \boldsymbol{e}_i \sum_j \underline{v}_j \underline{v}_{i}{}'_j$$

again since $\boldsymbol{e} \bullet \underline{v}$ is constant, and $(\underline{v} \bullet \underline{\nabla}) \underline{v} = \underline{\mathrm{D} v}\, \underline{v}$. Also $\mathbf{r}$ equals

$$\begin{aligned} \mathbf{r} &= \sum_i \boldsymbol{e}_i \Big(\underline{\mathbf{f}}_i + \sum_{p,q} C_i^{pq} T_{pq} \Big) + \sum_{i,j} \boldsymbol{e}_{i}{}'_j T_{ij} \\ &= \sum_i \boldsymbol{e}_i \underline{\mathbf{f}}_i + \sum_{p,q} \big(\boldsymbol{e}_i C_i^{pq} + \boldsymbol{e}_{p}{}'_q \big) T_{pq} \,, \end{aligned}$$

whish gives the second representation of $\mathbf{r}$. □

It is now clear that the mass equation is a part of the 4-momentum system. A question is how one formulates in general the momentum system without the mass equation.

6.6 Reduced momentum equation. This equation reads

$$\int_{\mathbb{R}^4} \big(\sum_{i,j=0}^{3} \partial_{y_j} \widetilde{\zeta}_i \cdot T_{ij} + \sum_{i=0}^{3} \widetilde{\zeta}_i \mathbf{f}_i \big) \, \mathrm{dL}^4 = 0 \quad \text{ for } \widetilde{\zeta} \bullet e_0 = 0.$$

Here $\mathbf{f}$ is the remaining force (without reaction term)

$$\mathbf{f} := \underline{r} - \underline{r} \bullet e_0' e_0 \,.$$

Remark: So this differential equation together with (6.15) is equivalent to the 4-momentum system (6.2).

Proof. The equation $\zeta = \eta e_0'$ implies $\eta = \zeta \bullet e_0$. Or $\widetilde{\zeta} := \zeta - \eta e_0'$ with $\eta := \zeta \bullet e_0$ gives $\widetilde{\zeta} \bullet e_0 = 0$. And the definition of $\mathbf{f}$ gives $\widetilde{\zeta} \bullet \underline{r} = \widetilde{\zeta} \bullet \mathbf{f}$. Therefore

$$\zeta \bullet \underline{r} = (\widetilde{\zeta} + \eta e_0') \bullet \underline{r} = \eta e_0' \bullet \underline{r} + \widetilde{\zeta} \bullet \mathbf{f} \,,$$

where $e_0' \bullet \underline{r} = \mathbf{r} - \varrho e_0' \bullet \underline{\mathrm{D} v}\, \underline{v}$. The η-term is the contribution in the mass equation and the $\widetilde{\zeta}$-term the contribution in the reduced system. □

7 Moving particle

The mass-momentum equation of section 6 should be consistent with the law of moving particles. Therefore we consider the differential equation (6.1)

$$\underline{\operatorname{div}}\, T = \underline{r}$$

now for distributions based on an evolving point $\Gamma \subset \mathbb{R}^4$. We refer to 7.4 and 7.5 for the distributional version of differential equations on Γ.

By an ***evolving point*** $\Gamma \subset \mathbb{R}^4$ we mean that for $y \in \Gamma$ the scalar product $\boldsymbol{e}(y) \bullet \tau > 0$ for some $\tau \in \mathbf{T}_y(\Gamma) \setminus \{0\}$, where $\mathbf{T}(\Gamma)$ denotes the spacetime tangent space of Γ. Multiplying τ by a positive constant we obtain $\boldsymbol{e}(y) \bullet \tau = 1$. This way, since Γ is one-dimensional, the tangential vector becomes unique. Thus we obtain a 4-vector

$$\underline{\mathrm{v}}_\Gamma(y) \in \mathbf{T}_y(\Gamma) \quad \text{with} \quad \boldsymbol{e}(y) \bullet \underline{\mathrm{v}}_\Gamma(y) = 1, \tag{7.1}$$

which we call the ***4-velocity*** of Γ, a notion which we already introduced in 5.2.

7.1 Remark. It follows from (7.1) that $\underline{\mathrm{v}}_\Gamma = e_0 + \mathrm{v}_\Gamma$ with $\mathrm{v}_\Gamma \in \boldsymbol{W} = \{\boldsymbol{e}\}^\perp$. And $\underline{\mathrm{v}}_\Gamma$ is a contravariant vector, i.e. it satisfies the transformation rule

$$\underline{\mathrm{v}}_\Gamma \circ Y = \mathrm{D}Y \, \underline{\mathrm{v}}_{\Gamma^*} \,,$$

where Y is the observer transformation.

Proof. If $\Gamma = Y(\Gamma^*)$ it follows for $y = Y(y^*)$ that $\underline{\mathrm{D}}Y(y^*)$ maps $\mathbf{T}_{y^*}(\Gamma^*)$ into $\mathbf{T}_y(\Gamma)$. □

In order to get an impression of a differential equation on Γ we prove the following lemma.

7.2 Lemma (Objectivity on Γ). For an evolving point Γ the following holds:

$$\int_\Gamma \frac{g}{|\underline{\mathrm{v}}_\Gamma|} \,\mathrm{dH}^1 = \int_{\Gamma^*} \frac{g^*}{|\underline{\mathrm{v}}_{\Gamma^*}|} \,\mathrm{dH}^1 \,,$$

if $g : \Gamma \to \mathbb{R}$ is an objective scalar. The velocity $\underline{\mathrm{v}}_\Gamma$ is the 4-velocity as in 7.1. Here $|\underline{\mathrm{v}}_\Gamma|$ denotes the Euclidean norm in spacetime of $\underline{\mathrm{v}}_\Gamma$.

Proof. Let $y = Y(y^*)$ be an observer transformation and $\Gamma = Y(\Gamma^*)$. Then the transformation formula from Mathematics says that for every local function $f : \Gamma \to \mathbb{R}$

$$\int_\Gamma f \,\mathrm{dH}^1 = \int_{\Gamma^*} f \circ Y \,|\det \mathrm{D}Y|_{\mathbf{T}_{y^*}(\Gamma^*)}| \,\mathrm{dH}^1 \,.$$

We have to bring this in our statement. If we choose tangent vectors

$$\tau(y) = \mathrm{D}Y(y^*)\,\tau^*(y^*) \in \mathbf{T}_y(\Gamma) \quad \text{for} \quad \tau^*(y^*) \in \mathbf{T}_{y^*}(\Gamma^*)\,,$$

and if we choose $\tau^*(y^*)$ as unit vector $|\tau^*(y^*)| = 1$ then

$$|\det \mathrm{D}Y|_{\mathbf{T}_{y^*}(\Gamma^*)}| = |\tau \circ Y| \,.$$

Since $\underline{\mathrm{v}}_\Gamma(y) = \mathrm{D}Y(y^*)\underline{\mathrm{v}}_{\Gamma^*}(y^*)$ by 7.1 we can choose

$$\tau^*(y^*) = \frac{\underline{\mathrm{v}}_{\Gamma^*}(y^*)}{|\underline{\mathrm{v}}_{\Gamma^*}(y^*)|}\,, \quad \tau(y) = \frac{\underline{\mathrm{v}}_\Gamma(y)}{|\underline{\mathrm{v}}_{\Gamma^*}(y^*)|}$$

and obtain

$$|\det \mathrm{D}Y|_{\mathbf{T}_{y^*}(\Gamma^*)}| = \frac{|\underline{\mathrm{v}}_\Gamma \circ Y|}{|\underline{\mathrm{v}}_{\Gamma^*}|}\,.$$

Therefore the above formula reads

$$\int_\Gamma f\,\mathrm{dH}^1 = \int_{\Gamma^*} (f\,|\underline{\mathrm{v}}_\Gamma|)\circ Y \frac{\mathrm{dH}^1}{|\underline{\mathrm{v}}_{\Gamma^*}|}\,.$$

Setting $g(y) := f(y)|\underline{\mathrm{v}}_\Gamma(y)|$ we obtain

$$\int_\Gamma \frac{g}{|\underline{\mathrm{v}}_\Gamma|}\,\mathrm{dH}^1 = \int_{\Gamma^*} \frac{g\circ Y}{|\underline{\mathrm{v}}_{\Gamma^*}|}\,\mathrm{dH}^1\,.$$

If g is an objective scalar, that is $g\circ Y = g^*$, the assertion follows. □

The 4-velocity $\underline{\mathrm{v}}_\Gamma$ is a contravariant vector, but the transformation rule for $|\underline{\mathrm{v}}_\Gamma|$ is not so easy, it is more convenient to consider the measure

$$\boldsymbol{\mu}_\Gamma := \frac{1}{|\underline{\mathrm{v}}_\Gamma|}\mathrm{H}^1 \llcorner \Gamma \tag{7.2}$$

and therefore one can write the result of 7.2 as

$$\int_{\mathbb{R}^4} g\,\mathrm{d}\boldsymbol{\mu}_\Gamma = \int_{\mathbb{R}^4} g^*\,\mathrm{d}\boldsymbol{\mu}_{\Gamma^*}\,. \tag{7.3}$$

We can view this also as a transformation rule for $\boldsymbol{\mu}_\Gamma$:

7.3 Remark. If Y is an observer transformation and $\Gamma = Y(\Gamma^*)$ then

$$\boldsymbol{\mu}_\Gamma(B) = \boldsymbol{\mu}_{\Gamma^*}(B^*) \text{ for } B = Y(B^*).$$

We now write down differential equations on the curve Γ in order to describe the movement of a "tiny mass". First let us use the distributional version of the mass equation (5.5) taking 7.2 into account. We mention that here we only consider a single mass point.

7.4 Mass equation. We define the distributional mass equation by

$$\underline{\mathrm{div}} q = \mathbf{r}\,,$$
$$q = m\underline{v}\boldsymbol{\mu}_\Gamma\,, \quad \mathbf{r} = r\boldsymbol{\mu}_\Gamma\,,$$

where the mass m of the particle is an objective scalar and the velocity $\underline{v} = \underline{\mathrm{v}}_\Gamma$ and also the rate r is an objective scalar. For scalar test functions η this reads

$$\begin{aligned} 0 = \langle\, \eta\,, -\underline{\mathrm{div}} q + \mathbf{r}\,\rangle_{\mathscr{D}(\mathcal{U})} &= \int_{\mathbb{R}^4} \big(\underline{\nabla}\eta\bullet(m\underline{v}) + \eta r\big)\,\mathrm{d}\boldsymbol{\mu}_\Gamma \\ = \int_\Gamma \Big(\underline{\nabla}\eta\bullet(m\underline{\mathrm{v}}_\Gamma) + \eta r\Big)\frac{\mathrm{dH}^1}{|\underline{\mathrm{v}}_\Gamma|} &= \int_\Gamma \eta\Big(- \underline{\mathrm{div}}^\Gamma\Big(\frac{m\underline{\mathrm{v}}_\Gamma}{|\underline{\mathrm{v}}_\Gamma|}\Big) + \frac{r}{|\underline{\mathrm{v}}_\Gamma|}\Big)\,\mathrm{dH}^1 \end{aligned}$$

hence

$$\underline{\mathrm{div}}^{\Gamma}\Big(\frac{m\underline{\mathrm{v}}_{\Gamma}}{|\underline{\mathrm{v}}_{\Gamma}|}\Big) = \frac{r}{|\underline{\mathrm{v}}_{\Gamma}|}\,.$$

For the last integral the regularity of Γ and m are required.
Hint: We refer to section 8 where the notion of the scalar m is motivated.

Proof. Since η is an objective scalar, that is $\eta\circ Y = \eta^*$, we deduce that $\nabla\eta$ satisfies $\underline{\nabla}\eta^* = (\underline{\mathrm{D}}Y)^{\mathrm{T}}\,\underline{\nabla}\eta\circ Y$. Then in the first integral the $\boldsymbol{\mu}_\Gamma$-integrand

$$g := \underline{\nabla}\eta\bullet(m\underline{v}) + \eta r$$

satisfies

$$\begin{aligned} g^* &= \underline{\nabla}\eta^*\bullet(m^*\underline{v}^*) + \eta^* r^* = ((\underline{\mathrm{D}}Y)^{\mathrm{T}}\,\underline{\nabla}\eta\circ Y)\bullet(m^*\underline{v}^*) + (\eta\circ Y)r^* \\ &= (\underline{\nabla}\eta\circ Y)\bullet(m^*\underline{\mathrm{D}}Y v^*) + (\eta\circ Y)r^* = (\underline{\nabla}\eta\bullet(m\underline{v}) + \eta r)\circ Y = g\circ Y\,, \end{aligned}$$

that is, g is an objective scalar. Hence by (7.3)

$$\langle\,\eta\,,\,-\,\mathrm{div}q + \mathbf{r}\,\rangle_{\mathscr{D}(\mathcal{U})} = \int_\Gamma g\,\mathrm{d}\boldsymbol{\mu}_\Gamma = \int_{\Gamma^*} g^*\,\mathrm{d}\boldsymbol{\mu}_{\Gamma^*} = \langle\,\eta^*\,,\,-\,\mathrm{div}q^* + \mathbf{r}^*\,\rangle_{\mathscr{D}(\mathcal{U}^*)}\,.$$

This shows that the distributional equation is the distributional mass equation. To derive the strong version note that $m\underline{v} = m\underline{\mathrm{v}}_\Gamma \in \mathbf{T}(\Gamma)$. □

Similarly, we treat the momentum equation for a single particle.

7.5 Momentum equation. The distributional 4-momentum system is defined by

$$\begin{gathered} \underline{\mathrm{div}}T = \underline{r} \quad \text{with} \\ T_{ij} = m\underline{v}_i\underline{v}_j\boldsymbol{\mu}_\Gamma\,, \quad \underline{r}_i = r_i\boldsymbol{\mu}_\Gamma\,. \end{gathered}$$

Here the mass m of the particle is an objective scalar and the velocity $\underline{v} = \underline{\mathrm{v}}_\Gamma$ and on the right-hand side $\underline{r}$ satisfies the transformation rule

$$r_i\circ Y = \sum_{\bar{i},\bar{j}\geq 0} mY_{i\,'\bar{i}\bar{j}}\,\underline{v}^*_{\bar{i}}\underline{v}^*_{\bar{j}} + \sum_{\bar{i}\geq 0} Y_{i\,'\bar{i}}\,r^*_{\bar{i}}\,.$$

With this matrix distribution T and this vector distribution $\underline{r}$ the system reads for covariant vector valued test functions ζ

$$\begin{aligned} 0 &= \langle\,\zeta\,,\,-\underline{\mathrm{div}}\,T + \underline{r}\,\rangle_{\mathscr{D}(\mathcal{U})} = \langle\,\underline{\mathrm{D}}\zeta\,,\,T\,\rangle_{\mathscr{D}(\mathcal{U})} + \langle\,\zeta\,,\,\underline{r}\,\rangle_{\mathscr{D}(\mathcal{U})} \\ &= \int_{\mathbb{R}^4}\big(\underline{\mathrm{D}}\zeta\,\boldsymbol{:}\,(m\underline{v}\,\underline{v}^{\mathrm{T}}) + \zeta\bullet r\big)\,\mathrm{d}\boldsymbol{\mu}_\Gamma \\ &= \int_{\mathbb{R}^4}\Big(\sum_{i,j\geq 0}\partial_j\zeta_i m\underline{\mathrm{v}}_{\Gamma i}\underline{\mathrm{v}}_{\Gamma j} + \sum_{i\geq 0}\zeta_i r_i\Big)\,\mathrm{d}\boldsymbol{\mu}_\Gamma \\ &= \int_\Gamma\sum_{i\geq 0}\zeta_i\Big(-\,\mathrm{div}^\Gamma\Big(m\underline{\mathrm{v}}_{\Gamma i}\frac{\underline{\mathrm{v}}_\Gamma}{|\underline{\mathrm{v}}_\Gamma|}\Big) + \frac{r_i}{|\underline{\mathrm{v}}_\Gamma|}\Big)\,\mathrm{dH}^1\,. \end{aligned}$$

For the second integral the regularity of Γ and m are required, and for the last identity $\underline{\mathrm{v}}_\Gamma \in \mathbf{T}(\Gamma)$ is used. (See also section 8.)

Proof. The test function ζ is a covariant vector, that is $\zeta^* = \underline{\mathrm{D}}Y^{\mathrm{T}}\,\zeta\circ Y$, hence

$$\zeta^*_{\bar{i}\,'\bar{j}} = \sum_{i,j} Y_{i\,'\bar{i}}Y_{j\,'\bar{j}}\,\zeta_{i\,'j}\circ Y + \sum_i Y_{i\,'\bar{i}\bar{j}}\,\zeta_i\circ Y\,.$$

Then $g := \underline{\mathrm{D}}\zeta\boldsymbol{:}T + \zeta\bullet\underline{r}$ satisfies

$$\begin{aligned} g^* &= \underline{\mathrm{D}}\zeta^*\boldsymbol{:}T^* + \zeta^*\bullet\mathbf{r}^* = \sum_{\bar{i},\bar{j}} \zeta^*_{\bar{i}\,'\bar{j}}T^*_{\bar{i}\bar{j}} + \sum_{\bar{i}} \zeta^*_{\bar{i}}r^*_{\bar{i}} \\ &= \sum_{i,j} \zeta_{i\,'j}\circ Y \sum_{\bar{i},\bar{j}} Y_{i\,'\bar{i}}Y_{j\,'\bar{j}}T^*_{\bar{i}\bar{j}} + \sum_i \zeta_i\circ Y \sum_{\bar{i}\bar{j}} Y_{i\,'\bar{i},\bar{j}}T^*_{\bar{i}\bar{j}} \\ &\qquad\qquad + \sum_i \zeta_i\circ Y \sum_{\bar{i}} Y_{i\,'\bar{i}}r^*_{\bar{i}} \\ &= \sum_{i,j} \zeta_{i\,'j}\circ Y\,T_{ij}\circ Y + \sum_i \zeta_i\circ Y\,r_i\circ Y \\ &= (\underline{\mathrm{D}}\zeta\boldsymbol{:}T + \zeta\bullet\mathbf{r})\circ Y = g\circ Y\,, \end{aligned}$$

that is, g is an objective scalar. Hence by (7.3)

$$\langle\,\zeta\,,\,-\operatorname{div}T + \underline{r}\,\rangle_{\mathscr{D}(\mathcal{U})} = \int_\Gamma g\,\mathrm{d}\boldsymbol{\mu}_\Gamma = \int_{\Gamma^*} g^*\,\mathrm{d}\boldsymbol{\mu}_{\Gamma^*} = \langle\,\zeta^*\,,\,-\operatorname{div}T^* + \underline{r}^*\,\rangle_{\mathscr{D}(\mathcal{U}^*)}\;.$$

Altogether this shows that we deal with the 4-momentum equation. □

We remark that the mass equation is a special case of the 4-momentum equation as shown in 6.5.

7.6 Example. Let the coordinates be $y = (t,x) \in \mathbb{R}^4$ and let the moving point be given by $\Gamma := \{(t,\xi(t))\,;\; t\in\mathbb{R}\}$ and define $\underline{\xi}(t) := (t,\xi(t))$. Then $\dot{\underline{\xi}} = (1,\dot{\xi}) \in \mathbf{T}(\Gamma)$, and we assume that the derivative points in the same direction as v_Γ, that is, for $y = \underline{\xi}(t)$

$$\underline{\mathrm{v}}_\Gamma(y) = \lambda(y)\frac{\mathrm{d}}{\mathrm{d}t}\underline{\xi}(t), \quad \lambda(y) > 0\,.$$

Then the 4-momentum equation for the evolving point Γ is with $\underline{v} = \underline{\mathrm{v}}_\Gamma$ for all test functions ζ

$$\begin{aligned} 0 &= \big\langle\,\zeta\,,\,-\operatorname{div}(m\underline{v}\,\underline{v}^{\mathrm{T}}) + \underline{r}\,\big\rangle_{\mathscr{D}(\mathcal{U})} = m\underline{v}\,\underline{v}^{\mathrm{T}}\int_{\mathbb{R}^4}\big(\underline{\mathrm{D}}\zeta\boldsymbol{:}(m\underline{v}\,\underline{v}^{\mathrm{T}}) + \zeta\bullet r\big)\,\mathrm{d}\boldsymbol{\mu}_\Gamma \\ &= \int_{\mathbb{R}}\big(\sum_{ij}\partial_j\zeta_i m\underline{v}_i\underline{v}_j + \sum_i\zeta_i r_i\big)(t,\xi(t))\frac{\mathrm{d}t}{\lambda(t,\xi(t))} \\ &= \int_{\mathbb{R}}\sum_i\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(\zeta_i(t,\xi(t))\big)\,m\underline{v}_i(t,\xi(t)) + \zeta_i\frac{r_i}{\lambda}(t,\xi(t))\Big)\,\mathrm{d}t \\ &= \int_{\mathbb{R}}\sum_i\zeta_i\Big(-\lambda(t,\xi(t))\frac{\mathrm{d}}{\mathrm{d}t}(m\underline{v}_i(t,\xi(t))) + r_i(t,\xi(t))\Big)\frac{\mathrm{d}t}{\lambda(t,\xi(t))} \\ &= \int_{\mathbb{R}}\zeta\bullet\Big(-\lambda\frac{\mathrm{d}}{\mathrm{d}t}\Big(m\lambda\frac{\mathrm{d}}{\mathrm{d}t}\underline{\xi}(t)\Big) + r\Big)\frac{\mathrm{d}t}{\lambda}\,. \end{aligned}$$

This is with respect to an arbitrary $\boldsymbol{e}\colon\mathbb{R}^4\to\mathbb{R}^4$.

Proof. Since $\Gamma \subset \mathbb{R}^4$ is one-dimensional we have $\underline{v} = \underline{\mathrm{v}}_\Gamma = \lambda(1, \dot{\xi})$, and $\lambda > 0$ by assumption, therefore

$$\boldsymbol{\mu}_\Gamma = \frac{\mathrm{H}^1 \llcorner \Gamma}{|\underline{\mathrm{v}}_\Gamma|} = \frac{\mathrm{H}^1 \llcorner \Gamma}{\lambda\sqrt{1 + |\dot{\xi}|^2}},$$

hence for every function g

$$\int g \,\mathrm{d}\boldsymbol{\mu}_\Gamma = \int_{\mathbb{R}} \frac{g(t, \xi(t))}{\lambda(t, \xi(t))} \,\mathrm{d}t\,.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\zeta_i(t, \xi(t))\big) = \underline{\nabla}\zeta_i \bullet (1, \dot{\xi}) = \frac{1}{\lambda}\underline{\nabla}\zeta_i \bullet \underline{v} = \frac{1}{\lambda}\sum_j \partial_j \zeta_i \cdot \underline{v}_j\,,$$

we obtain the result. □

This shows that the 4-momentum equation for a moving point $t \mapsto \xi(t)$ is equivalent to the ODE

$$\lambda \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\lambda \frac{\mathrm{d}}{\mathrm{d}t}\underline{\xi}(t)\Big) = \underline{r}\,, \tag{7.4}$$

where $\lambda > 0$ is given by $\lambda\dot{\underline{\xi}} = \underline{\mathrm{v}}_\Gamma$.

7.7 Case 1. If $\boldsymbol{e} = \underline{\mathrm{e}}_0$, that is, t is the "normal" time variable, then $\lambda = 1$ and the ODE reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(m \frac{\mathrm{d}}{\mathrm{d}t}\underline{\xi}(t)\Big) = \underline{r}\,.$$

Writing mass and momentum equations separately we get

$$\frac{\mathrm{d}}{\mathrm{d}t} m = \mathbf{r}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(m \frac{\mathrm{d}}{\mathrm{d}t}\xi\Big) = \mathbf{f}, \quad \underline{r} = \begin{bmatrix} \mathbf{r} \\ \mathbf{f} \end{bmatrix}.$$

Proof. It is $1 = \boldsymbol{e} \bullet \mathrm{v}_\Gamma = \lambda \underline{\mathrm{e}}_0 \bullet (1, \dot{\xi}) = \lambda$ and therefore

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{f} \end{bmatrix} := \underline{r} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(m \frac{\mathrm{d}}{\mathrm{d}t}\underline{\xi}(t)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(m \begin{bmatrix} 1 \\ \dot{\xi} \end{bmatrix}\Big) = \begin{bmatrix} \dot{m} \\ \big(m\dot{\xi}\big)^{\cdot} \end{bmatrix}.$$

□

7.8 Case 2. If $\boldsymbol{e}$ is arbitrary and Γ is an evolving curve, that is $\lambda > 0$, then t is a time variable which is specific to the observer.

(1) If the factor λ is known, the observer can choose to a variable s defined by $t = \boldsymbol{t}(s)$ and

$$\boldsymbol{t}'(s) = \lambda(\underline{\xi}(\boldsymbol{t}(s)))\,.$$

Then $\Gamma = \{\underline{\xi}(\boldsymbol{t}(s))\,;\; s \in \mathbb{R}\}$ and we obtain the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(m \frac{\mathrm{d}}{\mathrm{d}s}\underline{\xi}(\boldsymbol{t}(s))\Big) = \underline{r}\,.$$

(2) If as in 3.9

$$e_0 = \begin{bmatrix} \gamma \\ \gamma V \end{bmatrix}, \quad \boldsymbol{e} = e_0' = \begin{bmatrix} \gamma \\ -\frac{\gamma}{\mathrm{c}^2} V \end{bmatrix}, \quad |V| < \mathrm{c}\,,$$

then $\dot{\xi} \bullet V < \mathrm{c}^2$, which is satisfied if $|\dot{\xi}| < \mathrm{c}$.

(3) The ODE cannot be split as in 7.7. But we have, if $\boldsymbol{e}$ is a constant,

$$\underline{r} = \mathbf{r} e_0 + \mathbf{f}$$

according to 6.6, a formula which is well known in classical physics.

Proof (1). For every function $t \mapsto h(t)$

$$\frac{\mathrm{d}}{\mathrm{d}s} h(\boldsymbol{t}(s)) = \boldsymbol{t}'(s) \frac{\mathrm{d}}{\mathrm{d}t} h(t) = \lambda(\underline{\xi}) \frac{\mathrm{d}}{\mathrm{d}t} h(t)\,.$$

□

Proof (2). It is

$$1 = \boldsymbol{e} \bullet \underline{\mathrm{v}}_\Gamma = \lambda \boldsymbol{e} \bullet \underline{\dot{\xi}} = \begin{bmatrix} \gamma \\ -\frac{\gamma}{\mathrm{c}^2} V \end{bmatrix} \bullet \begin{bmatrix} 1 \\ \dot{\xi} \end{bmatrix} = \lambda\gamma\big(1 - \frac{1}{\mathrm{c}^2} V \bullet \dot{\xi}\big)$$

hence

$$0 < \frac{1}{\lambda\gamma} = \big(1 - \frac{1}{\mathrm{c}^2} V \bullet \dot{\xi}\big)\,.$$

□

Proof (3). The pure force $\mathbf{f} := \underline{r} - \underline{r} \bullet e_0' e_0$ has been defined in 6.6. And $\underline{r} \bullet e_0' = \boldsymbol{e}$ by 6.5 if $\boldsymbol{e}$ is constant. □

The following example is based on the coordinates in 7.8. You will find this in [4, I §5 Additionstheorem der Geschwindigkeiten].

7.9 Addition of velocities. Let $y^* = (t^*, x^*) \in \mathbb{R}^4$ and $y = (t, x) \in \mathbb{R}^4$ be connected by a Lorentz transformation $y = Y(y^*) := \mathbf{L}_\mathrm{c}(V, Q) y^*$ and consider two moving points $\Gamma := \{(t, \xi(t))\ ;\ t \in \mathbb{R}\}$ and $\Gamma^* := \{(t^*, \xi^*(t^*))\ ;\ t^* \in \mathbb{R}\}$ with $\Gamma = Y(\Gamma^*)$. Define $u(t, \xi(t)) := \dot{\xi}(t)$ and $u^*(t^*, \xi^*(t^*)) := \dot{\xi}^*(t^*)$.

(1) Then

$$u = \frac{V + \frac{1}{\gamma} \mathbf{B}_\mathrm{c}(V) Q u^*}{1 + \frac{1}{\mathrm{c}^2} V \bullet Q u^*}$$

if the denominator is positive.

(2) If $Q = \mathrm{Id}$ and $V \in \operatorname{span}\{u^*\}$ then

$$u = \frac{V + u^*}{1 + \frac{1}{\mathrm{c}^2} V \bullet u^*}$$

if the denominator is positive.

Proof (1). The identity $(t,\xi(t)) = Y(t^*,\xi^*(t^*))$ is

$$\begin{bmatrix} t \\ \xi(t) \end{bmatrix} = \begin{bmatrix} \gamma t^* + \frac{\gamma}{\mathrm{c}^2} V^{\mathrm{T}} Q\xi^*(t^*) \\ \gamma t^* V + \mathbf{B}_{\mathrm{c}}(V) Q\xi^*(t^*) \end{bmatrix}.$$

If s is a common parameter, that is $(t(s),\xi(t(s))) = Y(t^*(s),\xi^*(t^*(s)))$, then the derivative with respect to s gives the result. □

Proof (2). It is $u^* = (u^* \bullet \widehat{V})\,\widehat{V}$ and $\mathbf{B}_{\mathrm{c}}(V)\widehat{V} = \gamma\widehat{V}$. □

If one takes instead the velocities v_Γ and v_{Γ^*} of the moving points, one has at corresponding points $(t,\xi(t))$ and $(t^*,\xi^*(t^*))$ the relation $\mathrm{v}_\Gamma = \mathbf{L}_{\mathrm{c}}(V,Q)\mathrm{v}_{\Gamma^*}$ and

$$\mathrm{v}_\Gamma = \frac{\underline{u}}{\boldsymbol{e}\bullet\underline{u}}\,, \quad \underline{u} := \dot{\underline{\xi}}\,, \quad \underline{\xi}(t) := (t,\xi(t))\,, \quad \underline{u} = (1,u)\,,$$

similarly for v_{Γ^*}. Compare the identity in 5.4(2).

8 Approximation of particles

A particle is a mass concentrated at an evolving point $\Gamma \subset \mathbb{R}^4$. The aim is to approximate this particle by a mass density in a neighbourhood of this curve which converges to the particle mass on Γ. This is necessary in order to prove that the distributional equations of section 7 are really the 4-momentum equations.

In the general situation this means that we have functions $g_\varepsilon : \mathbb{R}^4 \to \mathbb{R}$ with support in a neighbourhood of Γ such that for a limit function $g_\Gamma : \Gamma \to \mathbb{R}$

$$g_\varepsilon \mathrm{L}^4 \to g_\Gamma\, \boldsymbol{\mu}_\Gamma \quad \text{as } \varepsilon \to 0, \text{ where} \quad \boldsymbol{\mu}_\Gamma := \frac{1}{|\underline{\mathrm{v}}_\Gamma|}\mathrm{H}^1 \llcorner \Gamma$$

is the measure from (7.2). This means that for test functions $\eta \in C_0^\infty(\mathbb{R}^4)$

$$\int_{\mathbb{R}^4} \eta g_\varepsilon \,\mathrm{dL}^4 \longrightarrow \int_\Gamma \eta g_\Gamma \,\mathrm{d}\boldsymbol{\mu}_\Gamma \quad \text{as } \varepsilon \to 0. \tag{8.1}$$

In this situation we prove the following Theorem 8.3 for objective scalars g_ε in spacetime which as $\varepsilon \to 0$ behave like a Dirac sequence around Γ. Before we do so it is useful to prove the following quite general statements.

8.1 Lemma. The vector $\boldsymbol{e} = e_0'$ satisfies

(1) $|e_0 \wedge \cdots \wedge e_3| = \dfrac{|e_1 \wedge \cdots \wedge e_3|}{|e_0'|}$.

(2) $|e_0 \wedge \cdots \wedge e_3|$ is an objective scalar.

Remark: It is $|e_0 \wedge \cdots \wedge e_3| = |\underline{\mathbf{e}}_0 \wedge \cdots \wedge \underline{\mathbf{e}}_3| = 1$ if the observer is connected with the standard Lorentz observer.

Proof (1). $\{e_0', e_1, \dots, e_3\}$ is a basis of $\mathbb{R}^4$ and hence $e_0 = \mu e_0' + \sum_{j=1}^n \nu_j e_j$ and $1 = e_0' \bullet e_0 = \mu |e_0'|^2$. Therefore the vector $\bar{e} := e_0 - \sum_{j=1}^n \nu_j e_j = \mu e_0'$ satisfies $\bar{e} \bullet e_i = 0$ for $i \geq 1$. We conclude

$$|e_0 \wedge \cdots \wedge e_3| = |\bar{e} \wedge e_1 \wedge \cdots \wedge e_3| = |\bar{e}| \cdot |e_1 \wedge \cdots \wedge e_3|$$

where $|\bar{e}| = |\mu e_0'| = |e_0'|^{-1}$ (see the proof of 3.3) $\square$

Proof (2). By 4.2 we have the transformation rule $e_k \circ Y = \underline{\mathrm{D}} Y e_k^*$. Hence

$$\begin{aligned} |e_0 \wedge \cdots \wedge e_3| \circ Y &= |\partial_{e_0^*} Y \wedge \cdots \wedge \partial_{e_3^*} Y| \\ &= |e_0^* \wedge \cdots \wedge e_3^*| \cdot |\partial_0 Y \wedge \cdots \wedge \partial_3 Y| \end{aligned}$$

and $|\partial_0 Y \wedge \cdots \wedge \partial_3 Y| = |\det \underline{\mathrm{D}} Y| = 1$. $\square$

8.2 Lemma. For any function $h\colon \{(y,z)\,;\ z \in \boldsymbol{W}(y)\} \to \mathbb{R}$

$$\begin{aligned} &\int_{\boldsymbol{W}(y)} h(y,z) \frac{\mathrm{dH}^3(z)}{|e_1(y) \wedge \cdots \wedge e_3(y)|} \\ &= \int_{\boldsymbol{W}^*(y^*)} h(Y(y^*), \mathrm{D}Y z^*) \frac{\mathrm{dH}^3(z^*)}{|e_1^*(y^*) \wedge \cdots \wedge e_3^*(y^*)|} \,. \end{aligned}$$

Remark: This lemma is applied to the case that $h^*(y^*, z^*) := h(Y(y^*), \mathrm{D}Y z^*)$.

Proof. It follows with the transformation $\mathrm{D}Y(y^*)\colon \boldsymbol{W}^*(y^*) \to \boldsymbol{W}(y)$

$$\int_{\boldsymbol{W}(y)} h(y,z)\, \mathrm{dH}^3(z) = |\det\ \mathrm{D}Y(y^*)|_{\boldsymbol{W}^*(y^*)}| \int_{\boldsymbol{W}^*(y^*)} h(Y(y^*), \mathrm{D}Y z^*)\, \mathrm{dH}^3(z^*) \,.$$

If $\{e_1^\perp, e_2^\perp, e_3^\perp\}$ is an orthonormal basis of $\boldsymbol{W}^*$ then

$$|\det\ \mathrm{D}Y|_{\boldsymbol{W}^*}| = |\partial_{e_1^\perp} Y \wedge \cdots \wedge \partial_{e_3^\perp} Y|$$

Since $\{e_1^*, e_2^*, e_3^*\}$ is a basis of the same space $\boldsymbol{W}^*$ we get

$$|\partial_{e_1^*} Y \wedge \cdots \wedge \partial_{e_3^*} Y| = |e_1^* \wedge \cdots \wedge e_3^*| \cdot |\partial_{e_1^\perp} Y \wedge \cdots \wedge \partial_{e_3^\perp} Y| \,,$$

and since $\mathrm{D}Y e_i^* = e_i \circ Y$ for $i \geq 1$ by 4.2 we see that

$$|\partial_{e_1^*} Y \wedge \cdots \wedge \partial_{e_3^*} Y| = |e_1 \circ Y \wedge \cdots \wedge e_3 \circ Y| = |e_1 \wedge \cdots \wedge e_3| \circ Y \,.$$

This gives

$$|\det\ \mathrm{D}Y|_{\boldsymbol{W}^*}| = \frac{|e_1 \wedge \cdots \wedge e_3| \circ Y}{|e_1^* \wedge \cdots \wedge e_3^*|}$$

and finishes the proof. $\square$

We use this in order show that a mass point is the limit of an distributed objective mass density.

8.3 Theorem (Convergence to Γ). We assume that Γ is an evolving point and let $g_\varepsilon : \mathbb{R}^4 \to \mathbb{R}$ be objective scalars whose support is in the ε-neighbourhood of Γ. We assume that for $y \in \Gamma$ and $z_\varepsilon \to z \in \boldsymbol{W}(y)$

$$\varepsilon^3 |e_1(y) \wedge \cdots \wedge e_3(y)|\, g_\varepsilon(y + \varepsilon z_\varepsilon) \to |\boldsymbol{e}(y)|\, g(y,z) \quad \text{ as } \varepsilon \to 0. \tag{8.2}$$

Then for $\eta \in C_0^\infty(\mathbb{R}^4)$

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^4} \eta g_\varepsilon \,\mathrm{dL}^4 = \int_\Gamma \eta g_\Gamma \,\mathrm{d}\boldsymbol{\mu}_\Gamma\,, \tag{8.3}$$

where

$$g_\Gamma(y) := \int_{\boldsymbol{W}(y)} g(y,z) \frac{\mathrm{dH}^3(z)}{|e_1(y) \wedge \cdots \wedge e_3(y)|}\,. \tag{8.4}$$

The function $g_\Gamma : \Gamma \to \mathbb{R}$ is an objective scalar.

Proof. We let $\Gamma = \{\underline{\xi}(s)\,;\ s \in \mathbb{R}\}$ where the parameter s can be chosen so that $\underline{\xi}'(s) = \underline{\mathrm{v}}_\Gamma(\xi(s))$. It follows that for every function $h : \Gamma \to \mathbb{R}$

$$\int_\Gamma h(y)\,\mathrm{d}\boldsymbol{\mu}_\Gamma(y) = \int_\Gamma h(y) \frac{\mathrm{dH}^1(y)}{|\underline{\mathrm{v}}_\Gamma(y)|} = \int_\mathbb{R} h(\underline{\xi}(s))\,\mathrm{dL}^1(s)\,. \tag{8.5}$$

Now we use the transformation

$$(s,z) \mapsto y = \boldsymbol{y}(s,z) := \underline{\xi}(s) + \sum_{i \geq 1} z_i e_i^\perp(\underline{\xi}(s)) \in \mathbb{R}^4\,,$$

where $\{e_1^\perp(y), e_2^\perp(y), e_3^\perp(y)\}$ is an orthonormal basis of $\boldsymbol{W}(y)$. We compute its determinant, since $\{\widehat{\boldsymbol{e}}, e_1^\perp, e_2^\perp, e_3^\perp\}$ is an orthonormal basis of $\mathbb{R}^4$, in an ε-neighbourhood of Γ as

$$\begin{aligned}
|\det \mathrm{D}\boldsymbol{y}(s,z)| &= |(\underline{\xi}' + \sum_{i\geq 1} z_i \mathrm{D}e_i^\perp(\underline{\xi})\underline{\xi}') \wedge e_1^\perp \wedge \cdots \wedge e_3^\perp| \\
&= |\underline{\xi}' \wedge e_1^\perp \wedge \cdots \wedge e_3^\perp| + \mathcal{O}(\varepsilon) = \underline{\xi}' \bullet \widehat{\boldsymbol{e}} + \mathcal{O}(\varepsilon)\,,
\end{aligned}$$

$$\underline{\xi}'(s) \bullet \widehat{\boldsymbol{e}}(\underline{\xi}(s)) = \underline{\mathrm{v}}_\Gamma(\xi(s)) \bullet \widehat{\boldsymbol{e}}(\underline{\xi}(s)) = \frac{1}{|\boldsymbol{e}(\underline{\xi}(s))|}\,,$$

hence

$$|\det \mathrm{D}\boldsymbol{y}(s,z)| = \frac{1}{|\boldsymbol{e}(\underline{\xi}(s))|} + \mathcal{O}(\varepsilon).$$

It then follows that for $\varepsilon \to 0$

$$\begin{aligned}
\int_{\mathbb{R}^4} \eta g_\varepsilon \,\mathrm{dL}^4 &= \int_\mathbb{R} \int_{\mathbb{R}^3} (\eta g_\varepsilon)(\boldsymbol{y}(s,z)) |\det \mathrm{D}\boldsymbol{y}(s,z)|\,\mathrm{dL}^3(z)\,\mathrm{dL}^1(s) \\
&= \int_\mathbb{R} \int_{\mathbb{R}^3} (\eta g_\varepsilon)(\boldsymbol{y}(s,z)) \frac{\mathrm{dL}^3(z)}{|\boldsymbol{e}(\underline{\xi}(s))|}\,\mathrm{dL}^1(s) + \mathcal{O}(\varepsilon)\,.
\end{aligned}$$

Therefore we consider the following function

$$\begin{aligned}
h_\varepsilon(y) &:= \int_{\mathbb{R}^3} (\eta g_\varepsilon)(\boldsymbol{y}(s,z)) \frac{\mathrm{dL}^3(z)}{|\boldsymbol{e}(\underline{\xi}(s))|}\bigg|_{\underline{\xi}(s)=y} \\
&= \int_{\boldsymbol{W}(y)} (\eta g_\varepsilon)(y+z) \frac{\mathrm{dH}^3(z)}{|\boldsymbol{e}(y)|} = \int_{\boldsymbol{W}(y)} \varepsilon^3 (\eta g_\varepsilon)(y + \varepsilon z) \frac{\mathrm{dH}^3(z)}{|\boldsymbol{e}(y)|} \\
&= \eta(y) \int_{\boldsymbol{W}(y)} \varepsilon^3 g_\varepsilon(y + \varepsilon z) \frac{\mathrm{dH}^3(z)}{|\boldsymbol{e}(y)|} + \mathcal{O}(\varepsilon)\,,
\end{aligned}$$

and (8.5) gives

$$\int_{\mathbb{R}^4} \eta g_\varepsilon \, \mathrm{dL}^4 = \int_\Gamma h_\varepsilon(y) \, \mathrm{d}\boldsymbol{\mu}_\Gamma(y) + \mathcal{O}(\varepsilon) \, .$$

By assumption (8.2) as $\varepsilon \to 0$ for $z \in \boldsymbol{W}(y)$

$$\frac{\varepsilon^3 g_\varepsilon(y + \varepsilon z)}{|\boldsymbol{e}(y)|} \to \frac{g(y,z)}{|e_1(y) \wedge \cdots \wedge e_3(y)|} \, ,$$

and this is why we obtain

$$h_\varepsilon(y) \to \eta(y) \int_{\boldsymbol{W}(y)} g(y,z) \frac{\mathrm{dH}^3(z)}{|e_1(y) \wedge \cdots \wedge e_3(y)|} = \eta(y) g_\Gamma(y) =: h(y) \, ,$$

and this is the convergence as $\varepsilon \to 0$ of (8.3).

It remains to prove that g_Γ is an objective scalar. We know that g_ε is an objective scalar, hence $g_\varepsilon(y) = g_\varepsilon^*(y^*)$ for $y = Y(y^*)$. By 8.1 the value

$$\lambda(y) := |e_0(y) \wedge \cdots \wedge e_3(y)| = \frac{|e_1(y) \wedge \cdots \wedge e_3(y)|}{|\boldsymbol{e}(y)|}$$

is an objective scalar, and therefore also $\lambda(y) g_\varepsilon(y) = \lambda^*(y^*) g_\varepsilon^*(y^*)$ for $y = Y(y^*)$. Thus, if in addition $z = \mathrm{D}Y(y^*) z^*$, $z^* \in \boldsymbol{W}^*(y^*)$, from assumption (8.2)

$$\begin{aligned}
& g^*(y^*, z^*) \leftarrow \varepsilon^3 \lambda^*(y^*) g_\varepsilon^*(y^* + \varepsilon z^*) = \varepsilon^3 \lambda(y) g_\varepsilon(Y(y^* + \varepsilon z^*)) \\
& = \varepsilon^3 \lambda(y) g_\varepsilon(y + \varepsilon z_\varepsilon) \, , \quad z_\varepsilon := \frac{1}{\varepsilon}(Y(y^* + \varepsilon z^*) - Y(y^*)) \, , \\
& \to g(y,z) \, , \text{ since } z_\varepsilon \to \mathrm{D}Y(y^*) z^* =: z \in \boldsymbol{W}(y) \, ,
\end{aligned}$$

that is,

$$g^*(y^*, z^*) = g(y,z) \text{ for } y = Y(y^*), \ z = \mathrm{D}Y(y^*) z^*. \tag{8.6}$$

From this it follows $g_{\Gamma^*}(y^*) = g_\Gamma(y)$, since we prove in 8.2 that the integral which defines g_Γ is frame independent. ☐

We wanted to clarify the connection between the mass $m : \Gamma \to \mathbb{R}$ of a moving point Γ and the mass density $\varrho_\varepsilon : \mathbb{R}^4 \to \mathbb{R}$ in spacetime, which is concentrated near Γ. We have shown in 8.3 that a certain convergence of the usually called mass in the point $y \in \mathbb{R}^4$

$$z \mapsto \varepsilon^3 |e_1(y) \wedge \cdots \wedge e_3(y)| \, \varrho_\varepsilon(y + \varepsilon z)$$

implies that $\varrho_\varepsilon \mathrm{L}^4 \to m \boldsymbol{\mu}_\Gamma$ in distributional sense where the mass of the particle in $y \in \Gamma$

$$m(y) = \frac{1}{|\boldsymbol{e}(y)|} \int_{\boldsymbol{W}(y)} \lim_{\varepsilon \to 0} \big(\varepsilon^3 \varrho_\varepsilon(y + \varepsilon z) \big) \, \mathrm{dH}^3(z)$$

is a mean value of the mass density across $\{y + z ; \ z \in \boldsymbol{W}(y)\}$. This mass m is an objective scalar. Accordingly, if the 4-velocity $\underline{v}_\varepsilon$ converges strongly to $\underline{\mathrm{v}}_\Gamma$ we have convergence of the mass and the momentum equation.

9 Fluid equations

In this section we consider the 4-momentum equation (6.2)

$$\underline{\operatorname{div}} T = \tau \quad \text{with tensor} \quad T = \varrho \underline{v} \otimes \underline{v} + \underline{\Pi} \, .$$

The characteristic behaviour of T for a fluid is that it depends on the gradients of the velocity $\underline{\mathrm{D}}\,\underline{v}$, where here we take a 4-velocity $\underline{v}$ as defined in 5.2. We prove the following theorem for the tensor $\underline{\Pi}$, which is the generalization of the fact, that in classical physics the dependence on the symmetric part of the velocity gradient is the only objective version of such a dependence.

9.1 Theorem. Let $\underline{v}$ be a 4-velocity (as in 5.2).

(1) The tensor $\widetilde{S} = \underline{\mathrm{D}v}\,\mathrm{G} + (\underline{\mathrm{D}v}\,\mathrm{G})^{\mathrm{T}} - (\underline{v} \bullet \underline{\nabla})\mathrm{G}$ is a contravariant tensor.

(2) Also $S := \underline{\mathrm{D}v}\,\mathrm{G}^{\mathrm{sp}} + (\underline{\mathrm{D}v}\,\mathrm{G}^{\mathrm{sp}})^{\mathrm{T}} - (\underline{v} \bullet \underline{\nabla})\mathrm{G}^{\mathrm{sp}}$ is a contravariant tensor.

Proof of contravariance. Let G be an arbitrary symmetric contravariant tensor, that is a tensor with the property

$$G \circ Y = \underline{\mathrm{D}}Y G^* \underline{\mathrm{D}}Y^{\mathrm{T}} \tag{9.1}$$

for observer transformations $y = Y(y^*)$. The 4-velocity $\underline{v}$ satisfies

$$\underline{v}_i \circ Y = \sum_{\bar{i}} Y_{i\,\prime\bar{i}}\, \underline{v}^*_{\bar{i}} \, .$$

Therefore one obtains for the derivative

$$\partial_{\bar{j}}(\underline{v}_i \circ Y) = \sum_{\bar{i}} Y_{i\,\prime\bar{i}\bar{j}}\, \underline{v}^*_{\bar{i}} + \sum_{\bar{i}} Y_{i\,\prime\bar{i}} \partial_{\bar{j}} \underline{v}^*_{\bar{i}}$$

and from the chain rule $\partial_{\bar{j}}(\underline{v}_i \circ Y) = \sum_j (\partial_j \underline{v}_i) \circ Y\, Y_{j\,\prime\bar{j}}$, that is

$$\sum_j (\partial_j \underline{v}_i) \circ Y\, Y_{j\,\prime\bar{j}} = \sum_{\bar{i}} Y_{i\,\prime\bar{i}\bar{j}}\, \underline{v}^*_{\bar{i}} + \sum_{\bar{i}} Y_{i\,\prime\bar{i}} \partial_{\bar{j}} \underline{v}^*_{\bar{i}}$$

or in matrix notation

$$(\underline{\mathrm{D}}v \circ Y)\underline{\mathrm{D}}Y = \sum_{\bar{i}} \underline{v}^*_{\bar{i}} \underline{\mathrm{D}}Y_{\prime\bar{i}} + \underline{\mathrm{D}}Y \underline{\mathrm{D}v}^* \, .$$

Multiplying this identity from the right side by $G^* \underline{\mathrm{D}}Y^{\mathrm{T}}$ one obtains using the property (9.1)

$$(\underline{\mathrm{D}}v\, G) \circ Y = \sum_{\bar{i}} \underline{v}^*_{\bar{i}} \underline{\mathrm{D}}Y_{\prime\bar{i}} G^* \underline{\mathrm{D}}Y^{\mathrm{T}} + \underline{\mathrm{D}}Y (\underline{\mathrm{D}v}^* G^*) \underline{\mathrm{D}}Y^{\mathrm{T}} \, .$$

This is the transformation rule for $\underline{\mathrm{D}}v\, G$ (and it is, for $G = \mathrm{G}_{\mathrm{c}}$ and $c \to \infty$, identical with the classical formula). From this we obtain the transposed version (G^* is symmetric)

$$(\underline{\mathrm{D}}v\, G)^{\mathrm{T}} \circ Y = \sum_{\bar{i}} \underline{v}^*_{\bar{i}} \underline{\mathrm{D}}Y G^* \underline{\mathrm{D}}Y^{\mathrm{T}}_{\prime\bar{i}} + \underline{\mathrm{D}}Y\, (\underline{\mathrm{D}v}^* G^*)^{\mathrm{T}} \underline{\mathrm{D}}Y^{\mathrm{T}} \, .$$

The sum of both equations has as inhomogeneous term

$$M := \sum_{\bar{i}} \underline{v}^*_{\bar{i}} \underline{\mathrm{D}}Y_{\prime\bar{i}} G^* \underline{\mathrm{D}}Y^{\mathrm{T}} + \sum_{\bar{i}} \underline{v}^*_{\bar{i}} \underline{\mathrm{D}}Y G^* \underline{\mathrm{D}}Y_{\prime\bar{i}}^{\mathrm{T}} \, ,$$

and reads

$$(\underline{\mathrm{D}}v\,G)\circ Y + (\underline{\mathrm{D}}v\,G)^{\mathrm{T}}\circ Y = M + \underline{\mathrm{D}}Y\big(\underline{\mathrm{D}v}^{*}G^{*} + (\underline{\mathrm{D}v}^{*}G^{*})^{\mathrm{T}}\,\big)\,\underline{\mathrm{D}}Y^{\mathrm{T}}\ .$$

The M-term also occurs in the transformation rule of $\sum_i \underline{v}_i G{}_{\prime i}$, since

$$\begin{aligned}
&\Big(\sum_i \underline{v}_i G{}_{\prime i}\Big)\circ Y = \sum_{i\bar{i}} Y_{i\,\prime\bar{i}}\underline{v}^{*}_{\bar{i}}(G{}_{\prime i}\circ Y)\\
&= \sum_{\bar{i}} \underline{v}^{*}_{\bar{i}}\Big(\sum_i Y_{i\,\prime\bar{i}} G{}_{\prime i}\circ Y\Big) = \sum_{\bar{i}} \underline{v}^{*}_{\bar{i}}(G\circ Y)_{\prime\bar{i}} = \sum_{\bar{i}} \underline{v}^{*}_{\bar{i}}(\underline{\mathrm{D}}Y G^{*}\,\underline{\mathrm{D}}Y^{\mathrm{T}})_{\prime\bar{i}}\\
&= \underline{\mathrm{D}}Y\Big(\sum_{\bar{i}} \underline{v}^{*}_{\bar{i}} G^{*}_{\prime\bar{i}}\Big)\,\underline{\mathrm{D}}Y^{\mathrm{T}} + \underbrace{\sum_{\bar{i}} \underline{v}^{*}_{\bar{i}}\Big(\underline{\mathrm{D}}Y_{\prime\bar{i}}G^{*}\,\underline{\mathrm{D}}Y^{\mathrm{T}} + \underline{\mathrm{D}}Y G^{*}\,\underline{\mathrm{D}}Y^{\mathrm{T}}_{\prime\bar{i}}\Big)}_{=M}\quad .
\end{aligned}$$

Hence subtracting both equations gives $S := \underline{\mathrm{D}}v\,G + (\underline{\mathrm{D}}v\,G)^{\mathrm{T}} - (\underline{v}\bullet\nabla)G$ and this matrix satisfies $S\circ Y = \underline{\mathrm{D}}Y S^{*}\,\underline{\mathrm{D}}Y^{\mathrm{T}}$. □

Proof (1). Because $G := \mathrm{G}$ satisfies (9.1). □

Proof (2). Also $G := \mathrm{G}^{\mathrm{sp}} = \mathrm{G} - \mathrm{G}^{\mathrm{ti}}$ satisfies (9.1), since $\mathrm{G}^{\mathrm{ti}} = -\frac{1}{\mathrm{c}^2}e_0\,e_0{}^{\mathrm{T}}$ and e_0 is a contravariant vector. □

9.2 Theorem. Let $\underline{v}$ be a 4-velocity as in 5.2.

(1) Define $\mathbf{J} := \boldsymbol{e}^{\mathrm{T}}\,S$ for the tensor S in 9.1(2). The vector satisfies

$$\mathbf{J}_l = \sum_{ik} \boldsymbol{e}_{i\,\prime k}(\underline{v}_k \mathrm{G}^{\mathrm{sp}}_{il} - \underline{v}_i \mathrm{G}^{\mathrm{sp}}_{kl})$$

and is a contravariant vector.

(2) It is $\boldsymbol{e}\bullet\mathbf{J} = 0$.

Proof (1). It is

$$\begin{aligned}
&(\boldsymbol{e}^{\mathrm{T}}\,S)_j = \sum_i \boldsymbol{e}_i S_{ij}\\
&= \sum_{ik} \boldsymbol{e}_i \underline{v}_{i\,\prime k}\mathrm{G}^{\mathrm{sp}}_{kj} + \sum_k \underbrace{\sum_i \boldsymbol{e}_i \mathrm{G}^{\mathrm{sp}}_{ik}}_{=0}\ \underline{v}_{j\,\prime k} - \sum_{ik} \underline{v}_k \boldsymbol{e}_i \mathrm{G}^{\mathrm{sp}}_{ij\,\prime k}\\
&= \sum_k \partial_k\ \underbrace{\Big(\sum_i \boldsymbol{e}_i \underline{v}_i\Big)}_{=1}\ \cdot\mathrm{G}^{\mathrm{sp}}_{kj} - \sum_{ki} \boldsymbol{e}_{i\,\prime k}\underline{v}_i \mathrm{G}^{\mathrm{sp}}_{kj}\\
&- \sum_k \underline{v}_k \partial_k\ \underbrace{\Big(\sum_i \boldsymbol{e}_i \mathrm{G}^{\mathrm{sp}}_{ij}\Big)}_{=0}\ + \sum_{ki} \underline{v}_k \boldsymbol{e}_{i\,\prime k}\mathrm{G}^{\mathrm{sp}}_{ij}\\
&= \sum_{ki} \boldsymbol{e}_{i\,\prime k}(\underline{v}_k \mathrm{G}^{\mathrm{sp}}_{ij} - \underline{v}_i \mathrm{G}^{\mathrm{sp}}_{kj})\,.
\end{aligned}$$

Thus
$$(\boldsymbol{e}^{\mathrm{T}}\,S)_j = \textstyle\sum_{ki}\boldsymbol{e}_{i\,'k}(\underline{v}_k\mathrm{G}^{\mathrm{sp}}_{ij} - \underline{v}_i\mathrm{G}^{\mathrm{sp}}_{kj})\,. \tag{9.2}$$
Now
$$\boldsymbol{e}_{i\,'k} = \frac{\boldsymbol{e}_{i\,'k}+\boldsymbol{e}_{k\,'i}}{2} + E_{ik}\,,\quad E_{ik} = \frac{\boldsymbol{e}_{i\,'k}-\boldsymbol{e}_{k\,'i}}{2}\,,$$
and since the bracket in (9.2) is antisymmetric in (i,k), it follows that
$$\mathbf{J}_j := (\boldsymbol{e}^{\mathrm{T}}\,S)_j = \sum_{ki} E_{ik}(\underline{v}_k\mathrm{G}^{\mathrm{sp}}_{ij} - \underline{v}_i\mathrm{G}^{\mathrm{sp}}_{kj})\,,$$
a term which we handled already in 5.6. This is because $\boldsymbol{e}^*_{\bar{i}} = \sum_i Y_{i\,'\bar{i}}\boldsymbol{e}_i\circ Y$, and hence
$$\boldsymbol{e}^*_{\bar{i}\,'\bar{k}} = \sum_i Y_{i\,'\bar{i}\bar{k}}\boldsymbol{e}_i\circ Y + \sum_{ik} Y_{i\,'\bar{i}}Y_{k\,'\bar{k}}\boldsymbol{e}_{i\,'k}\circ Y\,.$$
Since $Y_{i\,'\bar{i}\bar{k}}$ is symmetric in $(\bar{i},\bar{k})$, we obtain
$$E^*_{\bar{i}\bar{k}} = \sum_{ik} Y_{i\,'\bar{i}}Y_{k\,'\bar{k}}E_{ik}\circ Y\,,$$
a property which was assumed in 5.6. □

Proof (2). Since $\sum_j \boldsymbol{e}_j\mathrm{G}^{\mathrm{sp}}_{kj} = 0$ for every k. □

For fluids one has the following momentum system on the basis of 9.1(2)
$$\underline{\mathrm{div}}(\varrho\underline{v}\,\underline{v}^{\mathrm{T}} + \underline{\Pi}) = \tau\,,\quad \underline{\Pi} = p\mathrm{G}^{\mathrm{sp}} - S\,,$$
$$S = \mu(\underline{\mathrm{D}v}\,\mathrm{G}^{\mathrm{sp}} + (\underline{\mathrm{D}v}\,\mathrm{G}^{\mathrm{sp}})^{\mathrm{T}} - (\underline{v}\bullet\underline{\nabla})\mathrm{G}^{\mathrm{sp}}) + \lambda\,\underline{\mathrm{div}v}\,\mathrm{G}^{\mathrm{sp}}\,,$$
where p, μ, and λ are objective scalars. It contains the mass equation
$$\underline{\mathrm{div}}(\varrho\underline{v} + \mathbf{J}) = \mathbf{r}\,,\quad \mathbf{J} = \boldsymbol{e}\bullet\underline{\Pi} = \boldsymbol{e}\bullet S\,,$$
where $\boldsymbol{e}\bullet\mathbf{J} = 0$ and $\mathbf{r} = \boldsymbol{e}\bullet\tau + \mathrm{D}\boldsymbol{e}\,\boldsymbol{:}\big(\underline{v}\,(\varrho\underline{v}+\mathbf{J})^{\mathrm{T}} + \underline{\Pi}\big)$.

10 Higher moments

We now present the relativistic version of higher moments. We consider moments of order N with flux $T = (T_\beta)_{\beta\in\{0,\dots,3\}^{N+1}}$. Writing $T_\beta = T_{\alpha j}$ with $\beta = (\alpha, j)$, where $\alpha \in \{0,\dots,3\}^N$ and $j \in \{0,\dots,3\}$, the system of N^{th}-moments reads in the version for test functions $\zeta = (\zeta_\alpha)_{\alpha\in\{0,\dots,3\}^N}$ with $\zeta_\alpha \in C^\infty_0(\mathcal{U};\mathbb{R})$ in a domain $\mathcal{U}\subset\mathbb{R}^4$
$$\int_{\mathbb{R}^4}\sum_\alpha\Big(\sum_{j\geq 0}\partial_{y_j}\zeta_\alpha\cdot T_{\alpha j} + \zeta_\alpha\cdot g_\alpha\Big)\,\mathrm{dL}^4 = 0\,. \tag{10.1}$$
The strong version of this system is $\underline{\mathrm{div}}\,T = g$ or
$$\sum_{j\geq 0}\partial_{y_j}T_{\alpha j} = g_\alpha \quad \text{for } \alpha\in\{0,\dots,3\}^N. \tag{10.2}$$

The definition of the physical quantities of this ***system of N^{th}-moments*** is the following: We demand from the test functions that they satisfy the transformation rule

$$\zeta^*_{\bar{k}_1\cdots\bar{k}_N} = \sum_{k_1,\ldots,k_N\geq 0} Y_{k_1\,\prime\bar{k}_1}\cdots Y_{k_N\,\prime\bar{k}_N}\zeta_{k_1\cdots k_N}\circ Y$$

for all $\bar{k}_1,\ldots,\bar{k}_N\in\{0,\ldots,3\}$, where Y is a relativistic observer transformation. Hence these test functions ζ are covariant N-tensors. Here y, T, g, and ζ are the quantities for one observer and similarly y^*, T^*, g^*, and ζ^* are these quantities for another observer, and $y=Y(y^*)$ is the observer transformation where we as always assume that $\det \underline{\mathrm{D}}Y=1$. Therefore it holds

$$\int_{\mathbb{R}^4}\sum_\alpha\Big(\sum_{j\geq 0}\partial_{y_j}\zeta_\alpha\cdot T_{\alpha j}+\zeta_\alpha\cdot g_\alpha\Big)\,\mathrm{dL}^4=\int_{\mathbb{R}^4}\sum_\alpha\Big(\sum_{\bar{j}\geq 0}\partial_{y^*_{\bar{j}}}\zeta^*_\alpha\cdot T^*_{\alpha\bar{j}}+\zeta^*_\alpha\cdot g^*_\alpha\Big)\,\mathrm{dL}^4$$

and this is satisfied if the physical quantities T and g fulfill the following transformation rule (see the result in 11.1 below)

$$T_{k_1\cdots k_M}\circ Y=\sum_{\bar{k}_1,\ldots,\bar{k}_M\geq 0} Y_{k_1\,\prime\bar{k}_1}\cdots Y_{k_M\,\prime\bar{k}_M}T^*_{\bar{k}_1\cdots\bar{k}_M} \tag{10.3}$$

for $k_1,\ldots,k_M\in\{0,\ldots,3\}$ and $M=N+1$, and

$$\begin{aligned} g_{k_1\cdots k_N}\circ Y=&\sum_{\bar{k}_1,\ldots,\bar{k}_N,j\geq 0}\big(Y_{k_1\,\prime\bar{k}_1}\cdots Y_{k_N\,\prime\bar{k}_N}\big)_{\prime j}T^*_{\bar{k}_1\cdots\bar{k}_N j}\\ &+\sum_{\bar{k}_1,\ldots,\bar{k}_N\geq 0}Y_{k_1\,\prime\bar{k}_1}\cdots Y_{k_N\,\prime\bar{k}_N}g^*_{\bar{k}_1\cdots\bar{k}_N}\end{aligned} \tag{10.4}$$

for $k_1,\ldots,k_N\in\{0,\ldots,3\}$. If one wants to express a high moment in terms of another observer, one needs all moments of the other observer up to this order. As symmetry condition one might assume that $T_{\alpha j}$ and g_α are symmetric in the components of α, but in general there is no symmetry with respect to the last index j. Thus for $N=1$ the relativistic Navier-Stokes equations are included. And it is important to say that also for arbitrary N we do not prescribe a constitutive relation for T_β, we only assume that they are a solution of system (10.1). The form of this system is the only connection to section 2.

So far no special relativistic argument has occurred. But obviously the question arises, how the lower order momentum equations are contained in this presentation, are the $(N-1)^{\mathrm{th}}$-moments part of the N^{th}-moments as in the non-relativistic case? So we have to find a relativistic version of the reduction in 2.1. On the other hand what is clear is that the usual representation of the tensor

$$T_\beta=\varrho\underline{v}_{\beta_1}\cdots\underline{v}_{\beta_M}+\underline{\Pi}_\beta \tag{10.5}$$

with $M=N+1$ can be used also in the relativistic case. Here we choose the 4-velocity $\underline{v}$ as defined in 5.2 and ϱ as the mass density which is an objective scalar. Then the tensor T satisfies (10.3), if $\underline{\Pi}$ does it, because $\underline{v}$ is a contravariant vector, that is,

$$\underline{v}_i\circ Y=\sum_{\bar{i}=0}^{3}Y_{i\,\prime\bar{i}}\underline{v}^*_{\bar{i}}\,.$$

This implies

$$\Big(\varrho\prod_{i=1}^{M}\underline{v}_{k_i}\Big)\circ Y=\varrho\circ Y\prod_{i=1}^{M}\underline{v}_{k_i}\circ Y=\varrho^*\prod_{i=1}^{M}\Big(\sum_{\bar{k}_i=0}^{3}Y_{k_i}{}'_{\bar{k}_i}\underline{v}^*_{\bar{k}_i}\Big)$$
$$=\varrho^*\sum_{\bar{k}_1,\dots,\bar{k}_M=0}^{3}Y_{k_1}{}'_{\bar{k}_1}\cdots Y_{k_M}{}'_{\bar{k}_M}\underline{v}^*_{\bar{k}_1}\cdots\underline{v}^*_{\bar{k}_M}$$
$$=\sum_{\bar{k}_1,\dots,\bar{k}_M=0}^{3}Y_{k_1}{}'_{\bar{k}_1}\cdots Y_{k_M}{}'_{\bar{k}_M}\Big(\varrho^*\prod_{i=1}^{M}\underline{v}^*_{\bar{k}_i}\Big)$$

and gives (10.3) for T.

Reduction of the system

The system of N^{th}-moments (10.2) consists of 4^N differential equations (without taking symmetry of $T_{\alpha j}$ and g_α in $(\alpha_1,\dots,\alpha_N)$ into account) and it should contain the 4^{N-1} differential equations of the $(N-1)^{\text{th}}$-moments equation. We realize this the same way as we did for $N=1$ in section 6. There we considered the 4-moment system and we showed in 6.6 that it contains the mass equation. Here we present a generalization. We use as special test function

$$\zeta_{\alpha_1\cdots\alpha_N}:=\boldsymbol{e}_{\alpha_1}\eta_{\alpha_2\cdots\alpha_N}\tag{10.6}$$

where the vector $\boldsymbol{e}$ is the time vector from section 3. The function η is a covariant $(N-1)$-tensor.

10.1 Reduction lemma. The system of N^{th}-moments (10.2) contains as part the system of $(N-1)^{\text{th}}$-moments

$$\sum_{j\geq 0}\partial_{y_j}T_{\alpha j}=g_\alpha\qquad\text{for }\alpha\in\{0,\dots,3\}^{N-1}$$

with

$$T_{\alpha j}:=\sum_i\boldsymbol{e}_iT_{i\alpha j}\,,\qquad g_\alpha:=\sum_i\left(\boldsymbol{e}_i{}'_j\,T_{i\alpha j}+\boldsymbol{e}_i\,g_{i\alpha}\right).$$

Remark: The case $N=1$ is included by writing $\sum_{j\geq 0}\partial_{y_j}T_j=g$. Since $\sum_i\boldsymbol{e}_i\underline{v}_i=1$, this is consistent with (10.5), that is, (10.5) holds for all orders of moments.

Proof. We choose the test function as in (10.6). If η is a covariant $(N-1)$-tensor then since $\boldsymbol{e}$ is a covariant vector

$$\zeta^*_{\bar{\alpha}_1\cdots\bar{\alpha}_N}=\boldsymbol{e}^*_{\bar{\alpha}_1}\eta^*_{\bar{\alpha}_2\cdots\bar{\alpha}_N}$$
$$=\sum_{\alpha_1\geq 0}Y_{\alpha_1}{}'_{\bar{\alpha}_1}\boldsymbol{e}_{\alpha_1}\circ Y\cdot\sum_{\alpha_2,\dots,\alpha_N\geq 0}Y_{\alpha_2}{}'_{\bar{\alpha}_2}\cdots Y_{\alpha_N}{}'_{\bar{\alpha}_N}\eta_{\alpha_2\cdots\alpha_N}\circ Y$$
$$=\sum_{\alpha_1,\dots,\alpha_N\geq 0}Y_{\alpha_1}{}'_{\bar{\alpha}_1}\cdots Y_{\alpha_N}{}'_{\bar{\alpha}_N}\boldsymbol{e}_{\alpha_1}\circ Y\eta_{\alpha_2\cdots\alpha_N}\circ Y$$
$$=\sum_{\alpha_1,\dots,\alpha_N\geq 0}Y_{\alpha_1}{}'_{\bar{\alpha}_1}\cdots Y_{\alpha_N}{}'_{\bar{\alpha}_N}\zeta_{\alpha_1\cdots\alpha_N}\circ Y\,.$$

This means that ζ is an allowed test function and it follows from (10.1) writing $\alpha = (i, \gamma)$

$$\begin{aligned} 0 &= \int_{\mathbb{R}^4} \sum_{\alpha} \Big(\sum_j \partial_{y_j} \zeta_\alpha \cdot T_{\alpha j} + \zeta_\alpha \cdot g_\alpha \Big) \, \mathrm{dL}^4 \\ &= \int_{\mathbb{R}^4} \sum_{i\gamma} \Big(\sum_j \partial_{y_j} (\eta_\gamma \boldsymbol{e}_i) T_{i\gamma j} + \eta_\gamma \boldsymbol{e}_i \cdot g_{i\gamma} \Big) \, \mathrm{dL}^4 \\ &= \int_{\mathbb{R}^4} \sum_{\gamma} \Big(\sum_j \partial_{y_j} \eta_\gamma \cdot \underbrace{\sum_i \boldsymbol{e}_i T_{i\gamma j}}_{=:\, T_{\gamma j}} + \eta_\gamma \underbrace{\sum_i \Big(\partial_{y_j} \boldsymbol{e}_i T_{i\gamma j} + \boldsymbol{e}_i g_{i\gamma} \Big)}_{=:\, g_\gamma} \Big) \, \mathrm{dL}^4 \,. \end{aligned}$$

This gives the result. □

Coriolis coefficients
The transformation formula (10.4) gives rise to the following definition of the coefficients $C_\alpha = \big(C_\alpha^\beta \big)_{\beta \in \{0,\dots,3\}^{N+1}}$ (this is a generalization of 6.1)

$$g_\alpha = \underline{\mathbf{f}}_\alpha + \sum_{\beta \in \{0,\dots,3\}^{N+1}} C_\alpha^\beta T_\beta \quad \text{for } \alpha \in \{0, \dots, 3\}^N. \tag{10.7}$$

With these Coriolis coefficients the system (10.2) has the form

$$\sum_{j \geq 0} \partial_{y_j} T_{\alpha j} - \sum_{\beta \in \{0,\dots,3\}^{N+1}} C_\alpha^\beta T_\beta = \underline{\mathbf{f}}_\alpha \quad \text{for } \alpha \in \{0, \dots, 3\}^N \tag{10.8}$$

with transformation rule (10.3) for the tensor T, that is, T is a contravariant M-tensor ($M = N + 1$), and

$$\underline{\mathbf{f}}_{k_1 \cdots k_N} \circ Y = \sum_{\bar{k}_1, \dots, \bar{k}_N \geq 0} Y_{k_1}{}'_{\bar{k}_1} \cdots Y_{k_N}{}'_{\bar{k}_N} \, \underline{\mathbf{f}}^*_{\bar{k}_1 \cdots \bar{k}_N} \tag{10.9}$$

for the force (e.g. the Newton force, that is gravity, or the Lorentz force), that is, the entire force $\underline{\mathbf{f}} := (\underline{\mathbf{f}}_\alpha)_{\alpha \in \{0,\dots,3\}^N}$ is a contravariant N-tensor. (See the equation [9, Chap.2 (3.15)] for a comparison with the classical case.) The Coriolis coefficients satisfy the following transformation rule.

10.2 Rule for the Coriolis coefficients. The rule (10.4) for g is equivalent to the fact that $\underline{\mathbf{f}}$ is a contravariant N-tensor and

$$\begin{aligned} &\sum_{m_1, \dots, m_{N+1} \geq 0} Y_{m_1}{}'_{\bar{m}_1} \cdots Y_{m_{N+1}}{}'_{\bar{m}_{N+1}} C_{k_1 \cdots k_N}^{m_1 \cdots m_{N+1}} \circ Y \\ &= \sum_{\bar{k}_1, \dots, \bar{k}_N \geq 0} Y_{k_1}{}'_{\bar{k}_1} \cdots Y_{k_N}{}'_{\bar{k}_N} C^{*\bar{m}_1 \cdots \bar{m}_{N+1}}_{\bar{k}_1 \cdots \bar{k}_N} + \big(Y_{k_1}{}'_{\bar{m}_1} \cdots Y_{k_N}{}'_{\bar{m}_N} \big)_{,\bar{m}_{N+1}} \end{aligned}$$

for all $k_1, \dots, k_N$ and $\bar{m}_1, \dots, \bar{m}_{N+1}$.

Proof. We take the equation (10.4). Using the definition (10.7) and the above transformation rule (10.9) for $\underline{\mathbf{f}}$ this equation becomes for $k_1, \ldots, k_N \in \{0, \ldots, 3\}$

$$\begin{aligned}
&\Big(\sum_{m_1,\ldots,m_{N+1}\geq 0} C^{m_1\cdots m_{N+1}}_{k_1\cdots k_N} T_{m_1\cdots m_{N+1}} \Big) \circ Y \\
&= \sum_{\bar{k}_1,\ldots,\bar{k}_N,j\geq 0} \big(Y_{k_1\,'\bar{k}_1} \cdots Y_{k_N\,'\bar{k}_N} \big)_{\,'j} T^*_{\bar{k}_1\cdots\bar{k}_N j} \\
&+ \sum_{\bar{k}_1,\ldots,\bar{k}_N\geq 0} Y_{k_1\,'\bar{k}_1} \cdots Y_{k_N\,'\bar{k}_N} \sum_{\bar{m}_1,\ldots,\bar{m}_{N+1}\geq 0} C^{*\bar{m}_1\cdots\bar{m}_{N+1}}_{\bar{k}_1\cdots\bar{k}_N} T^*_{\bar{m}_1\cdots\bar{m}_{N+1}}
\end{aligned} \tag{10.10}$$

Using (10.3), that is

$$T_{m_1\cdots m_{N+1}} \circ Y = \sum_{\bar{m}_1,\ldots,\bar{m}_{N+1}\geq 0} Y_{m_1\,'\bar{m}_1} \cdots Y_{m_{N+1}\,'\bar{m}_{N+1}} T^*_{\bar{m}_1\cdots\bar{m}_{N+1}} \,,$$

the left-hand side of (10.10) becomes

$$\sum_{\substack{\bar{m}_1,\ldots,\bar{m}_{N+1}\geq 0,\\ m_1,\ldots,m_{N+1}\geq 0}} Y_{m_1\,'\bar{m}_1} \cdots Y_{m_{N+1}\,'\bar{m}_{N+1}} C^{m_1\cdots m_{N+1}}_{k_1\cdots k_N} \circ Y \, T^*_{\bar{m}_1\cdots\bar{m}_{N+1}} \,.$$

Now compare the coefficients of T^* with the one of the right-hand side of (10.10) and obtain

$$\begin{aligned}
&\sum_{m_1,\ldots,m_{N+1}\geq 0} Y_{m_1\,'\bar{m}_1} \cdots Y_{m_{N+1}\,'\bar{m}_{N+1}} C^{m_1\cdots m_{N+1}}_{k_1\cdots k_N} \circ Y \\
&= \big(Y_{k_1\,'\bar{m}_1} \cdots Y_{k_N\,'\bar{m}_N} \big)_{\,'\bar{m}_{N+1}} + \sum_{\bar{k}_1,\ldots,\bar{k}_N\geq 0} Y_{k_1\,'\bar{k}_1} \cdots Y_{k_N\,'\bar{k}_N} C^{*\bar{m}_1\cdots\bar{m}_{N+1}}_{\bar{k}_1\cdots\bar{k}_N} \,,
\end{aligned}$$

which is the assertion. □

11 Appendix: Divergence systems

We consider a spacetime domain $\mathcal{U} \subset \mathbb{R}^{n+1}$, $n = 3$, and in $\mathcal{U}$ integrable fluxes q^k and functions r^k, $k = 0, \ldots, M$, which solve the divergence system in $\mathcal{U}$

$$\sum_{i=0}^{n} \partial_{y_i} q_i^k = r^k \text{ for } k = 0, \ldots, M. \tag{11.1}$$

Further, we suppose that an invertible matrix $Z = Z(y^*)$,

$$Z(y^*) = (Z_{kl}(y^*))_{k,l=0,\ldots,M} \,, \tag{11.2}$$

is given. We consider the following transformation rule for observer transformations $y = Y(y^*)$

$$\begin{aligned}
&q_i^k \circ Y = \tfrac{1}{J} \textstyle\sum_{jl} Y_{i\,'j} Z_{kl} q_j^{*l} \,, \quad J := \det \mathrm{D}_{y^*} Y > 0 \,, \\
&r^k \circ Y = \tfrac{1}{J} \Big(\textstyle\sum_{jl} Z_{kl\,'j} q_j^{*l} + \sum_l Z_{kl} r^{*l} \Big) \,, \\
&\text{for all } i = 0, \ldots, n \text{ and } k = 0, \ldots, M \,, \\
&\text{where } j \text{ runs from 0 to } n, \text{ and } l \text{ from 0 to } M.
\end{aligned} \tag{11.3}$$

In [2, Section I.5] it has been proved that the system (11.1) is invariant under observer transformations if (11.3) is satisfied:

11.1 Theorem. If the quantities q^k, r^k, $k = 0,\ldots,M$, satisfy the transformation rule (11.3) for a matrix Z as in (11.2), then with $\mathcal{U} = Y(\mathcal{U}^*)$

$$\begin{aligned}&\sum_{l=0}^{M}\int_{\mathcal{U}^*}\Big(\sum_{j=0}^{n}\partial_{y_j^*}\zeta_l^*\,q_j^{*l} + \zeta_l^* r^{*l}\Big)\,\mathrm{dL}^{n+1}\\ &= \sum_{k=0}^{M}\int_{\mathcal{U}}\Big(\sum_{i=0}^{n}\partial_{y_i}\zeta_k\,q_i^k + \zeta_k r^k\Big)\,\mathrm{dL}^{n+1}\end{aligned} \tag{11.4}$$

where the test functions satisfy

$$\zeta^* = Z^{\mathrm{T}}\,\zeta\circ Y\,. \tag{11.5}$$

In the case $Z = \underline{\mathrm{D}}Y$ this theorem is used in this paper for the relativistic theory special for the 4-momentum system. In this case the condition (11.3) on the fluxes are for $i,k = 0,\ldots,n$

$$q_i^k\circ Y = \frac{1}{J}\sum_{j,l=0}^{n} Y_{i\,'j}Y_{k\,'l}q_j^{*l}\,,$$

and the test function ζ is a covariant vector. For the hierarchical theory the matrix Z is $Z = \big(Z_{(i_1,\ldots,i_N)(\bar{i}_1,\ldots,\bar{i}_N)}\big)_{i_1,\ldots,i_N,\bar{i}_1,\ldots,\bar{i}_N=0,\ldots,n}$ with

$$Z_{(i_1,\ldots,i_N)(\bar{i}_1,\ldots,\bar{i}_N)} = Y_{i_1\,'\bar{i}_1}\cdots Y_{i_N\,'\bar{i}_N}\,.$$

In this case the property (11.5) says that the test function is a covariant N-tensor. In the special case $Z = \mathrm{Id}$ this theorem can be used for the introduction to elasticity theory, see e.g. [2, Section I.6].

12 Appendix: Theorem on Lorentz matrix

The following is a well known theorem.

12.1 Theorem. The following sets of matrices are the same.

(1) The set of all matrices M satisfying

$$\mathrm{G_c} = M\mathrm{G_c}M^{\mathrm{T}}$$

with the normalization, that $M_{00} \geq 0$ and $\det M > 0$.

(2) The set of all matrices

$$M = \mathbf{L}_{\mathrm{c}}(V,Q)$$

with $V \in \mathbb{R}^3$, $|V| < \mathrm{c}$, and Q an orthonormal matrix with determinant 1.

The Lorentz matrices $\mathbf{L}_{\mathrm{c}}(V,Q)$ are given by

$$\mathbf{L}_{\mathrm{c}}(V,Q) = \begin{bmatrix}\gamma & \dfrac{\gamma}{\mathrm{c}^2}V^{\mathrm{T}}Q\\ \gamma V & \mathbf{B}_{\mathrm{c}}(V)Q\end{bmatrix},$$

where $\mathbf{B}_{\mathrm{c}}(V) := \mathrm{Id} + \frac{\gamma^2}{\mathrm{c}^2(\gamma+1)}VV^{\mathrm{T}}$ and $\gamma = \big(1 - \frac{|V|^2}{\mathrm{c}^2}\big)^{-\frac{1}{2}}$ for $|V| < \mathrm{c}$.

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