

ON (p, q) – EQUATIONS WITH CONCAVE TERMS

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Abstract. We consider a (p, q) – equation ($1 < q < p$, $p \geq 2$) with a parametric concave term and a $(p - 1)$ – linear perturbation. We show that the problem has five nontrivial smooth solutions: four of constant sign and the fifth nodal. When $q = 2$ (i.e., $(p, 2)$ equation) we show that the problem has six nontrivial smooth solutions, but we do not specify the sign of the sixth solution. Our approach uses variational methods, together with truncation and comparison techniques and Morse theory.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 boundary $\partial\Omega$. In this paper we study the following nonlinear Dirichlet problem:

$$-\Delta_p u(z) - \mu \Delta_q u(z) = \lambda |u(z)|^{\tau-2} u(z) + f(z, u(z)) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (P_\lambda)$$

Here $1 < q < p < \infty$, $p \geq 2$, $1 < \tau \leq q$, $\mu \geq 0$, $\lambda > 0$ is a parameter and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory perturbation (i.e., for all $x \in \mathbb{R}$, $z \rightarrow f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z, x)$ is continuous). For every $r \in (1, +\infty)$ by Δ_r we denote the r -Laplace differential operator defined by

$$\Delta_r u(z) = \operatorname{div} (\|Du(z)\|^{r-2} Du(z)) \text{ for all } u \in W_0^{1,r}(\Omega).$$

The aim of this work is to prove a multiplicity theorem for problem (P_λ) when the parameter $\lambda > 0$ belongs to an interval $(0, \lambda^*)$ (small values of the parameter) and when the "concave" $((p-1)$ -sublinear) term $\lambda |x|^{\tau-2} x$ is perturbed by an asymptotically at $\pm\infty$, $(p-1)$ -linear term $f(z, \cdot)$. The multiplicity theorem provides precise sign information for all the solutions.

In the past problems with reaction involving "concave" terms, were studied in the context of equations driven by the Laplacian (i.e., $p = 2$ and $\mu = 0$), see Ambrosetti-Brezis-Cerami [5], de Paiva-Massa [13], Li-Wu-Zhou [28], Perera [33], Wu-Yang [39]. Extensions to equations driven by the p -Laplacian, can be found in the works of Garcia Azorero-Manfredi-Peral Alonso [18], Guo-Zhang [21], Filippakis-Kristaly-Papageorgiou [17], Hu-Papageorgiou [23]. With the exception of [17], the other works do not provide sign information for all the solutions. In [17] the concave term is perturbed by a $(p-1)$ -superlinear nonlinearity (equation with concave and convex nonlinearities). We also mention the recent work of Aizicovici-Papageorgiou-Staicu [3], on periodic equations driven by the scalar p -Laplacian.

The (p, q) -differential operator $u \rightarrow -\Delta_p u - \mu \Delta_q u$, $u \in W_0^{1,p}(\Omega)$, is an important operator occurring in quantum physics (see, for example, Benci-Fortunato-Pisani [8]). Recently, (p, q) -equations were studied by Cingolani-Degiovanni [12], Figueiredo [16], Li-Zhang [27], Medeiros-Perera [31] and Sun [37]. None of the aforementioned works treats equations with concave terms, they do not examine the regularity of the solutions and those that prove multiplicity theorems, do not provide sign information for them.

We stress that the (p, q) -differential operator is not homogeneous. This creates serious technical difficulties and the techniques used in the context of p -Laplacian equations fail (see, for example, Filippakis-Kristaly-Papageorgiou [17]).

Our approach is variational, based on the critical point theory. The variational methods are coupled with suitable truncation and comparison techniques and with the use of Morse theory (critical groups).

In the next section, for easy reference, we review the main mathematical tools which we will use in the sequel.

2 Mathematical background

We start with the critical point theory. So, let $(X, \|\cdot\|)$ be a Banach space and let $(X^*, \|\cdot\|_*)$ be its dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X) . Let $\varphi \in C^1(X)$. A number $c \in \mathbb{R}$ is said to be a critical value of φ if there exists $x^* \in X$ such that $\varphi'(x^*) = 0$ and $\varphi(x^*) = c$.

We say that φ satisfies the *Palais-Smale condition* (*PS-condition*, for short), if the following is true:

”every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(x_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow \infty$$

admits a strongly convergent subsequence.”

Using this compactness-type condition, we can have the following minimax theorem, known in the literature as the ”mountain pass theorem”.

Theorem 1 *If $\varphi \in C^1(X)$ satisfies the PS- condition, $x_0, x_1 \in X$ and $\rho > 0$ are such that $\|x_1 - x_0\| > \rho$, $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} =: \eta_\rho$, and $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, then $c \geq \eta_\rho$ and c is a critical value of φ .*

The following notion from the theory of nonlinear operators of monotone type will help us verify the *PS* condition. Here and in the sequel, \xrightarrow{w} designates weak convergence in X .

Definition 1 *A map $A : X \rightarrow X^*$ is said to be of type $(S)_+$, if for every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and*

$$\limsup_{n \rightarrow \infty} \langle A(x_n), x_n - x \rangle \leq 0,$$

one has

$$x_n \rightarrow x \text{ in } X \text{ as } n \rightarrow \infty.$$

Throughout this work, by $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$, i.e.,

$$\|u\| = \|Du\|_p$$

(by virtue of the Poincaré inequality), where $\|\cdot\|_p$ stands for the norm in $L^p(\Omega)$ or $L^p(\Omega, \mathbb{R}^N)$. We mention that the notation $\|\cdot\|$ will be also used to denote the \mathbb{R}^N -norm. It will always be clear from the context, which norm we use.

Also, for $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and for every $u \in W_0^{1,p}(\Omega)$ we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in H_0^1(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^- \text{ (see [19]).}$$

By $|\cdot|_N$ we will denote the Lebesgue measure on \mathbb{R}^N . If $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function, then the corresponding Nemytskii map, N_h , is defined by

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W_0^{1,p}(\Omega).$$

In the analysis of problem (P_λ) , in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega \right\},$$

where by $n(\cdot)$ we denote the outward unit normal on $\partial\Omega$.

We will also use some basic facts about the spectrum of the negative p -Laplacian with Dirichlet boundary conditions, henceforth denoted by $-\Delta_p^D$. So, let $m \in L^\infty(\Omega)_+ := \{h \in L^\infty(\Omega) : h(z) \geq 0 \text{ for a.a. } z \in \Omega\}$, $m \neq 0$ and consider the following nonlinear weighted eigenvalue problem

$$-\Delta_p u(z) = \widehat{\lambda} m(z) |u(z)|^{p-2} u(z) \text{ in } \Omega, u|_{\partial\Omega} = 0 \text{ (} 1 < p < \infty \text{)}. \quad (2.1)$$

In the sequel we will refer to the eigenvalue problem (2.1) by $(-\Delta_p^D, m)$.

By an eigenvalue of $-\Delta_p^D$, we mean a number $\widehat{\lambda}(p, m) \in \mathbb{R}$ such that (2.1) has a nontrivial solution \widehat{u} . Nonlinear regularity theory (see for example, Gasinski-Papageorgiou [19], pp. 737-738) implies that $\widehat{u} \in C_0^1(\overline{\Omega})$. The least $\widehat{\lambda} \in \mathbb{R}$ for which (2.1) has a nontrivial solution, is the first eigenvalue of $-\Delta_p^D$ and is denoted by $\widehat{\lambda}_1(p, m)$. We recall the following properties of $\widehat{\lambda}_1(p, m)$:

- $\widehat{\lambda}_1(p, m) > 0$;
- $\widehat{\lambda}_1(p, m)$ is isolated, i.e., we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1(p, m), \widehat{\lambda}_1(p, m) + \varepsilon)$ contains no eigenvalues;
- $\widehat{\lambda}_1(p, m)$ is simple, i.e., if \widehat{u}, \widehat{v} are two eigenfunctions corresponding to $\widehat{\lambda}_1(p, m)$, then $\widehat{u} = c\widehat{v}$, with $c \in \mathbb{R}$.

We also have the following variational characterization of $\widehat{\lambda}_1(p, m) > 0$:

$$\widehat{\lambda}_1(p, m) = \inf \left\{ \frac{\|Du\|_p^p}{\int_\Omega m |u|^p dz} : u \in W_0^{1,p}(\Omega), u \neq 0 \right\}. \quad (2.2)$$

The infimum in (2.2) is attained on the one dimensional eigenspace of $\widehat{\lambda}_1(p, m)$. If $m \equiv 1$, then we set $\widehat{\lambda}_1(p) := \widehat{\lambda}_1(p, 1)$. Let $\widehat{u}_{1,p}$ be the L^p -normalized (i.e., $\|\widehat{u}_{1,p}\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(p) > 0$. It is clear from (2.2) that $\widehat{u}_{1,p}$ does not change sign. Hence we may assume that $\widehat{u}_{1,p} \in C_+$, and in fact the nonlinear strong maximum principle of Vazquez [38] implies that $\widehat{u}_{1,p} \in \text{int } C_+$.

It is easy to see that the set $\widehat{\sigma}(p, m)$ of eigenvalues of $(-\Delta_p^D, m)$ is closed. This and the fact that $\widehat{\lambda}_1(p, m) > 0$ is isolated, imply that the second eigenvalue

$$\widehat{\lambda}_2(p, m) := \inf \left\{ \widehat{\lambda} \in \widehat{\sigma}(p, m) : \widehat{\lambda} > \widehat{\lambda}_1(p, m) \right\}$$

is also well defined.

If $N = 1$ (ordinary differential equations), then $\widehat{\sigma}(p, m)$ is a sequence $\left\{ \widehat{\lambda}_k(p, m) \right\}_{k \geq 1} \subset (0, \infty)$ of simple eigenvalues of $-\Delta_p^D$ such that $\widehat{\lambda}_k(p, m) \rightarrow \infty$ as $k \rightarrow \infty$, and the corresponding eigenfunctions $\{\widehat{u}_{k,p}\}_{k \geq 1}$ have exactly $(k-1)$ -zeros (see Gasinski-Papageorgiou [19], p. 761).

If $N \geq 2$ (partial differential equations), then using the *Ljusternik-Schnirelmann* min-max scheme, we obtain an increasing sequence $\left\{ \widehat{\lambda}_n(p, m) \right\}_{n \geq 1}$ of eigenvalues of $-\Delta_p^D$. If $p = 2$ (linear eigenvalue problem) then these are all the eigenvalues. If $p \neq 2$ then we do not know if this is the case.

Viewed as functions of the weight $m \in L^\infty(\Omega)_+$, the eigenvalues $\widehat{\lambda}_1(p, m)$ and $\widehat{\lambda}_2(p, m)$, exhibit certain monotonicities properties. More precisely, we have:

Proposition 1 (a) *If $m, m' \in L^\infty(\Omega)_+ \setminus \{0\}$, $m(z) \leq m'(z)$ a.e. in Ω and $m \neq m'$, then $\widehat{\lambda}_1(p, m') < \widehat{\lambda}_1(p, m)$.*

(b) *If $m, m' \in L^\infty(\Omega)_+ \setminus \{0\}$, $m(z) < m'(z)$ a.e. in Ω then $\widehat{\lambda}_2(p, m') < \widehat{\lambda}_2(p, m)$.*

(c) *If $\theta \in L^\infty(\Omega)_+$, $\theta(z) \leq \widehat{\lambda}_1(p)$ a.e. in Ω , $\theta \neq \widehat{\lambda}_1(p)$, then there exists $\xi_0 > 0$ such that*

$$\|Du\|_p^p - \int_{\Omega} \theta |u|^p dz \geq \xi_0 \|Du\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

Next, from Morse theory let us recall the definition of critical groups. So, let X be a Banach space and $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets

$$\varphi^c = \{x \in X : \varphi(x) \leq c\}, \quad K_\varphi = \{x \in X : \varphi'(x) = 0\}, \quad K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subset Y_1 \subset X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th} -singular homology group of the pair (Y_1, Y_2) with integer coefficients. The *critical groups of φ* at an isolated $x_0 \in K_\varphi^c$ are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\}), \text{ for all } k \geq 0,$$

where U is a neighborhood of x_0 such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of the singular homology implies that this definition is independent of the particular choice of the neighborhood U .

Suppose that $\varphi \in C^1(X)$ satisfies the *PS*-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. Then, the *critical groups of φ at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0.$$

The second deformation theorem (see, for example, Gasinski-Papageorgiou [19], p. 628) implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. If for some integer $k \geq 0$, $C_k(\varphi, \infty) \neq 0$, then there exists $x \in K_\varphi$ such that $C_k(\varphi, x) \neq 0$.

In the analysis of problem (P_λ) we will need some auxiliary results, which are actually of independent interest and for this reason we state them in greater generality than we will need them. So, we consider a differential operator $\operatorname{div} a(Du)$, with $a(\cdot) : \mathbb{R}^N \rightarrow \mathbb{R}$ being a map satisfying the following hypotheses:

H(a): $a(y) = a_0(\|y\|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, $a_0 \in C^1(0, \infty)$, $\lim_{t \rightarrow 0^+} a_0(t) = 0$ such that

(i) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{g(\|y\|)}{\|y\|} \|\xi\|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$, where $g \in C^1(0, \infty)$ is such that

$$0 < \widehat{C} \leq \frac{tg'(t)}{g(t)} \leq C_0$$

(ii) $\|\nabla a(y)\| \leq C_1 \frac{g(\|y\|)}{\|y\|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$ and some $C_1 > 0$.

Example: Consider the map $a(y) = \|y\|^{p-2}y + \mu \|y\|^{q-2}y$ for all $y \in \mathbb{R}^N$, with $\mu > 0$, $1 < q \leq p < \infty$. We will show that $a(\cdot)$ (which corresponds to the (p, q) operator) satisfies hypotheses **H**(a). Note that $a(y) = a_0(\|y\|)y$ with $a_0(t) = t^{p-2} + \mu t^{q-2}$, $t > 0$, hence $a_0 \in C^1(0, \infty)$. When $2 \leq q \leq p < \infty$, we have

$$\begin{aligned} \nabla a(y) &= \|y\|^{p-2} \left(I + (p-2) \frac{y \otimes y}{\|y\|^2} \right) + \mu \|y\|^{q-2} \left(I + (q-2) \frac{y \otimes y}{\|y\|^2} \right) \\ &\text{for all } y \in \mathbb{R}^N \setminus \{0\}. \end{aligned}$$

Then

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq (\|y\|^{p-2} + \mu \|y\|^{q-2}) \|\xi\|^2 \text{ for all } \xi \in \mathbb{R}^N$$

and

$$\begin{aligned} \|\nabla a(y)\| &\leq (p-1) \|y\|^{p-2} + (q-1) \mu \|y\|^{q-2} \\ &\leq (p-1) (\|y\|^{p-2} + \mu \|y\|^{q-2}) \text{ (since } q \leq p). \end{aligned}$$

So, if we set $g(t) = t^{p-1} + \mu t^{q-1}$, $t > 0$, then hypotheses **H**(a) are satisfied.

Similarly, if $1 < q < 2 \leq p < \infty$, then $g(t) = t^{p-1} + \mu(q-1)t^{q-1}$, $t > 0$.

Now let $G_0(t) = \int_0^t a_0(s) ds$ for all $t > 0$. Evidently, $G_0(\cdot)$ is strictly convex and increasing. We set $G(y) = G_0(\|y\|)$. Then $G(0) = 0$ and for all $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$\nabla G(y) = G'_0(\|y\|) \frac{y}{\|y\|} = a_0(\|y\|) \|y\| \frac{y}{\|y\|} = a_0(\|y\|) y = a(y).$$

Also, let $f_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z, x)| \leq \widehat{a}(z) + \widehat{c}|x|^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $\widehat{a} \in L^\infty(\Omega)_+$, $\widehat{c} > 0$ and $1 < r < p^*$, where

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N. \end{cases}$$

We set $F_0(t, x) = \int_0^x f_0(z, s) ds$ and consider the C^1 -functional $\varphi_0 : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \int_{\Omega} G(Du(z)) dz - \int_{\Omega} F_0(z, u(z)) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

The next result relates $C_0^1(\overline{\Omega})$ and $W_0^{1,p}(\Omega)$ local minimizers of φ_0 . Such a result was first proved by Brézis-Nirenberg [10], when $G(y) = \frac{1}{2}\|y\|^2$ (this corresponds to the Laplace differential operator) and generalized by Garcia Azorero-Manfredi-Peral Alonso [18] to the case $G(y) = \frac{1}{p}\|y\|^p$, $1 < p < \infty$ (this corresponds to the p -Laplace differential operator). Recently, Aizicovici-Papageorgiou-Staicu [4], extended the result to more general functions $G(\cdot)$, which correspond to nonhomogeneous differential operators. Their proof remains valid in the present setting, using this time the regularity result of Lieberman [29]. So we can state the following proposition:

Proposition 2 *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$ -minimizer of φ_0 (i.e., there exists $\rho_0 > 0$ such that $\varphi_0(u_0) \leq \varphi_0(u_0 + h)$ for all $h \in C_0^1(\overline{\Omega})$ with $\|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0$) then $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ with $\beta \in (0, 1)$ and it is a $W_0^{1,p}(\Omega)$ -minimizer of φ_0 (i.e., there exists $\rho_1 > 0$ such that $\varphi_0(u_0) \leq \varphi_0(u_0 + h)$ for all $h \in W_0^{1,p}(\Omega)$ with $\|h\|_{W_0^{1,p}(\Omega)} \leq \rho_1$).*

Next we consider the following auxiliary Dirichlet problem

$$-\operatorname{div} a(Du(z)) = \widehat{f}(z, u(z)) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (2.3)$$

The hypotheses on $\widehat{f}(z, x)$ are the following:

H (\widehat{f}): $\widehat{f} : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory function such that $\widehat{f}(z, 0) = 0$ a.e. in Ω and

- (i) $\left| \widehat{f}(z, x) \right| \leq \alpha(z) + C |z|^{r-1}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)_+$, $C > 0$ and $1 < r < p^*$;
- (ii) for a.a. $z \in \Omega$, $x \mapsto \frac{\widehat{f}(z, x)x}{|x|^q}$ is strictly decreasing on $\mathbb{R} \setminus \{0\}$;
- (iii) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that $\widehat{f}(z, x)x + \xi_\rho |x|^p \geq 0$ for a.a. $z \in \Omega$, all $|x| \leq \rho$.

Also, we strengthen hypotheses $\mathbf{H}(a)$ as follows:

$\mathbf{H}'(a)$: Hypotheses $\mathbf{H}(a)$ hold and for some $q \in (1, p]$ we have $t \rightarrow G_0\left(t^{\frac{1}{q}}\right)$ is convex on $(0, +\infty)$.

Remark: The (p, q) -differential operator satisfies this condition.

Proposition 3 *If hypotheses $\mathbf{H}'(a)$ and $\mathbf{H}(\widehat{f})$ hold, then problem (2.3) has at most one nontrivial positive solution in $\text{int } C_+$ and at most one nontrivial negative solution in $-\text{int } C_+$.*

Proof. We show the uniqueness of the nontrivial positive solution of (2.3), if it exists. The proof for the nontrivial negative solution being similar.

So, let $u \in W_0^{1,p}(\Omega)$ be a nontrivial positive solution of (2.3). Then

$$-\text{div } a(Du(z)) = \widehat{f}(z, u(z)) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Invoking Theorem 7.1 of Ladyzhenskaya-Uraltseva [25], we have $u \in L^\infty(\Omega)$. Then the regularity result of Lieberman ([29], p. 320) implies that $u \in C_+ \setminus \{0\}$.

Let $\rho = \|u\|_\infty$ and let ξ_ρ be as postulated by hypothesis $\mathbf{H}(\widehat{f})$ (iii). We have

$$-\text{div } a(Du(z)) + \xi_\rho u(z)^{p-1} = \widehat{f}(z, u(z)) + \xi_\rho u(z)^{p-1} \geq 0 \text{ a.e. in } \Omega$$

(see hypothesis $\mathbf{H}(\widehat{f})$ (iii)), hence

$$\text{div } a(Du(z)) \leq \xi_\rho u(z)^{p-1} \text{ a.e. in } \Omega.$$

By virtue of Theorem 5.4.1 of Pucci-Serrin ([36], p. 111), we have $u(z) > 0$ for all $z \in \Omega$. Finally we can apply Theorem 5.5.1 of Pucci-Serrin ([36], p. 120) and conclude that $u \in \text{int } C_+$.

Next we show the uniqueness of this solution. To this end, we consider the integral functional $\eta_+ : L^1(\Omega) \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ defined by

$$\eta_+(u) = \begin{cases} \int_{\Omega} G\left(Du^{\frac{1}{q}}\right) dz & \text{if } u \geq 0, u^{\frac{1}{q}} \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \text{dom } \eta_+$ and set

$$y = (tu_1 + (1-t)u_2)^{\frac{1}{q}} \text{ with } t \in [0, 1] \text{ and } v_1 = u_1^{\frac{1}{q}}, v_2 = u_2^{\frac{1}{q}}.$$

As in the proof of Lemma 1 of Diaz-Saa [14] (see also Lemma 4 of Benguria-Brezis-Lieb [9]), using Holder's inequality, we have

$$\|Dy(z)\| \leq (t\|Dv_1(z)\|^q + (1-t)\|Dv_2(z)\|^q)^{\frac{1}{q}} \text{ a.e. in } \Omega.$$

Since G_0 is increasing, we have

$$\begin{aligned} G_0(Dy(z)) &\leq G_0\left((t\|Dv_1(z)\|^q + (1-t)\|Dv_2(z)\|^q)^{\frac{1}{q}}\right) \\ &\leq tG_0(\|Dv_1(z)\|) + (1-t)G_0(\|Dv_2(z)\|) \end{aligned}$$

(see hypotheses $\mathbf{H}'(a)$). Recall that $G(y) = G_0(\|y\|)$. Hence

$$G(\|Dy(z)\|) \leq tG\left(Du_1^{\frac{1}{q}}(z)\right) + (1-t)G\left(Du_2^{\frac{1}{q}}(z)\right) \text{ a.e. in } \Omega,$$

therefore η_+ is convex. Also, an easy application of Fatou's lemma shows that η_+ is lower semicontinuous and of course η_+ is not identically $+\infty$ (i.e., $\eta_+ \in \Gamma_0(L^1(\Omega))$), see Gasinski-Papageorgiou [19], p. 488).

Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial positive solution of (2.3). From the first part of the proof, we have $u \in \text{int } C_+$. Then $u^q \geq 0$ and $(u^q)^{\frac{1}{q}} = u \in W_0^{1,p}(\Omega)$, hence $u^q \in \text{dom } \eta_+$. Let $h \in C_0^1(\Omega)$ and $r > 0$ small. We have $u^q + rh \in \text{int } C_+$ and so, the Gateaux derivatives of η_+ at u^q in the direction h exists. Moreover, using the chain rule we have

$$\eta'_+(u^q)(h) = \frac{1}{q} \int_{\Omega} \frac{-\text{div } a(Du)}{u^{q-1}} dz. \quad (2.4)$$

If $v \in W_0^{1,p}(\Omega)$ is another nontrivial positive solution of (2.3), then again we have $v \in \text{int } C_+$ and

$$\eta'_+(v^q)(h) = \frac{1}{q} \int_{\Omega} \frac{-\text{div } a(Dv)}{v^{q-1}} dz. \quad (2.5)$$

The convexity of η_+ implies that $y \rightarrow \eta'_+(y)$ is monotone. Hence

$$\begin{aligned} 0 &\leq \frac{1}{q} \int_{\Omega} \left[\frac{-\text{div } a(Du)}{u^{q-1}} + \frac{\text{div } a(Dv)}{v^{q-1}} \right] (u-v) dz \text{ (see (2.4), (2.5))} \\ &= \frac{1}{q} \int_{\Omega} \left[\frac{\widehat{f}(z, u)}{u^{q-1}} - \frac{\widehat{f}(z, v)}{v^{q-1}} \right] (u-v) dz \text{ (see } \mathbf{H}(\widehat{f})(ii)). \end{aligned}$$

The strict monotonicity of $x \mapsto \frac{\widehat{f}(z, x)}{x^{q-1}}$ on $(0, \infty)$ implies that $u = v$. Therefore, the nontrivial positive solution of (2.3) when it exists, it is unique and belongs to $\text{int } C_+$. Similarly for the nontrivial negative solution. \square

Let $h, \widehat{h} \in L^\infty(\Omega)$. We write $h \prec \widehat{h}$ if for every compact set $K \subseteq \Omega$, we can find $\varepsilon > 0$ such that

$$h(z) + \varepsilon \leq \widehat{h}(z) \text{ for a.a. } z \in K.$$

Clearly, if $h, \widehat{h} \in C(\Omega)$ and $h(z) < \widehat{h}(z)$ for all $z \in \Omega$, then $h \prec \widehat{h}$.

A straightforward modification of the proof of Proposition 2.6 of Arcoya-Ruiz [6] (see also [2]) in order to accommodate the extra linear term $-\Delta u$, produces the following useful strong comparison principle.

Proposition 4 *If $\xi \geq 0$, $h, \widehat{h} \in L^\infty(\Omega)$, $h \prec \widehat{h}$, $u, v \in C_0^1(\overline{\Omega})$ are solutions of*

$$\begin{aligned} -\Delta_p u(z) - \mu \Delta u(z) + \xi |u(z)|^{p-2} u(z) &= h(z) \text{ in } \Omega \\ -\Delta_p v(z) - \mu \Delta v(z) + \xi |v(z)|^{p-2} v(z) &= \widehat{h}(z) \text{ in } \Omega \end{aligned}$$

with $\mu \geq 0$ and $v \in \text{int } C_+$, then $v - u \in \text{int } C_+$

For $r \in (1, +\infty)$, let $A_r : W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* \left(\frac{1}{r} + \frac{1}{r'} = 1\right)$ be the nonlinear map defined by

$$\langle A_r(u), y \rangle = \int_{\Omega} \|Du\|^{r-2} (Du, Dy)_{\mathbb{R}^N} \text{ for all } u, y \in W_0^{1,r}(\Omega). \quad (2.6)$$

If $r = 2$, then we set $A := A_2 \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

The following result is well known and can be found, for example, in Gasinski-Papageorgiou ([19], pp. 745-746)

Proposition 5 *The nonlinear map $A_r : W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega)$ defined by (2.6) is bounded, continuous, strictly monotone (strongly monotone if $r \geq 2$), hence it is maximal monotone and of type of type $(S)_+$.*

Finally, given $u, v \in W_0^{1,p}(\Omega)$ with $u \leq v$, we define

$$[u, v] = \{y \in W_0^{1,p}(\Omega) : u(z) \leq y(z) \leq v(z) \text{ for a.a. } z \in \Omega\}.$$

3 Solutions of constant sign

In this section we look for nontrivial constant sign smooth solutions. To this end, we introduce the following hypotheses on the perturbation $f(z, x)$:

The hypotheses on the nonlinearity $f(z, x)$ are the following:

H(f)₁: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ a.e. in Ω and:

(i) for every $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leq a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) there exist $\eta_1, \eta_2 \in L^\infty(\Omega)_+$ such that

$$\begin{aligned} \eta_1(z) &\geq \widehat{\lambda}_1(p) \text{ a.e. in } \Omega, \quad \eta_1 \neq \widehat{\lambda}_1(p), \quad \eta_2(z) < \widehat{\lambda}_2(p) \text{ a.e. in } \Omega \\ \eta_1(z) &\leq \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \eta_2(z) \\ &\text{uniformly for a.a. } z \in \Omega; \end{aligned}$$

(iii) if $F(z, x) = \int_0^x f(z, s) ds$, then there exists $\theta \in L^\infty(\Omega)_+$ such that $\theta(z) \leq \widehat{\lambda}_1(p)$ a.e. in Ω , $\theta \neq \widehat{\lambda}_1(p)$, and

$$\limsup_{x \rightarrow 0} \frac{pF(z, x)}{|x|^p} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega$$

(iv) for every $\rho > 0$ there exists $\xi_\rho > 0$ such that

$$f(z, x)x + \xi_\rho |x|^p \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho.$$

We introduce the following truncations of the reaction in problem (P_λ) :

$$g_+^\lambda(z, x) = \lambda(x^+)^{\tau-1} + f(z, x^+) \text{ and } g_-^\lambda(z, x) = -\lambda(x^-)^{\tau-1} + f(z, -x^-).$$

Both are Carathéodory functions. We set $G_\pm^\lambda(z, x) = \int_0^x g_\pm^\lambda(z, s) ds$ and consider the C^1 -functionals $\varphi_\pm^\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_\pm^\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{\mu}{q} \|Du\|_q^q - \int_\Omega G_\pm^\lambda(z, u(z)) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

In addition $\varphi_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ is the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{\mu}{q} \|Du\|_q^q - \frac{\lambda}{\tau} \|u\|_\tau^\tau - \int_\Omega F(z, u(z)) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$.

Proposition 6 *If hypotheses $\mathbf{H}(f)_1$ hold and $\lambda > 0$, then the functional $\varphi_\pm^\lambda, \varphi_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ satisfy the PS-condition.*

Proof. We do the proof for φ_+^λ , the proofs for φ_-^λ and φ_λ being similar. So, let $\{u_n\}_{n \geq 1} \subset W_0^{1,p}(\Omega)$ be a sequence such that $\{\varphi_+^\lambda(u_n)\}_{n \geq 1} \subset \mathbb{R}$ is bounded

$$(\varphi_+^\lambda)'(u_n) \rightarrow 0 \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \text{ as } n \rightarrow \infty.$$

Denoting by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair $(W^{-1,p'}(\Omega), W_0^{1,p}(\Omega))$, we have

$$\left| \langle A_p(u_n), v \rangle + \mu \langle A_q(u_n), v \rangle - \int_{\Omega} g_+^{\lambda}(z, u_n) v dz \right| \leq \varepsilon_n \|v\| \quad (3.1)$$

for all $v \in W_0^{1,p}(\Omega)$

with $\varepsilon_n \rightarrow 0^+$ (recall that $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$, hence $W^{-1,q'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$). In (3.1) we choose $v = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_n^-\|_p^p + \mu \|Du_n^-\|_q^q \leq \varepsilon_n \|u_n^-\| \quad \text{for all } n \geq 1,$$

hence

$$u_n^- \rightarrow 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \rightarrow \infty. \quad (3.2)$$

Suppose that $\|u_n^+\| \rightarrow \infty$. We set

$$y_n = \frac{u_n^+}{\|u_n^+\|}, n \geq 1.$$

Then $y_n \geq 0$, $\|y_n\| = 1$ for all $n \geq 1$. By passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^p(\Omega) \text{ as } n \rightarrow \infty, y \geq 0. \quad (3.3)$$

From (3.1) we have

$$\left| \langle A_p(y_n), v \rangle + \left\langle \frac{A_p(-u_n^-)}{\|u_n^+\|^{p-1}}, v \right\rangle + \frac{\mu}{\|u_n^+\|^{p-q}} \langle A_q(y_n), v \rangle + \mu \left\langle \frac{A_q(-u_n^-)}{\|u_n^+\|^{p-1}}, v \right\rangle \right. \\ \left. - \int_{\Omega} \frac{g_+^{\lambda}(z, u_n^+)}{\|u_n^+\|^{p-1}} v dz \right| \leq \varepsilon_n \frac{\|v\|}{\|u_n^+\|^{p-1}} \text{ for all } n \geq 1. \quad (3.4)$$

From (3.2) it follows that

$$\frac{A_p(-u_n^-)}{\|u_n^+\|^{p-1}} \text{ and } \frac{A_q(-u_n^-)}{\|u_n^+\|^{p-1}} \rightarrow 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \rightarrow \infty. \quad (3.5)$$

Also, since $q < p$ and $\{A_q(y_n)\}_{n \geq 1} \subset W^{-1,q'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ is bounded, we have

$$\frac{A_q(y_n)}{\|u_n^+\|^{p-q}} \rightarrow 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \rightarrow \infty. \quad (3.6)$$

Hypotheses $\mathbf{H}(f)_1(i)$, (ii) , imply that

$$|f(z, x)| \leq C_2(1 + |x|^{p-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } C_2 > 0.$$

Since $\tau < q < p$, it follows that

$$\left\{ \frac{N_{g_+^{\lambda}}(u_n^+)}{\|u_n^+\|^{p-1}} \right\}_{n \geq 1} \subset L^{p'}(\Omega) \text{ is bounded.}$$

Using hypotheses $\mathbf{H}(f)_1(ii)$ and reasoning as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 13), by passing to a subsequence if necessary, we have

$$\frac{N_{g_+^\lambda}(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} hy^{p-1} \text{ in } L^{p'}(\Omega) \text{ and } \eta_1 \leq h \leq \eta_2. \quad (3.7)$$

In (3.4), we choose $v = y_n - y$, pass to the limit as $n \rightarrow \infty$ and use (3.5), (3.6) and (3.7). Then

$$\lim_{n \rightarrow \infty} \langle A_p(y_n), y_n - y \rangle = 0$$

hence

$$y_n \rightarrow y \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 6) and so } \|y\| = 1. \quad (3.8)$$

So, if in (3.4) we pass to the limit as $n \rightarrow \infty$ and use (3.5) – (3.8), then

$$\langle A_p(y), v \rangle = \int_{\Omega} hy^{p-1}v dz \text{ for all } z \in W_0^{1,p}(\Omega),$$

hence

$$A_p(y) = hy^{p-1},$$

and we conclude that

$$-\Delta_p y(z) = h(z)y(z)^{p-1} \text{ a.e. in } \Omega, \quad y|_{\partial\Omega} = 0. \quad (3.9)$$

Proposition 1 implies that

$$\begin{aligned} \widehat{\lambda}_1(p, h) &\leq \widehat{\lambda}_1(p, \eta_1) < \widehat{\lambda}_1\left(p, \widehat{\lambda}_1(p)\right) = 1 \text{ and} \\ \widehat{\lambda}_2(p, h) &\geq \widehat{\lambda}_1(p, \eta_2) > \widehat{\lambda}_2\left(p, \widehat{\lambda}_2(p)\right) = 1. \end{aligned} \quad (3.10)$$

Using (3.10) in (3.9), it follows that $y = 0$, which contradicts (3.8). This proves that $\{u_n^+\}_{n \geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. This fact together with (3.2) implies that $\{u_n\}_{n \geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^p(\Omega) \text{ as } n \rightarrow \infty. \quad (3.11)$$

In (3.1), we choose $v = u_n - u$, pass to the limit as $n \rightarrow \infty$ and use (3.11). Then

$$\lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \mu \langle A_q(u_n), u_n - u \rangle] = 0,$$

hence

$$\limsup_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \mu \langle A_q(u), u_n - u \rangle] \leq 0$$

(since $A_q(\cdot)$ is monotone) therefore

$$\limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle \leq 0, \text{ (see (3.11)),}$$

and by Proposition 5, we conclude that

$$u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega).$$

This proves that φ_+^λ satisfies the PS-condition. Similarly for φ_-^λ and φ_λ . \square

Proposition 7 *If hypotheses $\mathbf{H}(f)_1$ hold, then there exists $\lambda_+^* > 0$ such that for all $\lambda \in (0, \lambda_+^*)$ we can find $\rho_\lambda > 0$ for which we have*

$$\inf \{ \varphi_+^\lambda(u) : u \in \partial B_{\rho_\lambda} \} =: \eta_+^\lambda > 0$$

where $\partial B_{\rho_\lambda} = \{ u \in W_0^{1,p}(\Omega) : \|u\| = \rho_\lambda \}$.

Proof. Recall that from hypotheses $\mathbf{H}(f)_1(i), (ii)$, we have

$$|F(z, x)| \leq C_3(1 + |x|^p) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } C_3 > 0. \quad (3.12)$$

Then from hypothesis $\mathbf{H}(f)_1(iii)$ and (3.12) it follows that given $\varepsilon > 0$, we can find $C_4 = C_4(\varepsilon) > 0$ such that

$$F(z, x) \leq \frac{1}{p}(\theta(z) + \varepsilon)|x|^p + C_4|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } r \in (p, p^*). \quad (3.13)$$

For all $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \varphi_+^\lambda(u) &= \frac{1}{p} \|Du\|_p^p + \frac{\mu}{q} \|Du\|_q^q - \int_\Omega G_+^\lambda(z, u(z)) dz \\ &\geq \frac{1}{p} \left[\|Du\|_p^p - \int_\Omega \theta |u|^p dz \right] - \frac{\varepsilon}{p} \|u\|_p^p - \frac{\lambda}{\tau} \|u\|_\tau^\tau - C_5 \|u\|^r \text{ for some } C_5 > 0 \\ &\geq \frac{1}{p} \left[\widehat{\lambda}_1(p) \xi_0 - \varepsilon \right] \|u\|^p - \lambda C_6 \|u\|^\tau - C_5 \|u\|^r \text{ for some } C_6 > 0 \end{aligned} \quad (3.14)$$

(see Proposition 1 and (2.2)). Choosing $\varepsilon \in (0, \widehat{\lambda}_1(p) \xi_0)$, from (3.14) we have

$$\begin{aligned} \varphi_+^\lambda(u) &\geq C_7 \|u\|^p - \lambda C_6 \|u\|^\tau - C_5 \|u\|^r \text{ for some } C_7 > 0 \\ &= (C_7 - \lambda C_6 \|u\|^{\tau-p} - C_5 \|u\|^{r-p}) \|u\|^p. \end{aligned} \quad (3.15)$$

We consider the function

$$\gamma_\lambda(t) = \lambda C_6 t^{\tau-p} + C_5 t^{r-p}, \quad t > 0.$$

Evidently $\gamma_\lambda \in C^1(0, +\infty)$ and since $\tau < p < r$ (see (3.13)), we have

$$\gamma_\lambda(t) \rightarrow \infty \text{ as } t \rightarrow 0^+, t \rightarrow \infty.$$

So, we can find $t_0 \in (0, +\infty)$ such that

$$\gamma_\lambda(t_0) = \inf_{t>0} \gamma_\lambda(t).$$

Then $\gamma_\lambda'(t_0) = 0$, hence

$$t_0 = t_0(\lambda) = \left[\frac{\lambda C_6 (p - \tau)}{C_5 (r - p)} \right]^{\frac{1}{r-\tau}}.$$

Since $\tau < p < r$, it follows that

$$\gamma_\lambda(t_0(\lambda)) \rightarrow 0^+ \text{ as } \lambda \rightarrow 0^+.$$

Hence we can find $\lambda_+^* > 0$ such that for $\lambda \in (0, \lambda_+^*)$ we have $\gamma_\lambda(t_0(\lambda)) < C_6$ and so,

$$\varphi_+^\lambda(u) \geq \eta_+^\lambda > 0 \text{ for all } \|u\| = \rho_\lambda = t_0(\lambda)$$

(see (3.15)). □

In a similar fashion, we show an analogous result for the functional φ_-^λ .

Proposition 8 *If hypotheses $\mathbf{H}(f)_1$ hold then there exists $\lambda_-^* > 0$ such that for all $\lambda \in (0, \lambda_-^*)$ we can find $\widehat{\rho}_\lambda > 0$ for which we have*

$$\inf \{ \varphi_-^\lambda(u) : u \in \partial B_{\widehat{\rho}_\lambda} \} =: \eta_-^\lambda > 0$$

where $\partial B_{\widehat{\rho}_\lambda} = \{u \in W_0^{1,p}(\Omega) : \|u\| = \widehat{\rho}_\lambda\}$.

The next proposition completes the mountain pass geometry.

Proposition 9 *If hypotheses $\mathbf{H}(f)_1$ hold and $\lambda > 0$, then $\varphi_\pm^\lambda(t\widehat{u}_1(p)) \rightarrow -\infty$ as $t \rightarrow \pm\infty$.*

Proof. We do the proof for φ_+^λ , the proof for φ_-^λ being similar. hypotheses $\mathbf{H}(f)_1$ (i) (ii) imply that given $\varepsilon > 0$, we can find $C_8 = C_8(\varepsilon) > 0$ such that

$$F(z, x) \geq \frac{1}{p} (\eta_1(z) - \varepsilon) |x|^p - C_8 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \quad (3.16)$$

Then for all $t > 0$, we have

$$\begin{aligned} \varphi_+^\lambda(t\widehat{u}_{1,p}) &= \frac{t^p}{p} \widehat{\lambda}_1(p) + \frac{\mu t^q}{q} \|D\widehat{u}_{1,p}\|_q^q - \frac{\lambda t^\tau}{\tau} \|\widehat{u}_{1,p}\|_\tau^\tau - \int_\Omega F(z, t\widehat{u}_{1,p}(z)) dz \\ &\leq \frac{t^p}{p} \left[\int_\Omega (\widehat{\lambda}_1(p) - \eta_1(z)) \widehat{u}_{1,p}(z)^p dz + \varepsilon \right] + C_8 |\Omega|_N + \frac{\mu t^q}{q} \|D\widehat{u}_{1,p}\|_q^q \end{aligned} \quad (3.17)$$

(see (3.17) and recall that $\|\widehat{u}_{1,p}\|_p = 1$). Note that

$$\int_\Omega (\eta_1(z) - \widehat{\lambda}_1(p)) \widehat{u}_{1,p}(z)^p dz =: \beta^* > 0$$

and choose $\varepsilon \in (0, \beta^*)$. Since $q < p$ it follows from (3.17) that

$$\varphi_+^\lambda(t\widehat{u}_{1,p}) \rightarrow -\infty \text{ as } t \rightarrow +\infty.$$

Similarly we show that

$$\varphi_-^\lambda(t\widehat{u}_{1,p}) \rightarrow -\infty \text{ as } t \rightarrow -\infty.$$

□

Now we are ready to produce the first constant sign smooth solutions for problem (P_λ) . It what follows $\lambda_+^* > 0$ (respectively, $\lambda_-^* > 0$) is the critical parameter value produced in Proposition 7 (respectively, Proposition 8). Also, let

$$\lambda^* := \min \{ \lambda_+^*, \lambda_-^* \} > 0.$$

We have the following existence result for constant sign solutions for problem (P_λ) .

Proposition 10 *If hypotheses $\mathbf{H}(f)_1$ hold, then:*

- (a) *for every $\lambda \in (0, \lambda_+^*)$ problem (P_λ) has a positive solution $u_0 \in \text{int } C_+$;*
- (b) *for every $\lambda \in (0, \lambda_-^*)$ problem (P_λ) has a negative solution $v_0 \in -\text{int } C_+$;*
- (c) *for every $\lambda \in (0, \lambda^*)$ problem (P_λ) has two constant sign smooth solutions $u_0 \in \text{int } C_+$, $v_0 \in -\text{int } C_+$.*

Proof. Propositions 6, 7 and 9 allow the use of Theorem 1 (the mountain pass theorem). So, for $\lambda \in (0, \lambda_+^*)$, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_+^\lambda(0) = 0 < \eta_+^\lambda \leq \varphi_+^\lambda(u_0) \quad (3.18)$$

and

$$(\varphi_+^\lambda)'(u_0) = 0. \quad (3.19)$$

From (3.18) we see that $u_0 \neq 0$. From (3.19) we have

$$A_p(u_0) + \mu A_q(u_0) = N_{g_+^\lambda}(u_0). \quad (3.20)$$

Acting on (3.20) with $-u_0^- \in W_0^{1,p}(\Omega)$, we obtain $u_0 \geq 0$, $u_0 \neq 0$. So (3.20) becomes

$$A_p(u_0) + \mu A_q(u_0) = \lambda u_0^{\tau-1} + N_f(u_0),$$

hence

$$-\Delta_p u_0(z) - \mu \Delta_q u_0(z) = \lambda u_0(z)^{\tau-1} + f(z, u_0(z)) \quad \text{a.e. in } \Omega, u_0|_{\partial\Omega} = 0,$$

therefore $u_0 \in C_+ \setminus \{0\}$ (nonlinear regularity, see [25], [29]) solves problem (P_λ) .

Let $\rho = \|u_0\|_\infty$ and let ξ_ρ be as postulated by hypothesis $\mathbf{H}(f)_1(iv)$. Then,

$$\begin{aligned} & -\Delta_p u_0(z) - \mu \Delta_q(u_0) + \xi_\rho u_0(z)^{p-1} \\ & = \lambda u_0(z)^{\tau-1} + f(z, u_0(z)) + \xi_\rho u_0(z)^{p-1} \geq 0 \quad \text{a.e. in } \Omega, \end{aligned}$$

hence

$$\Delta_p u_0(z) + \mu \Delta_q(u_0) \leq \xi_\rho u_0(z)^{p-1} \quad \text{a.e. in } \Omega.$$

As before (see the proof of Proposition 3), using the results of Pucci-Serrin ([36], pp. 111, 120), we conclude that $u_0 \in \text{int } C_+$.

(b) The proof of this part is similar to that of (a).

(c) This is an immediate consequence of parts (a) and (b). \square

Next we look for additional nontrivial constant sign smooth solutions for problem (P_λ) . To this end we need the following proposition.

Proposition 11 *If hypotheses $\mathbf{H}(f)_1$ hold, then*

- (a) *for every $\lambda \in (0, \lambda_+^*)$ we have $\inf \{\varphi_+^\lambda(u) : \|u\| \leq \rho_\lambda\} < 0$;*
- (b) *for every $\lambda \in (0, \lambda_-^*)$ we have $\inf \{\varphi_-^\lambda(u) : \|u\| \leq \widehat{\rho}_\lambda\} < 0$.*

Proof. (a) Let $t \in (0, 1)$ be such that $t \|\widehat{u}_{1,p}\| \leq \rho_\lambda$. Let $\rho = \|\widehat{u}_{1,p}\|_\infty$ and let ξ_ρ be as postulated by hypothesis $\mathbf{H}(f)_1$ (iv). Then

$$f(z, x) + \xi_\rho x^{p-1} \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } x \in [0, \rho],$$

hence

$$F(z, t\widehat{u}_{1,p}(z)) + \frac{\xi_\rho}{p} t^p (\widehat{u}_{1,p}(z))^p \geq 0 \text{ a.e. in } \Omega. \quad (3.21)$$

So, we have

$$\begin{aligned} \varphi_+^\lambda(t\widehat{u}_{1,p}) &= \frac{t^p}{p} \|D\widehat{u}_{1,p}\| + \frac{\mu t^q}{q} \|D\widehat{u}_{1,p}\|_q^q - \frac{\lambda t^\tau}{\tau} \|\widehat{u}_{1,p}\|_\tau^\tau - \int_\Omega F(z, t\widehat{u}_{1,p}(z)) dz \\ &\leq \frac{t^p}{p} \left(\widehat{\lambda}_1(p) + \xi_\rho \right) + \frac{\mu t^q}{q} \|D\widehat{u}_{1,p}\|_q^q - \frac{\lambda t^\tau}{\tau} \|\widehat{u}_{1,p}\|_\tau^\tau. \end{aligned} \quad (3.22)$$

(see (3.21) and recall that $\|\widehat{u}_{1,p}\|_p = 1$). Since $\tau < q < p$, by choosing $t \in (0, 1)$ even smaller if necessary, from (3.22) we have

$$\varphi_+^\lambda(t\widehat{u}_{1,p}) < 0,$$

hence

$$\inf \{ \varphi_+^\lambda(u) : \|u\| \leq \rho_\lambda \} < 0.$$

(b) The proof of this part is similar to that of (a). \square

Using this proposition, we can produce two more nontrivial constant sign smooth solutions for problem (P_λ) .

Proposition 12 *If hypotheses $\mathbf{H}(f)_1$ hold, then:*

(a) *for every $\lambda \in (0, \lambda_+^*)$ problem (P_λ) has two nontrivial positive solutions $u_0, \widehat{u} \in \text{int } C_+$, with \widehat{u} being a local minimizer of φ_λ ;*

(b) *for every $\lambda \in (0, \lambda_-^*)$ problem (P_λ) has two nontrivial negative solutions $v_0, \widehat{v} \in -\text{int } C_+$, with \widehat{v} being a local minimizer of φ_λ ;*

(c) *for every $\lambda \in (0, \lambda^*)$ problem (P_λ) has four nontrivial constant sign smooth solutions $u_0, \widehat{u} \in \text{int } C_+, v_0, \widehat{v} \in -\text{int } C_+$, with \widehat{u}, \widehat{v} being a local minimizers of φ_λ ;*

Proof. (a) Let $d := \inf_{\partial B_{\rho_\lambda}} \varphi_+^\lambda - \inf_{\overline{B}_{\rho_\lambda}} \varphi_+^\lambda$ (see Propositions 7 and 11); here

$$\overline{B}_{\rho_\lambda} = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq \rho_\lambda\}.$$

Let $\varepsilon \in (0, d)$. Invoking the Ekeland variational principle (see, for example, Gasinski-Papageorgiou ([19], p. 579), we can find $u_\varepsilon \in \overline{B}_{\rho_\lambda}$ such that

$$\varphi_+^\lambda(u_\varepsilon) \leq \inf_{\overline{B}_{\rho_\lambda}} \varphi_+^\lambda + \varepsilon. \quad (3.23)$$

and

$$\varphi_+^\lambda(u_\varepsilon) \leq \varphi_+^\lambda(u) + \varepsilon \|u - u_\varepsilon\| \text{ for all } u \in \overline{B}_{\rho_\lambda}. \quad (3.24)$$

From (3.23) and since $\varepsilon < d$, we have

$$\varphi_+^\lambda(u_\varepsilon) < \inf_{\partial B_{\rho_\lambda}} \varphi_+^\lambda,$$

hence

$$u_\varepsilon \in B_{\rho_\lambda} = \{u \in W_0^{1,p}(\Omega) : \|u\| < \rho_\lambda\}, \quad u_\varepsilon \neq 0 \quad (3.25)$$

(see Proposition 11). Let $h \in W_0^{1,p}(\Omega)$ and $t \in (0, 1)$ small such that $u_\varepsilon + th \in \overline{B}_{\rho_\lambda}$ (see (3.25)). Then from (3.24) we have

$$-\varepsilon t \|h\| \leq \varphi_+^\lambda(u_\varepsilon + th) - \varphi_+^\lambda(u_\varepsilon)$$

hence

$$-\varepsilon \|h\| \leq \left\langle (\varphi_+^\lambda)'(u_\varepsilon), h \right\rangle,$$

therefore

$$\left\| (\varphi_+^\lambda)'(u_\varepsilon) \right\|_* \leq \varepsilon \quad (3.26)$$

(since $h \in W_0^{1,p}(\Omega)$ was arbitrary). Let $\varepsilon_n = \frac{1}{n}$ and $u_n = u_{\varepsilon_n}$, $n \geq 1$. Then

$$\varphi_+^\lambda(u_n) \rightarrow \inf_{\overline{B}_{\rho_\lambda}} \varphi_+^\lambda \text{ as } n \rightarrow \infty \text{ (see (3.23))}$$

and

$$(\varphi_+^\lambda)'(u_n) \rightarrow 0 \text{ in } W^{-1,p'}(\Omega) \text{ in as } n \rightarrow \infty \text{ (see (3.26))}.$$

From these convergences and Proposition 6, we infer that at least for a subsequence we have

$$u_n \rightarrow \widehat{u} \text{ in } W_0^{1,p}(\Omega) \text{ in as } n \rightarrow \infty,$$

hence (see Proposition 11)

$$\varphi_+^\lambda(\widehat{u}) = \inf_{\overline{B}_{\rho_\lambda}} \varphi_+^\lambda < 0 = \varphi_+^\lambda(0), \quad (\varphi_+^\lambda)'(\widehat{u}) = 0. \quad (3.27)$$

From (3.27) we have $\widehat{u} \in \overline{B}_{\rho_\lambda} \setminus \{0\}$ and

$$A_p(\widehat{u}) + \mu A_q(\widehat{u}) = N_{g_+^\lambda}(\widehat{u}). \quad (3.28)$$

Acting on (3.28) with $-\widehat{u}^- \in W_0^{1,p}(\Omega)$, we obtain $\widehat{u} \geq 0$, $\widehat{u} \neq 0$. Hence (3.20) becomes

$$A_p(\widehat{u}) + \mu A_q(\widehat{u}) = \lambda \widehat{u}^{\tau-1} + N_f(\widehat{u}),$$

hence $\widehat{u} \in \text{int } C_+$ solves problem (P_λ) (as before, see the proof of Proposition 10).

Since $\varphi_\lambda|_{C_+} = \varphi_+^\lambda|_{C_+}$, it follows that \widehat{u} is a local $C_0^1(\overline{\Omega})$ -minimizer of φ_λ , hence by virtue of Proposition 2, it is also a local $W_0^{1,p}(\Omega)$ -minimizer of φ_λ .

(b) The proof of this part is similar to that of (a).

(c) follows from parts (a) and (b). \square

4 Nodal solutions

In this section, we produce a nodal (sign-changing) solution for problem (P_λ) . In this way, we have the complete multiplicity result for problem (P_λ) producing at least five nontrivial smooth solutions, all with sign information.

To do this we need to strengthen a little the hypotheses on the perturbation $f(z, \cdot)$: The hypotheses on the nonlinearity $f(z, x)$ are the following:

$\mathbf{H}(f)_2$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ a.e. in Ω , and:

(i) for every $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leq a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) there exist $\eta_1, \eta_2 \in L^\infty(\Omega)_+$ such that

$$\begin{aligned} \eta_1(z) &\geq \widehat{\lambda}_1(p) \text{ a.e. in } \Omega, \eta_1 \neq \widehat{\lambda}_1(p), \eta_2(z) < \widehat{\lambda}_2(p) \text{ a.e. in } \Omega \\ \eta_1(z) &\leq \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \eta_2(z) \\ &\text{uniformly for a.a. } z \in \Omega; \end{aligned}$$

(iii) there exists $\theta \in L^\infty(\Omega)_+$, $\theta(z) \leq \widehat{\lambda}_1(p)$ a.e. in Ω , $\theta \neq \widehat{\lambda}_1(p)$ such that

$$\limsup_{x \rightarrow 0} \frac{f(z, x)}{|x|^{p-2}x} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega$$

(iv) $f(z, x)x \geq -\widehat{C}_0|x|^p$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, and with $\widehat{C}_0 > 0$.

Remarks: While $\mathbf{H}(f)_2$ (i), (ii) are the same as $\mathbf{H}(f)_1$ (i), (ii), both hypotheses $\mathbf{H}(f)_2$ (iii), (iv) are stronger than the corresponding hypotheses $\mathbf{H}(f)_1$ (iii), (iv). Hypothesis $\mathbf{H}(f)_2$ (iii) is in terms of the perturbation $f(z, x)$, in contrast with $\mathbf{H}(f)_1$ (iii) which is formulated in terms of the primitive $F(z, x)$. Therefore $\mathbf{H}(f)_2$ (iii) implies $\mathbf{H}(f)_1$ (iii). Also, hypothesis $\mathbf{H}(f)_2$ (iv) is global, hence is stronger than $\mathbf{H}(f)_1$ (iv).

We start by considering the following auxiliary Dirichlet (p, q) -problem:

$$-\Delta_p u(z) - \mu \Delta_q u(z) = \lambda |u(z)|^{\tau-2} u(z) - \widehat{C}_0 |u(z)|^{p-2} u(z) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (4.1)$$

Proposition 13 *For every $\lambda > 0$ problem (4.1) has a unique nontrivial positive solution $\underline{u} \in \text{int } C_+$ and a unique nontrivial negative solution $-\underline{u}$.*

Proof. Let $\psi_0^+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functional defined by

$$\psi_0^+(u) = \frac{1}{p} \|Du\|_p^p + \frac{\mu}{q} \|Du\|_q^q - \frac{\lambda}{\tau} \|u^+\|_\tau^\tau + \frac{\widehat{C}_0}{p} \|u^+\|_p^p.$$

Then

$$\psi_0^+(u) \geq \frac{1}{p} \|u\|^p - \lambda C_9 \|u\|^\tau \text{ for some } C_9 > 0.$$

Since $p > \tau$, it follows that ψ_0^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\underline{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi_0^+(\underline{u}) = \inf \{ \psi_0^+(u) : u \in W_0^{1,p}(\Omega) \} =: m_0^+. \quad (4.2)$$

Because $1 < \tau < q < p$, for $\xi \in (0, 1)$ small, we have

$$\psi_0^+(\xi \widehat{u}_{1,p}) < 0 = \psi_0^+(0).$$

Then

$$\psi_0^+(\underline{u}) < 0 = \psi_0^+(0)$$

hence $\underline{u} \neq 0$. From (4.2) we have

$$(\psi_0^+)'(\underline{u}) = 0$$

hence

$$A_p(\underline{u}) + \mu A_q(\underline{u}) = \lambda (u^+)^{\tau-1} - \widehat{C}_0 (u^+)^{p-1}.$$

Acting with $-\underline{u}^- \in W_0^{1,p}(\Omega)$ we obtain that $\underline{u} \geq 0$, $\underline{u} \neq 0$. Nonlinear regularity (see [25], [29]) and the results of Pucci-Serrin ([36], p.120), imply that $\underline{u} \in \text{int } C_+$ and solves problem (4.1).

Since $x \mapsto \frac{\lambda}{|x|^{q-\tau}} - \widehat{C}_0 |x|^{p-\tau}$ is strictly decreasing on $\mathbb{R} \setminus \{0\}$, we can apply Proposition 3 and conclude that $\underline{u} \in \text{int } C_+$ is the unique nontrivial positive solution of (4.1). The fact that (4.1) is odd implies that $-\underline{u} \in -\text{int } C_+$ is the unique nontrivial negative solution of (4.1). \square

With this proposition, we can establish the existence of extremal nontrivial constant sign solutions for problem (P_λ) with $\lambda \in (0, \lambda^*)$.

Proposition 14 *If hypotheses $\mathbf{H}(f)_2$ hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has a smallest nontrivial positive solution $u_* \in \text{int } C_+$ and a biggest nontrivial negative solution $v_* \in -\text{int } C_+$.*

Proof. Let \mathcal{S}_+ denote the set of nontrivial positive solutions of (P_λ) . From Proposition 12, we have $\mathcal{S}_+ \neq \emptyset$ and $\mathcal{S}_+ \subset \text{int } C_+$.

Claim: If $\tilde{u} \in \mathcal{S}_+$, then $\underline{u} \leq \tilde{u}$

Let $\zeta_+^\lambda : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be the Carathéodory function defined by

$$\zeta_+^\lambda(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda x^{\tau-1} - \widehat{C}_0 x^{p-1} & \text{if } 0 \leq x \leq \tilde{u}(z) \\ \lambda \tilde{u}(z)^{\tau-1} - \widehat{C}_0 \tilde{u}(z)^{p-1} & \text{if } \tilde{u}(z) < x \end{cases} \quad (4.3)$$

We set $Z_+^\lambda(z, x) = \int_0^x \zeta_+^\lambda(z, s) ds$ and consider the C^1 -functional $\xi_+^\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\xi_+^\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{\mu}{q} \|Du\|_q^q - \int_\Omega Z_+^\lambda(z, u(z)) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (4.3) it is clear that ξ_+^λ is coercive. Also, using the Sobolev embedding theorem, we can easily show that ξ_+^λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\xi_+^\lambda(\tilde{u}_0) = \inf \{ \xi_+^\lambda(u) : u \in W_0^{1,p}(\Omega) \} = m_+^\lambda. \quad (4.4)$$

As before (see the proof of Proposition 11), since $\tau < q < p$, we have

$$\xi_+^\lambda(t\tilde{u}_1) < 0 \text{ for } t \in (0, 1) \text{ small,}$$

hence

$$\xi_+^\lambda(\tilde{u}_0) = m_+^\lambda < 0 = \xi_+^\lambda(0), \text{ i.e., } \tilde{u}_0 \neq 0.$$

From (4.4) we have

$$A_p(\tilde{u}_0) + \mu A_q(\tilde{u}_0) = N_{\xi_+^\lambda}(\tilde{u}_0). \quad (4.5)$$

On (4.5) we act with $-\tilde{u}_0^- \in W_0^{1,p}(\Omega)$ we obtain $\tilde{u}_0 \geq 0$, $\tilde{u}_0 \neq 0$, while from Pucci-Serrin [36], we have $\tilde{u}_0 \in \text{int } C_+$. Next, on (4.5) we act with $(\tilde{u}_0 - \tilde{u})^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(\tilde{u}_0), (\tilde{u}_0 - \tilde{u})^+ \rangle + \mu \langle A_q(\tilde{u}_0), (\tilde{u}_0 - \tilde{u})^+ \rangle \\ &= \int_{\Omega} \zeta_+^\lambda(z, \tilde{u}_0(z)) (\tilde{u}_0 - \tilde{u})^+(z) dz \\ &= \int_{\Omega} \left(\lambda \tilde{u}(z)^{\tau-1} - \widehat{C}_0 \tilde{u}(z)^{p-1} \right) (\tilde{u}_0 - \tilde{u})^+(z) dz \text{ (see (4.3))} \\ &\leq \int_{\Omega} (\lambda \tilde{u}^{\tau-1} + f(z, \tilde{u})) (\tilde{u}_0 - \tilde{u})^+(z) dz \text{ (see } \mathbf{H}(f)_2(iv)) \\ &= \langle A_p(\tilde{u}), (\tilde{u}_0 - \tilde{u})^+ \rangle + \mu \langle A_q(\tilde{u}), (\tilde{u}_0 - \tilde{u})^+(z) \rangle, \end{aligned}$$

hence

$$\int_{\{\tilde{u}_0 > \tilde{u}\}} (\|D\tilde{u}_0\|^{p-2} D\tilde{u}_0 - \|D\tilde{u}\|^{p-2} D\tilde{u}, D\tilde{u}_0 - D\tilde{u})_{\mathbb{R}^N} \leq 0,$$

therefore

$$|\{\tilde{u}_0 > \tilde{u}\}|_N = 0, \text{ i.e., } \tilde{u}_0 \leq \tilde{u}.$$

So, we have proved that

$$\tilde{u}_0 \in [0, \tilde{u}] := \{u \in W_0^{1,p}(\Omega) : 0 \leq u(z) \leq \tilde{u}(z) \text{ a.e. in } \Omega\}$$

and

$$u_0 \in \text{int } C_+.$$

Then (4.5) becomes

$$A_p(\tilde{u}_0) + \mu A_q(\tilde{u}_0) = \lambda \tilde{u}_0^{\tau-1} - \widehat{C}_0 \tilde{u}_0^{p-1},$$

hence $\tilde{u}_0 \in \text{int } C_+$ is a solution of (4.1). We conclude that $\tilde{u}_0 = \underline{u}$ (see Proposition 13), therefore

$$\underline{u} \leq \tilde{u}.$$

This proves the Claim.

Now, let $C \subset \mathcal{S}_+$ be a chain (i.e., a totally ordered subset of \mathcal{S}_+). From Dunford-Schwartz ([15], p.336), we can find $\{u_n\}_{n \geq 1} \subset C$ such that

$$\inf C = \inf_{n \geq 1} u_n.$$

In fact, invoking Lemma 1.1.5 of Heikkila-Lakshmikantham [22], we may assume that $\{u_n\}_{n \geq 1} \subset C$ is decreasing. We have

$$A_p(u_n) + \mu A_q(u_n) = \lambda u_n^{\tau-1} + N_f(u_n), \quad \underline{u} \leq u_n \leq u_1 \text{ for all } n \geq 1 \quad (4.6)$$

(see the Claim), hence from (4.6), hypothesis $\mathbf{H}(f)_1(i)$ and the fact that $\tau < q < p$, it follows that

$$\{u_n\}_{n \geq 1} \subset W_0^{1,p}(\Omega) \text{ is bounded.}$$

Hence, by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^p(\Omega). \quad (4.7)$$

Acting on (4.6) with $u_n - u$, passing to the limit as $n \rightarrow \infty$ and using (4.7), as in the proof of Proposition 6, we obtain

$$\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

hence

$$u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 5)}$$

therefore

$$A_p(u) + \mu A_q(u) = \lambda u^{\tau-1} + N_f(u) \text{ and } \underline{u} \leq u \text{ (see (4.6))}, \quad (4.8)$$

and we conclude that $u \in \mathcal{S}_+$, and $u = \inf C$.

Since $C \subset \mathcal{S}_+$ was an arbitrary chain, invoking the Kuratowski-Zorn Lemma, we can find $u_* \in \mathcal{S}_+$, a minimal element of \mathcal{S}_+ . As in Filippakis-Kristaly-Papageorgiou [17] (see Lemma 4.3), we show that \mathcal{S}_+ is downward directed (i.e., if $u, y \in \mathcal{S}_+$, then there exists $v \in \mathcal{S}_+$ such that $v \leq u, v \leq y$). Hence $u_* \in \mathcal{S}_+$ is the smallest nontrivial positive solution of (P_λ) with $\lambda \in (0, \lambda^*)$.

Also let \mathcal{S}_- be the set of nontrivial negative solutions of (P_λ) . Again we have $\mathcal{S}_- \neq 0$ and $\mathcal{S}_+ \subset -\text{int } C_+$ (see Proposition 12). Moreover, \mathcal{S}_- is upward directed (i.e., if $u, y \in \mathcal{S}_-$, then there exists $v \in \mathcal{S}_-$ such that $u \leq v, y \leq v$; see Filippakis-Kristaly-Papageorgiou [17], Lemma 4.4). So, reasoning as above, via the Kuratowski-Zorn Lemma, we produce $v_* \in -\text{int } C_+$ the biggest nontrivial negative solution of (P_λ) with $\lambda \in (0, \lambda^*)$. \square

Now we are ready for the full multiplicity theorem for problem (P_λ) producing five nontrivial smooth solutions with precise sign information for all of them.

Theorem 2 *If hypotheses $\mathbf{H}(f)_2$ hold, then there exist $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least five nontrivial smooth solutions $u_0, \hat{u} \in \text{int } C_+, v_0, \hat{v} \in -\text{int } C_+$ and $y_0 \in C_0^1(\bar{\Omega})$ a nodal solution.*

Proof. From Proposition 12, we know that there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least four nontrivial smooth solutions $u_0, \widehat{u} \in \text{int } C_+$ and $v_0, \widehat{v} \in -\text{int } C_+$.

Let $u_* \in \text{int } C_+$ and $v_* \in -\text{int } C_+$ be the two extremal constant sign smooth solutions of (P_λ) produced in Proposition 14. We introduce the following truncation of the reaction in problem (P_λ) :

$$h_\lambda(z, x) = \begin{cases} \lambda |v_*(z)|^{\tau-2} v_*(z) + f(z, v_*(z)) & \text{if } x < v_*(z) \\ \lambda |x|^{\tau-2} x + f(z, x) & \text{if } v_*(z) \leq x \leq u_*(z) \\ \lambda u_*(z)^{\tau-1} + f(z, u_*(z)) & \text{if } u_*(z) < x \end{cases} \quad (4.9)$$

This is a Carathéodory function. We set $H_\lambda(z, x) = \int_0^x h_\lambda(z, s) ds$ and consider the C^1 -functional $\widehat{\psi}_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{\mu}{q} \|Du\|_q^q - \int_\Omega H_\lambda(z, u(z)) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also, we consider the positive and negative truncations of $h_\lambda(z, \cdot)$, namely

$$h_\lambda^\pm(z, x) = h_\lambda(z, \pm x^\pm).$$

Both are Carathéodory functions. We set $H_\lambda^\pm(z, x) = \int_0^x h_\lambda^\pm(z, s) ds$ and we consider the C^1 -functional $\widehat{\psi}_\lambda^\pm : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda^\pm(u) = \frac{1}{p} \|Du\|_p^p + \frac{\mu}{q} \|Du\|_q^q - \int_\Omega H_\lambda^\pm(z, u(z)) dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Reasoning as in the proof of Proposition 14, we show that

$$K_{\widehat{\psi}_\lambda^+} \subset [0, u_*], \quad K_{\widehat{\psi}_\lambda^-} \subset [v_*, 0] \text{ and } K_{\widehat{\psi}_\lambda} \subset [v_*, u_*].$$

The extremality of the solutions u_* and v_* implies that

$$K_{\widehat{\psi}_\lambda^+} = \{0, u_*\}, \quad K_{\widehat{\psi}_\lambda^-} = \{0, v_*\} \text{ and } K_{\widehat{\psi}_\lambda} \subset [v_*, u_*]. \quad (4.10)$$

Claim: u_* and v_* are local minimizers of $\widehat{\psi}_\lambda$.

From (4.9) it is clear that $\widehat{\psi}_\lambda^+$ is coercive. Also, it is sequentially weakly lower semi-continuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\lambda^+(\tilde{u}) = \inf \left\{ \widehat{\psi}_\lambda^+(u) : u \in W_0^{1,p}(\Omega) \right\}. \quad (4.11)$$

As in the proof of Proposition 11, since $\tau < q < p$, we have

$$\widehat{\psi}_\lambda^+(\tilde{u}) < 0 = \widehat{\psi}_\lambda^+(0), \text{ i.e. } \tilde{u} \neq 0,$$

hence

$$\tilde{u} = u_* \in \text{int } C_+ \text{ (see (4.10)).}$$

Since $\widehat{\psi}_\lambda^+|_{C_+} = \widehat{\psi}_\lambda|_{C_+}$, it follows that u_* is a local $C_0^1(\Omega)$ -minimizer of $\widehat{\psi}_\lambda$ (see (4.11)). So, by virtue of Proposition 2, u_* is also a local $W_0^{1,p}(\Omega)$ -minimizer of $\widehat{\psi}_\lambda$. Similarly for $v_* \in -\text{int } C_+$ using this time $\widehat{\psi}_\lambda^-$. This proves the Claim.

Without any loss of generality, we may assume that

$$\widehat{\psi}_\lambda(v_*) \leq \widehat{\psi}_\lambda(u_*) \quad (4.12)$$

(the analysis is similar if the opposite inequality holds). Because of the Claim and reasoning as in Aizicovici-Papageorgiou-Staicu [1] (the proof of Proposition 29), we can find $\rho_0^\lambda > 0$ small such that

$$\widehat{\psi}_\lambda(v_*) \leq \widehat{\psi}_\lambda(u_*) < \inf \left\{ \widehat{\psi}_\lambda(u) : \|u - u_*\| = \rho_0^\lambda \right\} =: \eta_0^\lambda, \quad \|v_* - u_*\| > \rho_0^\lambda. \quad (4.13)$$

Since $\widehat{\psi}_\lambda$ is coercive (see (4.9)), it satisfies the PS-condition. This fact and (4.13) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $y_0 \in K_{\widehat{\psi}_\lambda}$ such that

$$\widehat{\psi}_\lambda(v_*) \leq \widehat{\psi}_\lambda(u_*) < \eta_0^\lambda \leq \widehat{\psi}_\lambda(y_0),$$

hence

$$y_0 \notin \{v_*, u_*\}. \quad (4.14)$$

Because $y_0 \in K_{\widehat{\psi}_\lambda}$ is of mountain pass type, we have

$$C_1(\widehat{\psi}_\lambda, y_0) \neq 0 \quad (4.15)$$

(see Chang [11], p.89). By virtue of hypothesis $\mathbf{H}(f)_2$ (iii), we can find $\widehat{C}_1 > 0$ and $\delta > 0$ such that

$$f(z, x) x \leq \widehat{C}_1 |x|^p \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta.$$

Hence, if $\mu \in (\tau, p)$ and by choosing $\delta \in (0, 1)$ even smaller if necessary, we have

$$\begin{aligned} \lambda \left(\frac{\mu}{\tau} - 1 \right) |x|^\tau &\geq \widehat{C}_2 |x|^p \geq f(z, x) x - \mu F(z, x) \text{ for a.a. } z \in \Omega, \\ &\text{all } |x| \leq \delta, \text{ and some } \widehat{C}_2 > 0, \end{aligned}$$

hence

$$\lambda \frac{\mu}{\tau} |x|^\tau + \mu F(z, x) \geq -1 |x|^\tau \geq \lambda |x|^\tau + f(z, x) x \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta.$$

From this and hypothesis $\mathbf{H}(f)_2$ (iv), we see that we can apply Proposition 2.1 of Jiu-Su [24] and infer that

$$C_k(\widehat{\psi}_\lambda, 0) = 0 \text{ for all } k \geq 0. \quad (4.16)$$

Comparing (4.15) and (4.16), we infer that $y_0 \neq 0$. Since $y_0 \in K_{\widehat{\psi}_\lambda} \subset [v_*, u_*]$ (see (4.15)), by virtue of the extremality of v_*, u_* , we infer that $y_0 \in C_0^1(\overline{\Omega})$ (nonlinear regularity), is a nodal solution of (P_λ) (see (4.10)). \square

A careful reading of the proof reveals that Theorem 2 remains valid under the following slightly different set of hypotheses:

$\mathbf{H}(f)_3$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ a.e. in Ω , hypotheses $\mathbf{H}(f)_3(i), (iii), (iv)$ are the same as the corresponding hypotheses $\mathbf{H}(f)_2(i), (iii), (iv)$ and:

(ii) there exist $\widehat{\eta} \notin \widehat{\sigma}(p, 1)$ (the spectrum of $(-\Delta_p^D, 1)$), $\widehat{\eta} > \widehat{\lambda}_1(p)$ such that

$$\lim_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2} x} = \widehat{\eta} \text{ uniformly for a.a. } z \in \Omega.$$

Remark: Concerning hypothesis $\mathbf{H}(f)_3(ii)$, we mention that from the point of view of the spectral theory of $-\Delta_p^D$, it is not excluded the possibility that, for some domains Ω , we will have $\widehat{\sigma}(p, 1) = \{\widehat{\lambda}_1(p)\} \cup [\widehat{\lambda}_2(p), +\infty)$. We mention that this can not happen if $p = 2$ or if $N = 1$ (ordinary differential equations). Nevertheless, such hypothesis has been used in the literature, see Cingolani-Degiovanni [12], Liu-Li [30], Medeiros-Perera [31].

We can state the following multiplicity theorem.

Theorem 3 *If hypotheses $\mathbf{H}(f)_3$ hold, then there $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least five nontrivial smooth solutions $u_0, \widehat{u} \in \text{int } C_+, v_0, \widehat{v} \in -\text{int } C_+$ and $y_0 \in C_0^1(\overline{\Omega})$ nodal.*

5 $(p, 2)$ –equations

In this section we deal with the following special case of problem (P_λ) :

$$-\Delta_p u(z) - \mu \Delta u(z) = \lambda |u(z)|^{\tau-2} u(z) + f(u(z)) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (\widehat{P}_\lambda)$$

Here $1 < \tau < 2 = q < p, \mu > 0$ and $\lambda > 0$ is a parameter. The hypotheses on the perturbation $f(x)$ are the following:

$\mathbf{H}(f)_4$: $f \in C^1(\mathbb{R}), f(0) = f'(0) = 0$ and

- (i) $|f'(x)| \leq C|x|^{p-2}$ for all $x \in \mathbb{R}$ with $C > 0$;
- (ii) there exists $\widehat{\eta} \notin \sigma(p, 1), \widehat{\eta} > \widehat{\lambda}_2(p)$ such that

$$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{|x|^{p-2} x} = \widehat{\eta};$$

- (iii) there exists $\theta < \widehat{\lambda}_1(p)$ such that

$$\limsup_{x \rightarrow 0} \frac{f(x)}{|x|^{p-2} x} \leq \theta;$$

- (iv) $f(x)x \geq -\widehat{C}_0|x|^p$ for all $x \in \mathbb{R}$ and some $\widehat{C}_0 > 0$, and for every $\rho > 0$ there exists $\xi_\rho > 0$ such that $x \mapsto f(x) + \xi_\rho|x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

We can prove the following multiplicity theorem producing six nontrivial smooth solutions. However, we are not able to specify the sign of the sixth solution.

Theorem 4 *If hypotheses $\mathbf{H}(f)_4$ hold, then exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (\widehat{P}_λ) has at least six nontrivial smooth solutions $u_0, \widehat{u} \in \text{int } C_+$, $v_0, \widehat{v} \in -\text{int } C_+$ and $y_0 \in C_0^1(\overline{\Omega})$ nodal, and $\widehat{v} \in C_0^1(\overline{\Omega}) \setminus \{0\}$.*

Proof. From Theorem 3, we know that we can find $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (\widehat{P}_λ) has at least five nontrivial smooth solutions $u_0, \widehat{u} \in \text{int } C_+$, $v_0, \widehat{v} \in -\text{int } C_+$ and $y_0 \in C_0^1(\overline{\Omega})$ nodal.

Recall that $y_0 \in [v_*, u_*]$ with v_*, u_* being the extremal constant sign solutions from Proposition 14. Let

$$\rho = \max \{ \|v_*\|_\infty, \|u_*\|_\infty \}$$

and let $\xi_\rho > 0$ be as postulated by hypothesis $\mathbf{H}(f)_4(iv)$. We have

$$\begin{aligned} & -\Delta_\rho u_*(z) - \mu \Delta u_*(z) + \xi_\rho u_*(z)^{p-1} \\ &= \lambda u_*(z)^{\tau-1} + f(u_*(z)) + \xi_\rho u_*(z)^{p-1} \\ &\geq \lambda |y_0(z)|^{\tau-2} y_0(z) + f(y_0(z)) + \xi_\rho |y_0(z)|^{p-2} y_0(z) \quad (5.1) \\ &\quad (\text{see } \mathbf{H}(f)_4(iv) \text{ and recall that } y_0 \leq u_*) \\ &= -\Delta_\rho y_0(z) - \mu \Delta y_0(z) + \xi_\rho |y_0(z)|^{p-2} y_0(z) \text{ a.e. in } \Omega. \end{aligned}$$

From the tangency principle of Pucci-Serrin ([36], p.35), we know that

$$y_0(z) < u_*(z) \text{ for all } z \in \Omega.$$

Hence, we have

$$\begin{aligned} \widehat{h}(z) &:= \lambda u_*(z)^{\tau-1} + f(u_*(z)) + \xi_\rho u_*(z)^{p-1} \\ &> \lambda |y_0(z)|^{\tau-2} y_0(z) + f(y_0(z)) + \xi_\rho |y_0(z)|^{p-2} y_0(z) =: h(z) \text{ for all } z \in \Omega, \end{aligned}$$

and $h, \widehat{h} \in C(\Omega)$. Therefore, because of (5.1), we can apply Proposition 4 and obtain

$$u_* - y_0 \in \text{int } C_+.$$

Similarly we show that

$$y_0 - v_* \in \text{int } C_+.$$

So, we see that

$$y_0 \in \text{int}_{C_0^1(\overline{\Omega})} [v_*, u_*],$$

that is, y_0 belongs to the interior of $[v_*, u_*] |_{C_0^1(\overline{\Omega})}$, hence there is $\varepsilon > 0$ such that

$$B_\varepsilon^{C_0^1(\overline{\Omega})}(y_0) := \left\{ h \in C_0^1(\overline{\Omega}) : \|h - y_0\|_{C_0^1(\overline{\Omega})} < \varepsilon \right\} \subseteq [v_*, u_*].$$

Let

$$\tilde{h}_\lambda(t, u) = t\varphi_\lambda(u) + (1-t)\widehat{\psi}_\lambda(u) \text{ for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Claim: We may assume that there exists $\rho > 0$ such that y_0 is the isolated critical point of $\left\{ \tilde{h}_\lambda(t, \cdot) \right\}_{t \in [0,1]}$ in $\overline{B}_\rho(y_0) := \{u \in W_0^{1,p}(\Omega) : \|u - y_0\| \leq \rho\}$.

Otherwise, we can find $\{t_n\}_{n \geq 1} \subset [0, 1]$ and $\{u_n\}_{n \geq 1} \subset W_0^{1,p}(\Omega)$ such that

$$t_n \rightarrow t, u_n \rightarrow y_0 \text{ in } W_0^{1,p}(\Omega) \text{ and } \frac{\partial}{\partial u} \tilde{h}_\lambda(t_n, u_n) = 0 \text{ for all } n \geq 1. \quad (5.2)$$

We have

$$A_p(u_n) + \mu A_2(u_n) = t_n \lambda u_n + t_n N_f(u_n) + (1-t_n) N_{h_\lambda}(u_n)$$

(see (5.2)). From Lieberman [29], we know that we can find $\beta \in (0, 1)$ and $M_1 > 0$ such that

$$u_n \in C_0^{1,\beta}(\overline{\Omega}) \text{ and } \|u_n\|_{C_0^{1,\beta}(\overline{\Omega})} \leq M_1 \text{ for all } n \geq 1. \quad (5.3)$$

Exploiting the compact embedding of $C_0^{1,\beta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and using (5.2) and (5.3) we obtain

$$u_n \rightarrow y_0 \text{ in } C_0^1(\overline{\Omega}).$$

Since $y_0 \in \text{int}_{C_0^1(\overline{\Omega})}[v_*, u_*]$, it follows that there exists $n_0 \geq 1$ such that $u_n \in [v_*, u_*]$ for all $n \geq n_0$. Then by virtue of (4.9), it follows that $\{u_n\}_{n \geq 1} \subset C_0^1(\overline{\Omega})$ are all distinct solutions of (\widehat{P}_λ) and so, we are done. Therefore the Claim holds.

By virtue of the Claim and the homotopy invariance of critical groups, we have

$$C_k(\varphi_\lambda, y_0) = C_k(\widehat{\psi}_\lambda, y_0) \text{ for all } k \geq 0,$$

hence

$$C_1(\varphi_\lambda, y_0) \neq 0 \text{ (see (4.15))},$$

therefore

$$C_k(\varphi_\lambda, y_0) = \delta_{k,1} \mathbb{Z} \text{ for all } k \geq 0 \quad (5.4)$$

(see Gasinski-Papageorgiou [20]). Let $\widehat{\eta} > \widehat{\lambda}_2(p)$ be as in $\mathbf{H}(f)_4(ii)$ and let $\psi_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{\mu}{2} \|Du\|_2^2 - \frac{\lambda}{q} \|u\|_q^q - \frac{\widehat{\eta}}{p} \|u\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

From [30] (Lemma 3.1) we know that there exist $R > 0$ and $\widehat{\varphi}_\lambda \in C^1(W_0^{1,p}(\Omega))$ such that

$$\widehat{\varphi}_\lambda(u) = \begin{cases} \psi_\lambda(u) & \text{if } \|u\| \geq 2^{\frac{1}{p}} R \\ \varphi_\lambda(u) & \text{if } \|u\| \leq R, \end{cases}$$

$$\inf \{ \widehat{\varphi}'_\lambda(u) : \|u\| \geq R \} > 0$$

and so,

$$K_{\widehat{\varphi}_\lambda} \subseteq K_{\varphi_\lambda}.$$

Hypothesis $\mathbf{H}(f)_4(ii)$ implies that

$$C_d(\widehat{\varphi}_\lambda, \infty) \neq 0 \text{ for some } d \geq 2. \quad (5.5)$$

(see Theorem 5.9 of Perera-Agarwal-O'Regan [35]). Alternatively, to see (5.5), let

$$\sigma(u) = \frac{1}{p} \|Du\|_p^p - \frac{\widehat{\eta}}{p} \|u\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega);$$

since $2 < p$, using Lemma 8 of O'Regan-Papageorgiou-Smyrlis [32], we have

$$C_k(\psi_\lambda, \infty) = C_k(\sigma, \infty) \text{ for all } k \geq 0$$

and

$$C_k(\sigma, \infty) = C_k(\sigma, 0) \quad (5.6)$$

since $K_\sigma = \{0\}$ (recall $\widehat{\eta} \notin \sigma(p, 1)$), with

$$C_d(\sigma, 0) \neq 0 \text{ for some } d \geq 2 \quad (5.7)$$

(since $\widehat{\eta} > \widehat{\lambda}(p)$, see Liu-Li [30], p. 85).

Recall that \widehat{u}, \widehat{v} are local minimizers of φ_λ (see Proposition 12). Hence

$$C_k(\varphi_\lambda, \widehat{u}) = C_k(\varphi_\lambda, \widehat{v}) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \quad (5.8)$$

Since $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$ and $\varphi_\lambda|_{C_+} = \varphi_+^\lambda|_{C_+}$, $\varphi_\lambda|_{-C_+} = \varphi_-^\lambda|_{-C_+}$, by virtue of the homotopy invariance of critical groups, we have

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_+^\lambda, u_0) \text{ for all } k \geq 0 \quad (5.9)$$

and

$$C_k(\varphi_\lambda, v_0) = C_k(\varphi_-^\lambda, v_0) \text{ for all } k \geq 0. \quad (5.10)$$

Since $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$ are critical points of mountain pass type for φ_+^λ and φ_-^λ , respectively, from Gasinski-Papageorgiou [20] (see the proof of Theorem 4.1) we have

$$C_k(\varphi_+^\lambda, u_0) = C_k(\varphi_-^\lambda, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0,$$

hence

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0, \quad (5.11)$$

(see (5.9), (5.10)). Finally, since $\varphi_\lambda|_{[v_*, u_*]} = \widehat{\psi}_\lambda|_{[v_*, u_*]}$ from (4.16) we have

$$C_k(\varphi_\lambda, 0) = 0 \text{ for all } k \geq 0. \quad (5.12)$$

From (5.5) it follows that there exists $\widehat{y} \in K_{\widehat{\varphi}_\lambda} \subseteq C_0^1(\overline{\Omega})$ such that

$$C_d(\widehat{\varphi}_\lambda, \widehat{y}) \neq 0,$$

hence

$$C_d(\varphi_\lambda, \hat{y}) \neq 0 \text{ and } \hat{y} \in K_{\hat{\varphi}_\lambda} \quad (5.13)$$

Comparing (5.13) with (5.4), (5.8), (5.11), (5.12), we see that

$$\hat{y} \notin \{0, u_0, \hat{u}, v_0, \hat{v}, y_0\}$$

hence $\hat{y} \in C_0^1(\overline{\Omega}) \setminus \{0\}$ (by nonlinear regularity) solves (\hat{P}_λ) . \square

Remark: An alternative approach to show that $C_d(\varphi_\lambda, \infty) \neq 0$ is the following. Approximate the concave term $\lambda|x|^{\tau-2}x$ uniformly near zero by a locally Lipschitz function (see Lasota-Yorke [26]). Then the corresponding energy functional $\hat{\varphi}_\lambda$ is a C^2 function with $u \rightarrow \hat{\varphi}'_\lambda(u)$ locally Lipschitz. Then using the homotopy

$$\hat{h}_\lambda(t, u) = t\hat{\varphi}_\lambda(u) + (1-t)\psi_\lambda(u) \text{ for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega)$$

with $\psi_\lambda(u)$ as above, and using Lemma 8 of O'Regan-Papageorgiou-Smyrlis [32], we have

$$C_k(\hat{\varphi}_\lambda, \infty) = C_k(\psi_\lambda, \infty) \text{ for all } k \geq 0.$$

hence

$$C_d(\hat{\varphi}_\lambda, \infty) \neq 0 \text{ for some } d \geq 2,$$

therefore

$$C_d(\varphi_\lambda, \infty) \neq 0$$

from the C^1 -continuity of critical groups (see [11]).

It is an interesting open problem whether the sixth solution \hat{y} is nodal too.

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